REPORT RESUMES

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THE MAN MADE WORLD, A HIGH SCHOOL COURSE ON THE THEORIES AND TECHNIQUES WHICH CONTRIBUTE TO OUR TECHNOLOGICAL CIVILIZATION.

COMMISSION ON ENGINEERING EDUC., WASHINGTON, D.C.

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THIS STUDENTS' MANUAL FOR THE ENGINEERING CONCEPTS CURRICULUM PROJECT'S (ECCP) HIGH SCHOOL COURSE, "THE MAN MADE WORLD," IS THE THIRD DRAFT OF THE EXPERIMENTAL VERSION. THE MATERIAL WRITTEN BY SCIENTISTS, ENGINEERS, AND EDUCATORS, EMPHASIZES THE THEORIES AND TECHNIQUES WHICH CONTRIBUTE TO OUR TECHNOLOGICAL CIVILIZATION. RESOURCES OF THE MAN-MADE WORLD ARE ALSO STRESSED--CONCEPTS, PHYSICAL PRINCIPLES, MODES OF THINKING, SCIENTIFIC METHODS, ARTS, SKILLS, AND INSPIRATIONS. THE INSIGHTS THAT ARE DEVELOPED THROUGHOUT THE COURSE ARE ENVISIONED AS BEING USEFUL IN HELPING THE STUDENT TO COPE WITH THE REAL WORLD OF SOCIAL, ECONOMIC, POLITICAL, AND TECHNICAL PROBLEMS. CENTRAL IN THE COURSE IS THE THEME OF MAN'S ABILITY TO SHAPE HIS OWN FUTURE, AND HIS ABILITY TO COMMUNICATE VAST AMOUNTS OF ACCUMULATED KNOWLEDGE EFFICIENTLY AND SPEEDILY. THE COMPUTER IS USED AS THE OBJECT OF EXAMINATION BECAUSE OF ITS INVOLVEMENT IN ALL PHASES OF MAN'S ENDEAVOR, AND ITS IMPORTANCE TO ALL PHASES OF FUTURE DEVELOPMENTS. TOPICS IN PART I ARRANGED ACCORDING TO CHAPTERS ARE (1) INTRODUCTION, (2) LOGICAL THOUGHT AND LOGIC CIRCUITS, (3) BINARY NUMBERS AND LOGIC AND CIRCUITS, (4) LOGIC CIRCUITS WITH MEMORY, (5) ORGANIZATION OF A COMPUTER, AND (6) PROGRAMING. PART II DEALS WITH (1) DECISION MAKING, (2) OPTIMIZATION (OPERATIONS RESEARCH), (3) MODELING, (4) MODELS AND THE ANALOG COMPUTER, (5) PATTERNS OF CHANGE, AND (6) CHANGE IN DRIVEN SYSTEMS. PART III INCLUDES (1) FEEDBACK, (2) AMPLIFICATION, AND (3) STABILITY. CHAPTERS INCLUDE PROBLEMS, BUT LABORATORY EXPERIMENTS ARE CONTAINED IN A SEPARATE LABORATORY MANUAL. (DH)

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
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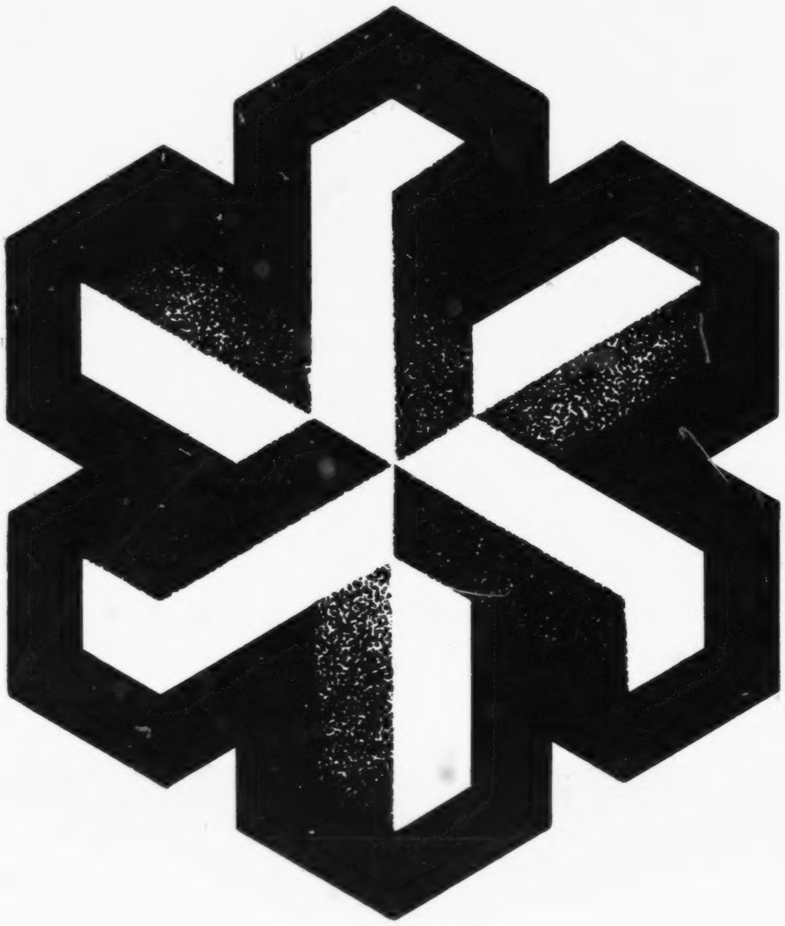

The
Man
Made
World

A high school course / developed by
Engineering Concepts Curriculum Project

A Program of the Commission on
Engineering Education / Washington, D.C.

SE004 216

The
Man
Made
World

Third Draft

A high school course on the theories and techniques which contribute to our technological civilization.

Developed by the Engineering Concepts Curriculum Project.

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PREFACE TO SECOND DRAFT

The ECCP course, entitled "The Man-Made World", is intended as a part of the cultural curriculum. It is a course for embryo journalists, businessmen, lawyers, medical doctors, executives, teachers, and in fact for all citizens who will take part in guiding the currents of our society. The authors are engineers, scientists, and educators who are convinced that the world is increasingly shaped by technical accomplishments. Indeed, the world we live in is largely man-made. Within the short space of only several dozen years man has discovered more about the world in which he lives than he had known from all the preceding generations since the beginning. With this knowledge came a new boldness. Man now has the faith that he can shape his own environment and that he need not leave his destiny to chance. His knowledge, ingenuity, and vigor give him the power to change the world toward what he wants it to be, rather than having to accept it as it is. This is the engineering viewpoint, for the engineer thinks of the world in terms of how it can be manipulated to serve man.

Resources used to create the man-made world are diverse. There are concepts, physical principles, modes of thinking, and the much heralded "scientific method" as well as arts, skills, and inspiration. This course brings these into focus by reference to vital technical accomplishments, but the course also strives to demonstrate the relevance of these resources for biology, economics, sociology, business, communication, psychology, and even the arts and humanities. In emphasizing the utility of precise thought and language the course does not overlook the importance of procedures and techniques for achieving concrete goals, nor the importance of value judgments in deciding "what to do" from the vast number of possibilities. Through this broad approach, the course aims to help students develop insights useful in coping with social, economic, political as well as purely technical problems.

The accompanying text reflects revisions of an earlier version along the lines indicated by the 1965-66 pilot trial in five high schools. This revised text, although nearer to the ultimate goal, will undoubtedly require further revision as a result of experience during the 28-school, 1966-67 trial. It is expected that students and teachers from these schools will make significant contributions to this revision.

An unusual combination of academic interests and professional viewpoints is being brought to bear on the creation of this course. So that it can be made to appeal to a broad spectrum of the student body and, at the same time, be academically authentic, the Commission on Engineering Education has brought together college professors, high school science teachers, engineers, and scientists. The Polytechnic Institute of Brooklyn, the Massachusetts Institute of Technology, and the Johns Hopkins University have been major contributors in cooperation with scientists and engineers from Bell Telephone Laboratories and the International Business Machines Corporation. Listed on the following page are the names of individuals who have participated to date.

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An additional year of test in 27 schools has resulted in some changes in the content and approach. These changes have not materially altered the direction or emphasis of the course, but do reflect the suggestions of the 1966-67 trial teachers. There are more references to the effects of technology on Society, and more examples of how the concepts which are taught in the course can be used in areas of endeavor other than engineering.

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TO THE STUDENTS

From time to time during this school year you will be asked various questions about your reasons for taking this course, how effective the text has been, how important the laboratory exercises are, etc. This is because the authors want to know what you are getting out of the course. It is only fair, therefore, that you know in advance why the course was written, and what it is hoped you will "get out of" the course.

The following quotation from Theodore J. Gordon of the Rand Corporation in the magazine ARCHITECTURAL DESIGN, Feb., 1967 might help you understand the reasons for developing this course.

"As scientific research progresses we gain greater control over our environment. Yet the power of these tools which give us this control, in the hands of an unprepared or indifferent people, faced with social pressures of unprecedented magnitude, may result not in greater control but self-extinction."

As citizens of both the 20th and 21st centuries, you will need to gain an insight into the nature of the tools and processes which have affected the society in which you now live and will help to shape the society in which you will be living. "Only through this kind of projection can we hope to avoid social calamities which may result from the sudden emergence of powerful mechanisms of control without previous preparation or understanding of the implications of their use. Prediction-making is clearly part of our everday life." A few of the effects of prediction-making devices are cited.

"Computers are used during political elections to predict results, based on a small (but carefully selected) sampling. In rocketry, a computer technique known as "Monte Carlo" is used to predict the probable path of a rocket. Operations analysis and operations research are very powerful tools. The technical, military, commercial, social and political planning of our world is becoming increasingly related to prediction-making."

During the 1970's (while you are still in college) while air traffic increases, the danger of mid-air collisions will be reduced because by the use of accurate sensing devices, coupled to computers, air traffic control will develop positive and predictive tracks on all aircraft.

You may have already noted a decrease in the use of cash as the credit card becomes more widespread; this will continue until there will be a direct link from stores to banks in order to check credit and record transactions.

At the present time teaching machines are in a minority of school classrooms, but by 1975 there will be widespread use of simple teaching machines.

As factories have become more automated there has been an influx of white collar workers to offices. Automation of office work and services will progress to the point that there will be a 25% displacement of the work force (though some economists think that these people will be absorbed by the general growth of business).

Libraries will become automated, looking up and reproducing copy on demand.

There will be automated looking up of legal information, automatic language translation (with correct grammar) and automated rapid transit for metropolitan areas.

At present we hear the weather reports given as odds, "there is a 40% chance of scattered showers this afternoon", by 1975 there will be accurate, reliable weather forecasts.

Other predictions suggest that, in the 1980's (when you are in your 30's) there will be:

Automatic decision-making at management level for industrial and national planning.

Radar for the blind, servo-mechanical arms and legs, etc.

Automated interpretation of medical symptons.

Widespread use of robot services for refuse collection, household chores, sewer inspections, etc.

In this course you will not learn how to design the equipment or processes which will make the above predictions come true, but rather you will learn something about such devices and processes so that as a citizen you might make wise decisions regarding them. For example, you will learn about logic and how logic can be represented in simple electric circuits and how many of these simple circuits can be combined to make a large computer. By actually programming a simple computer, you will learn something about how man communicates with machines. Understanding the process of decision-making based on a systematic procedure (algorithm) will help you in many areas of your personal and professional life. You will learn about the values and pitfalls of predictions based on models. The concept of feedback and how it controls what you do physically and intellectually as well as how it is used to control "automated" devices and processes is one which should be understood by all citizens today. Amplification of man's meager energy is the process by which our civilization has developed from the time of the cave man to the space flights of today and tomorrow. The understanding of the concept of stability and its effect on economic, social and medical problems as well as on bridges and autos going around curves is important today and will continue to be important tomorrow.

There are many engineering concepts which can be understood by all people who are interested in them. Many are in this course, many have been omitted owing to the lack of time and space in the school program. If you understand the concepts presented in this course, and apply them today, and tomorrow, your time will have been well spent.

Chapter A-1

INTRODUCTION

Some people say that the thing that distinguishes man from the animals is his use of tools to help him cope with nature and to live comfortably. Regardless of the truth of this idea, tools are important and apes do not have automobiles, bulldozers, or even picks and shovels. However, as important as tools are, man's language is perhaps even more important. Language is the key to communication between men, and it has permitted man to accumulate knowledge over the centuries so that each generation can stand on the shoulders of all earlier ones.

Basic to language are symbols; among them, numbers and letters. Numbers stand for a property of a collection of objects; namely, "how many". Letters stand for the sounds of speech. All symbols stand for, or "symbolize", something; and man uses them in his thought and communication. Among man's tools, those that deal with symbols, such as printing presses, typewriters, and copying machines, have been particularly influential. "The pen is mightier than the sword" reflects this importance of symbols and language.

During the past 25 years, the ultimate tool for dealing with symbols has emerged; it is the electronic digital computer. Bridging the gap between thought and action, computers not only reproduce and print letters, numbers, and drawings as presses and copying machines do, but they can change and manipulate these symbols. Manuscripts can be edited, word counts compiled, equations solved, and pictures rearranged. Computers can handle an almost unlimited range of symbols, and so can engage in "computer make-believe," commonly called simulation. Auto traffic flow, missile flight, and ballet choreography can be recreated and followed through in a computer, and valuable experience obtained, all without actual automobiles, missiles, or dancers. Computers keep track of airline reservations, stock transactions, telephone calls and charges, and the whereabouts of railroad cars. Such services are the key to keeping the increasing complexity of our society from making us all into robots. Computers are capable of providing us with the mass-produced, yet individualized product, and with individual treatment for each person in a mass society, providing of course we are wise enough to use computers in that way. It is not too much to say that you, no matter who you are, have a computer in your future.

Where did computers come from, and how were they created? They began with the idea of using mechanical devices to aid man in performing arithmetic. This notion can be traced back thousands of years. The first step was the idea of numbers and of counting, using objects like fingers and stones. Because men had ten fingers it was natural that the number ten had special importance, probably even to prehistoric men. The development of methods to express numbers greater than ten took many centuries. The "positional" notation(units, tens, hundreds, etc.) expressed by the positions of numerals in a number and the use of a special symbol for zero were key steps in the evolution of our own number system. Hindu-Arabic numerals appeared about the fourteenth century, but our numbers and the rules for using them were not completely accepted in Europe until a few centuries ago.

The origin of the abacus is lost in antiquity; it may be the first widely used computing device. Computers of some kind were apparently built by the ancient Greeks. John Napier (the inventor of logarithms) developed a device for multiplication about 1617. The first slide rule, based on adding logarithms for multiplication, appeared shortly afterward. Blaise Pascal in 1642 built a stylus-operated adding machine with numbered wheels geared together, in much the same way that automobile mileage indicators are, so that the "carry" digit from one column could be automatically added to the next; this was a particularly important development. Samuel Morland had a machine based on the same principle which could subtract as well as add. In 1694 Gottfried Wilhelm von Leibnitz introduced still another machine which also multiplied (by repeated addition), divided, and extracted roots.

From an entirely different kind of endeavor came an idea which was to be critical in the development of modern computers. In the seventeenth century the Frenchman, Jacquard, developed a loom which could automatically weave cloth in a pattern which was specified by information from punched cards fed to the loom one after another. If the pattern of holes in punched cards could control a loom, they could also control an arithmetic calculator, causing it to carry out a planned sequence of arithmetic operations in a long computation. This idea occurred to Charles Babbage, a mathematics professor from Cambridge University. In 1820, he had built a small model of a "difference engine" which evaluated polynomials by a method using differences among a series of numbers. About 1833 he proposed to build an even more elaborate "analytical engine" which had all of the essential features of modern electronic computers. It was to have a "store" or memory unit which was to hold numbers and the results of doing arithmetic on these numbers; a "mill", or arithmetic unit, which was to do the necessary succession of steps of adding, subtracting, multiplying, and dividing; and a "control" which translated and carried out instructions from the punched cards.

Lord Byron's daughter, the Countess of Lovelace, had always been fascinated by mathematics, and when she heard of Babbage's ideas she was enthusiastic about helping him fulfill this ambition to construct the machine which "weaves algebraic patterns just as the Jacquard loom weaves flowers and leaves." Together, she and Babbage conceived all sorts of wild fund-raising schemes. He got some financial backing from the British government, but the necessary gears and other mechanisms proved to be beyond the technology of the times. Only a very few artisans of Babbage's days could have made them accurately enough. Even so, it is possible that his grand plan for the "analytical engine" might have actually succeeded if he had based it on the binary instead of the decimal number system.

Sad to say, Babbage's work was forgotten. (It was "discovered" only sometime after the modern computer was well on the way). A new start was made by Howard Aiken of Harvard University and George Stibitz of the Bell Telephone Laboratories in the late 1930's when they developed automatic calculators using telephone relays. These were completed in the early 1940's.

The first automatic computer based on electronic rather than mechanical or electro-mechanical technology was the ENIAC (Electronic Numerical Integrator and Computer) built at the University of Pennsylvania. It was conceived and built by John Mauchly, who had been a professor of physics, and J. Presper Eckert, Jr., a graduate student of electrical engineering. Together they proposed, in 1943, to build an electronic digital computer to replace the huge "differential analyzer" at the University of Pennsylvania which was being used during the second World War for military purposes. This computer was completed in 1945. It could add or multiply two numbers in a fraction of a second, but it took tremendous effort just to keep it working. ENIAC had some 18, 000 vacuum tubes and they generated an enormous amount of heat which had to be carried away by fans and air-conditioning. The computer did not melt but it was always potentially a furnace.

The work of Eckert and Mauchly was elaborated further by the mathematician, John von Neumann. The ENIAC had to be rewired for each specific calculation; if another were to be done, extensive and tedious rewiring of a plug board had to be done. In a series of reports in 1945 and 1946 von Neumann and his colleagues presented the design of a computer which could use a program stored in its own memory and which could be changed without rewiring. The first working stored-program computers were demonstrated at about the same time in both the United States and Britain, in 1949. The first commercial electronic computer was the Eckert-Mauchly UNIVAC, put on the market in 1950.

The stored program concept was a critical one. It meant that a series of instructions could be stored in the computer memory in the same way that numerical data could. A program of instructions could, therefore, make modifications in another program, or even in another part of itself. A program could then be designed to modify itself automatically during the course of a computation in a way which depended upon the intermediate results which could not have been predicted before the computer started to work on the problem. The stored program idea also meant that the same computer could be used at various times on many different classes of problems without changing the wiring of the computer itself. A computer could now truly be "general purpose."

A digital computer acts fundamentally as a mechanism which transforms information. It takes sets of numbers, letters, or other symbols, processes them according to sets of rules and gives its results as other sets of numbers, letters, etc. Critical to the concept of modern computing is the representation of all information as two-valued (binary) signals. In effect each symbol in a digital computer is represented by a series of signals which can have only two possible values. When numbers are stored in the computer they are stored in sets of devices each of which can be in one of only two states, which we can think of as "off" or "on". Often, for example, a number is "remembered" by putting it into the form of a series of "spots" on magnetic tape; each spot location may be magnetized in either one of two directions.

Constructing a computer with "on" or "off" devices and signals has one great advantage. It is that circuits can be designed to handle these much more simply and surely than if the signals had a wide range of values. These circuits are often called "switching or logic circuits" (because switches are commonly either on or off just as logical propositions can be either true or false).

Modern electronic computers started with the realization that the electron flow in vacuum tubes, which had originally been developed for radios, could be switched on and off. Vacuum tube computers were, by our present standards, bulky, slow, and unreliable, but they did work and they handled problems with speeds which had not been possible before them. Computations which had required months took only minutes. Vacuum tubes made automatic computing practical. Nevertheless, the first electronic computer, ENIAC, with its 18,000 tubes, weighed 30 tons and took up 1500 square feet of floor space. Today, an equally capable computer could easily be fitted into a cabinet the size of a 21" TV set. What advances have made this possible?

During the 1950's two inventions, the transistor and the magnetic core memory, changed computers to almost as great an extent as they had earlier been changed by the exploitation of vacuum tubes.

Transistors are far smaller than vacuum tubes. They use less power and therefore generate less heat. They are more reliable and the electrons which flow in them can be switched on and off much more quickly. Transistors are examples of "solid state" devices, in which the flow of electrons in solid material can be controlled just as electron flow in a vacuum tube can be controlled. The material most commonly used for transistors is silicon (the main ingredient in sand) which is artificially grown in large crystals and then sliced into very small pieces.

The magnetic cores used in the memories of computers are minute doughnut-shaped rings, usually made of a magnetic ceramic material called ferrite. A typical core is smaller than the size of a printed "o" on this page. In a computer memory millions of cores may be used. Typically, they are woven into a sort of cloth or mesh made of small wires. By sending currents through selected wires of this mesh it is possible to magnetize a chosen core with one polarity or the other. In this way a core can "remember" the direction of the currents last sent through the wires. The first electronic computers used vacuum tubes, switched "on" and "off", as memory elements. Magnetic cores were more reliable, less costly, and occupied less space. They have become the standard method of constructing "memories" in today's computers.

More recent developments make it possible to create and connect together on a single tiny "chip" of silicon, measuring only a tenth of an inch across, dozens or hundreds of components, each of which is equivalent to one transistor. These "integrated" circuits will make it possible to build computers of enormous logical capacity, reliability, and complexity cheaply and in very little space.

The switching time of some of the latest computer circuits is measured in "nanoseconds". A nanosecond is one billionth of a second. There are as many nanoseconds in one second as there are seconds in 32 years. In one nanosecond light travels about twelve inches. Thus, today, it is important for a computer designer to take into account the time that it takes to send electrical signals over a wire from one part of the computer to another, even though these signals travel at nearly the velocity of light!

With all these developments has come the possibility of performing computations at negligible cost. Before electronic computers existed it cost about $30, 000 to make one million calculations. Now the same number of computations cost only 30 cents. It will be even less in the future.

In principle, present day electronic computers can do nothing which could not be done by Babbage's "analytical engine", had it been built. But there is an enormous difference between being able to do a computation "in principle" and actually doing it at a reasonable cost and at a speed which gives the answer quickly enough that it can be useful. There are now thousands of menial tasks which are done routinely by computers which would not have been done at all a decade ago. The computer is the contemporary counterpart of the steam engine which initiated the industrial revolution. The steam engine was the first economical way to convert the energy stored in coal to a form useful for production. Information is a commodity which is no less tangible than energy. Computers are the first economical way to process and manipulate information. Jast as our ability to control energy can be used or misused it will be possible to use, wisely or frivolously, our ability to manipulate information.

The impact on our society of the computer, for good or evil, will be limited only by our imagination and our sense of responsibility. Some of the things they do already were mentioned at the beginning of this introduction. The word "they" may be a bit misleading, for we must remember that it is human beings who tell computers what to do. If we can understand how to solve a problem by devising an appropriate set of rules and procedures then we can present these to a computer. The computer can then carry out the necessary individual steps with lightning speed. The result will only be as correct as the instructions we have given. We may find ourselves in the awkward position of a newly- commissioned army officer whose men obey him to the letter. If he omits a necessary command, or substitutes one for another by accident, or has no plan of action, chaos will result. When people speak to each other they may say, "You know what I mean". This, however, is not an appropriate computer command; a computer must be told explicitly.

The computer makes it impossible to substitute sloppy thinking or mere talk or bluff for a specific set of directions. The computer forces us to look at the consequences of our assumptions. If we do not like the results we have only ourselves to blame -- just as the new officer does.

The importance of the computer to society is hard to overstate. There is little doubt that we will become (if we are not already) a computerized society just as we have been for many years a mechanized society. We also believe that this should not be regretted or avoided. Computers give the promise of lifting from us the burden of routine and dulling mental labor. They promise also to give us the information necessary to make human decisions in an enlightened way. However they are used, they are going to be a major force in the world. It seems likely that soon almost everyone will be using computers for their own purposes.

We are not forced to use automobiles and airplanes to get where we want to go, yet many of us find it desirable to do so -- if only to get to a vacation spot where we can hike or swim or participate in strenuous sports. Similarly we shall not be forced to use computers, but most of us will, if only to free ourselves for strenuous but challenging and productive intellectual activity. The very real dangers of an uncontrolled automobile do not make us give up driving. The real dangers of unthinkingly accepting every "answer" reported by a computer will not make us give up computers.

Even if we admit the great importance of computers in our present and future society, why should we talk about them in this text which deals with engineering concepts and why do we talk about computers before we mention other topics? There are several reasons.

In ancient days only a limited number of people could read or write. Those who could occupied a special place in society, keeping official records and interpreting the law. Now we assume that every educated person can write and read and no special exalted position is given to those who can. At present, only a limited number of people can program a computer. They often act as a sort of "middlemen", and take the problems of scientists, engineers, businessmen and others and put them into languages which computers can understand. Development of newer, simpler computer languages will make computers more directly accessible to the people who can use them. In the future fewer and fewer people will be unable to communicate problems to a computer. No longer will the computer be an instrument clouded in mystery, about which a few elite from time to time make statements which are incomprehensible and therefore frightening to the majority of the population. The day of the "high priests" of computers is coming to an end. Scribes had their value only when the average citizen could not read or write.

The computer is a tool which can be understood. It is more important for the high school graduate to understand the computer than his automobile, for the citizenry will ultimately determine how computers will be used to shape our society.

The computer represents an outstanding example of the application of ideas to practical goals; it thus serves as an unusually good example of what engineering is all about. Who could have dreamed that the computer would provide a bridge between the abstract logic of the ancient philosophers and the practical needs of the industrial society of our century?

Computers also furnish a perfect example of how a complex man-made mechanism is synthesized by first combining small elements into larger units. By understanding each stage of this process we shall be able to understand the final result despite its apparent complexity. Engineering design is typified by the assembly of subunits into a single result.

One very convenient way of describing problem solutions in a way appropriate for programming on a computer is by "flow charts". These diagrams make it easy to organize one's thoughts about a problem in a form easy to visualize. Because flow charts are a convenient tool for the other parts of this text it is natural to mention them here first.

A final reason for our talking about computers first is that no mathematical background is needed. No complex equations have to be solved. All that is required is that the reader be willing to take a fresh look at what numbers are and how he adds and subtracts them. No background in physics is required. The student need only be able to trace paths along wires to see if a path exists from one part of a computer circuit to another.

In our man-made world there will be a partnership between man and his computers. In this partnership each party should perform the activity to which it is best suited. Man is good at organizing ideas, at invention, at making associations among apparently unrelated notions, and at recognizing patterns and ignoring irrelevant details. He is creative, unpredictable, capricious and acts on hunches, but he is sensitive to human values.

The computer is almost exactly what man is not. It gives its undivided attention to unlimited and intricate detail, it is immune to distraction, boredom, and fatigue. It needs to be told only once and it then remembers perfectly until it is commanded to forget; it then forgets, instantly and absolutely. It is precise, reliable, emotionless, and never complains.

Men are not machines and machines are not men. Each can do what the other cannot. The shortcomings of one are complemented by the strength of the other. The potential of such a partnership is greater than the sum of its parts. But since computers will not understand us it is up to us to understand them.

There are at least two paths to studying and understanding computers. One is to learn how to use computers; the other is to learn what is inside a computer. This course combines these approaches. In the next three chapters, we study the fundamental circuits of computers and see how they are related to logical thought. We see too how to organize many individual circuits to form a computer. Finally, you will learn how to use a digital computer. When you finish this section of The Man-Made World you will be able to make informed judgements about computers, and you will be in a position to undertake the further study necessary to use computers wisely and profitably in your future work.

Chapter A - 2

LOGICAL THOUGHT AND LOGIC CIRCUITS

1. INTRODUCTION

Computers are able to carry out millions of simple operations each second. The nature of each of these operations is determined by the way in which the smallest physical elements of the computer are connected together. The rules for correctly interconnecting the elements contained in the various sections of the computer are quite similar to the rules for expressing and solving problems in logic. Because of this fact, the procedures for deciding how to interconnect the smallest computer elements is called logical design. In this chapter we see that the logical design of the circuits required in a simple computer is really quite easy. The basic elements of computer circuits are switches, each of which can be either "off" or "on" in the same sense as the switch used to turn a light bulb off and on. For this reason, basic computer circuits are often called switching circuits.

The ideas from logic needed for assembling computer circuitry are merely those associated with the words "and", "or", and "not". In logic these words are used almost in the same sense that we use them in everyday conversation.

2. HOW TO MAKE ELECTRIC CIRCUITS SAY "AND" AND "OR"

Simple circuits with switches and lights.

Switching circuits enter constantly into our daily lives. We use them without thinking about it — flicking a light switch, dialing a telephone, controlling an automatic elevator or a pedestrian-operated traffic light. The inner circuits of a computer, though more complex, operate on exactly the same principles as do these simple examples.

When you walk into a dark room, you flick the light switch from "off" to "on". You know that you have succeeded when the light appears. What you have done is to establish a metallic connection from a source of electricity to the light bulb. The actual source of this energy may be a generator owned by the electric company, to which the wall outlet in your room is connected, or it might be a battery. In either case, the source has two terminals, which we label "+" and "-" for identification purposes, and the lamp must be connected between these [Fig. 1(a)] in order to be lighted. The connection is usually a copper wire and it conducts electrical current from one terminal of the battery or plug through the lamp and back to the other terminal.

For our immediate purpose, the only fact of importance is that an unbroken metallic connection from a source through a lamp and back to the source is necessary to light the lamp. Thus, to turn the light on we complete the path, and to turn it off we interrupt the path. These actions can be accomplished by inserting a switch, as shown in Fig. 1(b). When the switch is operated ("on"), a pair of metal points (the pair of points is called a contact) press against each other. This action completes the path for the current. When the switch is released ("off"), the metal points no longer press against each other and the path is interrupted. In this arrangement the switch contact is said to be in series with the lamp.

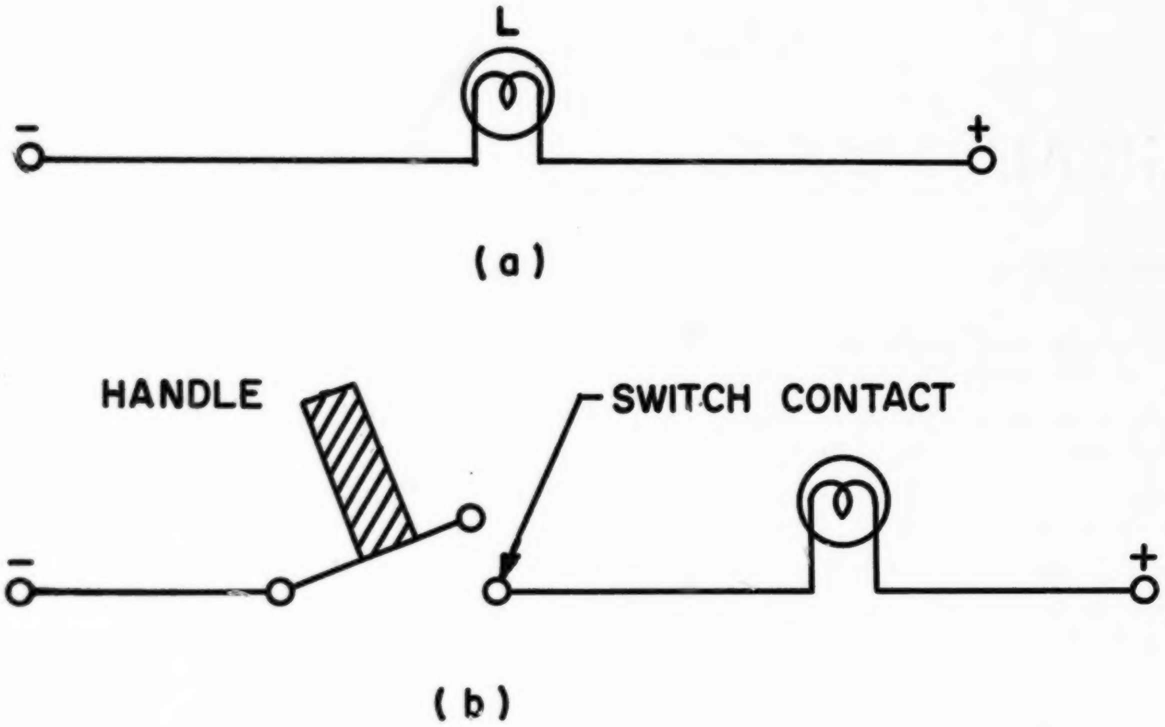


Fig. 1 An electric light, controlled by a switch.
(a) A lamp, with connecting wires.
(b) A lamp in series with a switch contact.

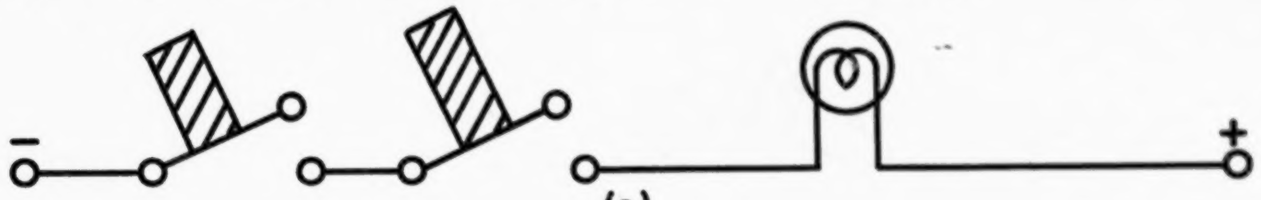
It is possible to control a light with two switches in such a way that the light is on only when both switches are operated. The circuit in Fig. 2(a) shows how this is done using a series arrangement of the contacts from the two switches. This arrangement is sometimes called an "and" circuit since the light is on only when switch A and switch B are both operated.

It is easy to imagine an important application for our "and" circuit. It could be used, for example, to activate the firing circuit for a rocket when both of two operators in the blockhouse must agree that the firing should take place. (How would you control the firing circuit when three, or more, operators in the blockhouse must all concur that the rocket should be fired?)

The operation of the "and" circuit can be summarized in the table shown in Fig. 2(b). Another way of describing this circuit is to concentrate on the condition of the two contacts and the condition of the resulting path through both of them in series. If we let "0" stand for an interrupted (open) contact or path and "1" stand for a completed (closed) contact or path, the table of combinations in Fig. 2(c) lists all possible combinations of conditions controlled by the two switches.

Another basic way of connecting contacts from two switches is to put them in parallel with each other, so that the operation of either switch will complete a path through the circuit. In Fig. 3(a) you can see that there is a path for electric current if either switch A or switch B is operated (or, of course, when both are operated). This parallel connection of contacts is, therefore, commonly called an "or" circuit.

Meanwhile, back at the launching site, imagine that each of two astronauts lying in the capsule on the top of the rocket has a switch which he controls. The first astronaut operates his switch (A) whenever he thinks there is a fuel leak. The second astronaut, also on the lookout for fuel leaks, operates his switch (B) when he thinks he has found one. The light, controlled by the "or" circuit, will go on whenever either astronaut senses trouble. (How would an astronaut feel if he knew that the warning light was controlled by an "and" circuit?)



(a)

Switch A	Switch B	Light
Released	Released	Off
Released	Operated	Off
Operated	Released	Off
Operated	Operated	On

(b)

Path through contact on Switch A	Path through contact on Switch B	Path through the contacts in series
0	0	0
0	1	0
1	0	0
1	1	1

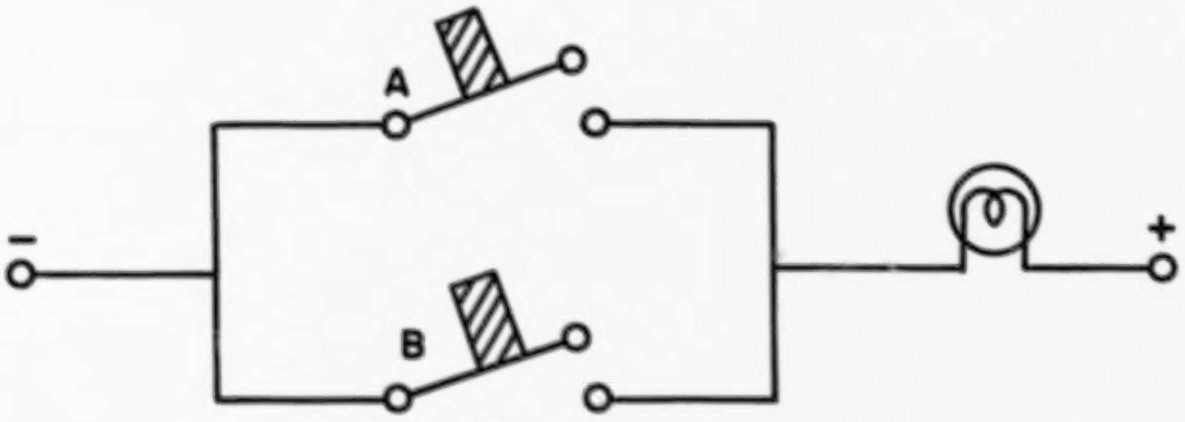
(c)

Fig. 2 The series (or "and") configuration of two contacts.
(a) A lamp controlled by two contacts in series.
(b) A tabular description of the operation of the circuit.
(c) A table of combinations for the "and" circuit.

As in the case of the "and" circuit, the "or" circuit can be summarized by a tabular description [see Fig. 3(b)]. Using 0's and 1's as before, we can write a table of combinations [see Fig. 3(c)] which shows how the condition of the parallel configuration of contacts depends upon the condition of the individual contacts.

Imagine now that the pilot of a high-performance jet airplane has just lost a large portion of his right wing. He must eject, but two distinct steps are necessary. First, the cockpit canopy must be blown off; second, the seat (and he) must be ejected by another explosive charge. A straightforward method would be to fire the canopy charge with one switch and the seat ejection charge with another. But then in the confusion of the emergency, he might fire the seat charge first, and it and he would then be projected through the closed canopy — an undesirable result. How can the switches be arranged so that regardless of the order in which the switches are operated, the first will fire the canopy charge and the second will fire the seat charge? There is a way, perhaps already guessed, since it uses only a combination of the "and" and "or" circuits. However, a slightly different type of switch is required.

In the switches discussed so far, a single contact is controlled by a single handle or lever. To solve the pilot's problem we need a switch in which two contacts are controlled by a single lever. The two contacts are linked together by nonmetallic material that prevents the flow of electricity from one to the other, so that they can be used in two electrically separate circuits. When the switch is operated, both contacts close simultaneously, and when the switch is released, both contacts open simultaneously.



(a)

Switch A	Switch B	Light
Released	Released	Off
Released	Operated	On
Operated	Released	On
Operated	Operated	On

(b)

Path through contacts on Switch A	Path through contact on Switch B	Path through the contacts in parallel
0	0	0
0	1	1
1	0	1
1	1	1

(c)

Fig. 3 The parallel (or "on") configuration of two contacts.
(a) A lamp controlled by two contacts in parallel.
(b) A tabular description of the operation of the circuit.
(c) A table of combinations for the "or" circuit.

Now return to the pilot's problem. We can restate it in terms of "and" and "or". The canopy blow-off charge should be fired when either switch A or switch B is operated. The seat ejection charge should be fired when switch A and switch B are operated. Consequently, the combinations of the two circuits in Fig. 4 will do the required job. When only one switch is operated, no matter which one it is, the canopy will blow off. Only when the other switch is operated will the seat be ejected. (Note that the two contacts controlled by switch A have both been marked "a" and the two contacts controlled by switch B have both been marked "b". The reason for the distinction between a switch and its contacts will be apparent later.)

Let us close this section with a summary of terms we have used to refer to switches. Switches may be in one of two states: released or operated. A contact controlled by a switch may also be in one of two states: open or closed. It is also true that a path through a set of interconnected contacts is open or closed, whether the path is through a single contact or through a complex configuration of contacts. At a given time a path is either interrupted (in the open state) or it is completed (in the closed state). In speaking of the condition of a contact or a more complex path, it is useful to use the pair of symbols, "0" and "1", to stand for "open" and "closed", respectively. Any other pair of symbols, such as "\$" and "¢", could have been used.

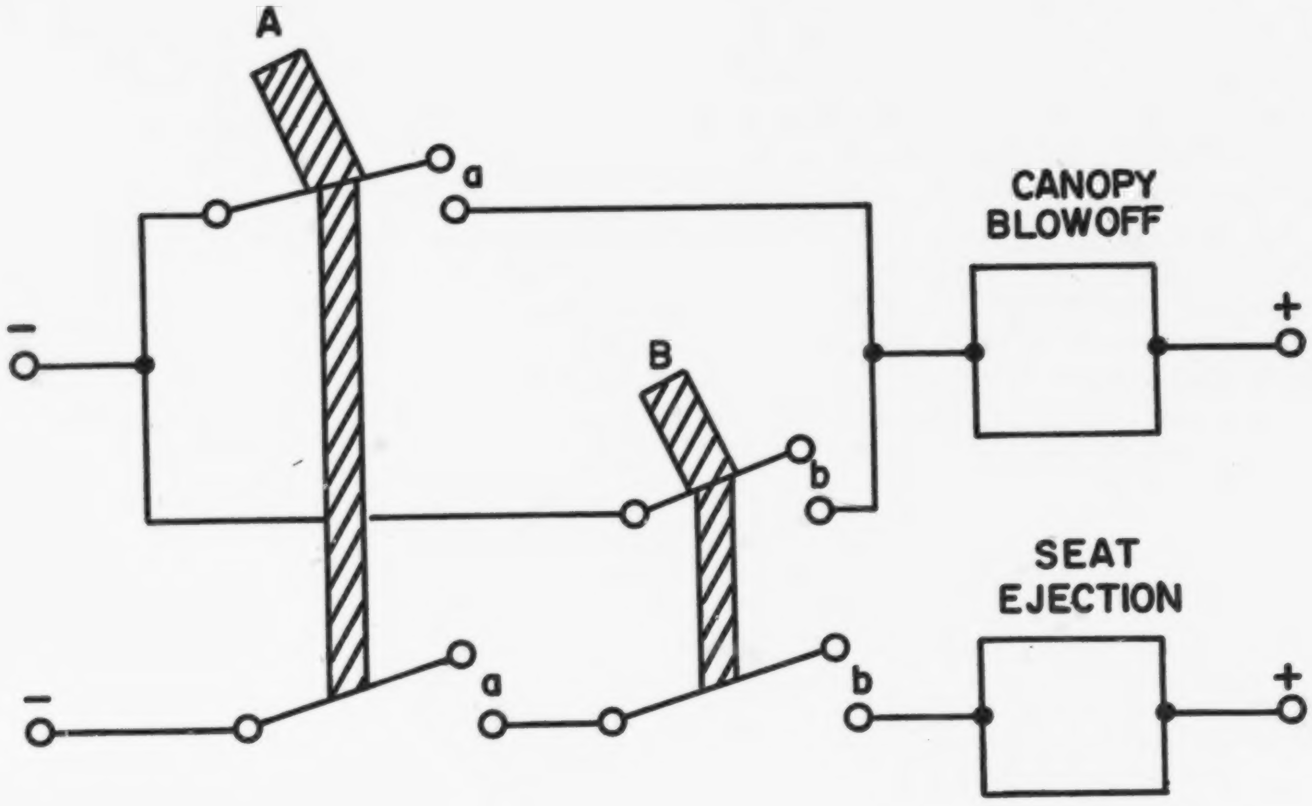


Fig. 4 A circuit with contacts both in series and in parallel.

However, it is conventional to use "0" and "1" and we do that here. A major source of difficulty can develop from failure to remember that they are merely a pair of arbitrary symbols and that they do not have numerical values. Switch contacts can be connected in series or parallel. The logical connective "and" corresponds to a "series" connection and "or" corresponds to a "parallel" connection.

3. AN EXAMPLE: THE MAJORITY VOTE PROBLEM

Three legislators wish to vote on a number of issues and have their votes anonymous. Each one is to control a switch which is labelled "No" in the released position and "Yes" in the operated position. A lamp indicating that the majority vote is favorable is to be lighted whenever two or three of the members vote "Yes". One way of restating these requirements using "and" and "or" is:

"The control circuit is to be closed when, and only when,
1) switches A and B are operated or
2) switches A and C are operated or
3) switches B and C are operated.
(If switches A and B and C are all operated, then conditions 1), 2) and 3) are all satisfied at once.)"

To design the circuit, we remember that "and" calls for a series connection and "or" calls for a parallel connection. Thus, statement 1) corresponds to a series connection of contacts a and b on switches A and B. Similarly, statement 2) requires contacts a and c to be placed in series, and 3) requires a series connection of contacts b and c. In addition, statements 1), 2), and 3) are connected by "or's"; thus, the above three series arrangements of contacts should themselves be connected in parallel, as shown in Fig. 5(a). (Switch contacts have been represented by the symbol —X— in this figure. Representations of the switch handles have been omitted.)

To check that the circuit really performs as advertised, we must trace through the circuit with the switches in all their possible positions to find when the overall circuit is closed and when it is open. First, construct a table as shown in Fig. 6 listing the switches A, B, C across the top and their various possible conditions; "1" for operated and "0" for released. For example, the first row, 0, 0, 0, means that all three switches are released; the second row, 0, 0, 1, means that A and B are released while C is operated. Now make an additional column corresponding to statements 1), 2), and 3) above. Put a "1" in that column wherever A and B, or A and C, or B and C are "1" in the table, otherwise put "0". Next construct a column corresponding to the state of the circuit just designed. Enter a "0" whenever the circuit is open and a "1" whenever it is closed. Do this for each of switches A, B, and C. For example, with A, B, and C all operated (1, 1, 1) the circuit is closed, "1", since all three series branches are closed. When A and B are operated and C released (1, 1, 0), the circuit is also closed, "1", since the uppermost branch is closed (the lower two branches are open). This column will agree with the previous one unless there has been an error in the circuit design.

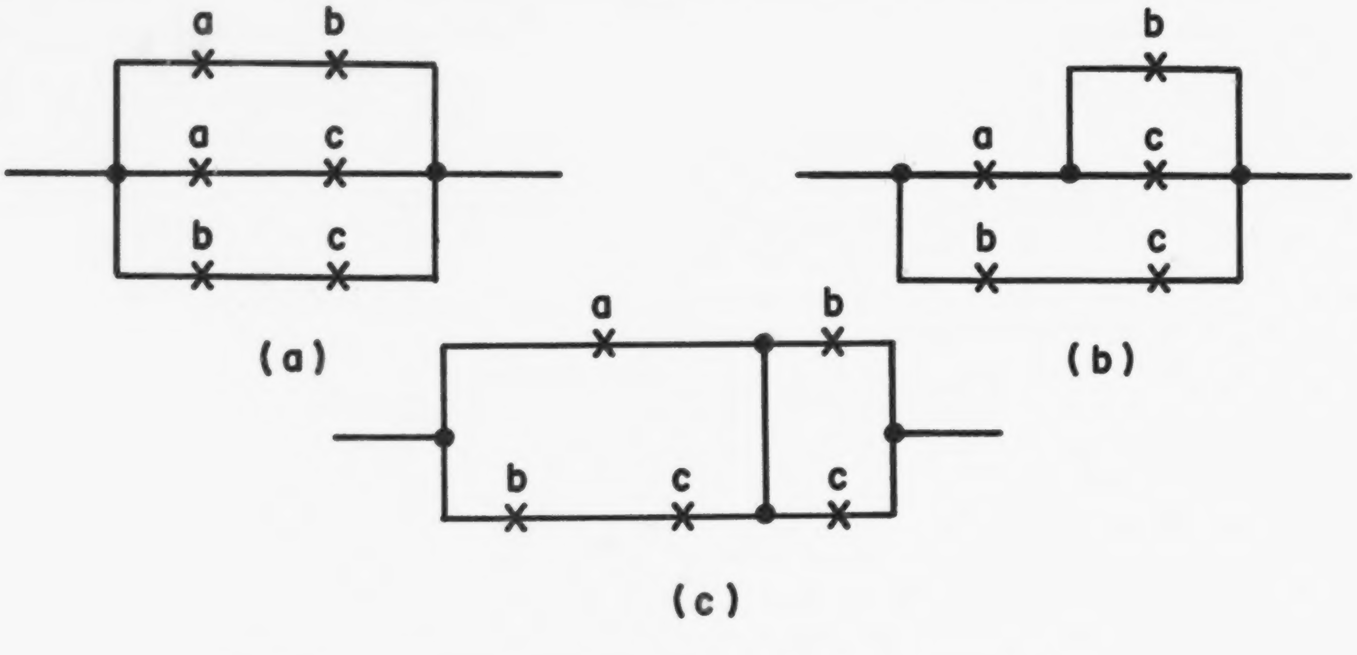


Fig. 5 Three equivalent majority circuits.

The logical statement of circuit requirements at the beginning of this section is only one of several ways to state those requirements. Another is:

"The control circuit is to be closed when, and only when,
1) A is operated and either B or C is operated, or
2) B and C are operated."

This would lead to the circuit in Fig. 5(b). Another way is:

"The circuit is to be closed when
1) A is operated, or B and C are operated, and
2) B or C is operated."

The resulting circuit is shown in Fig. 5(c). There are many other ways of connecting contacts controlled by switches A, B, and C to achieve the desired result;

namely, to indicate a majority vote. All, however, will have the same tabular description, called a truth table, indicated in Fig. 6, since they must all give the same result when a particular set of switches is operated. It is always true that two or more circuits with the same table of combinations are logically equivalent.

A	B	C	Logical Statements	State of Network
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Fig. 6 Table of Combinations for the Majority Vote Problem.

4. HOW TO MAKE AN ELECTRIC CIRCUIT SAY "NOT"

"Make" and "break" contacts.

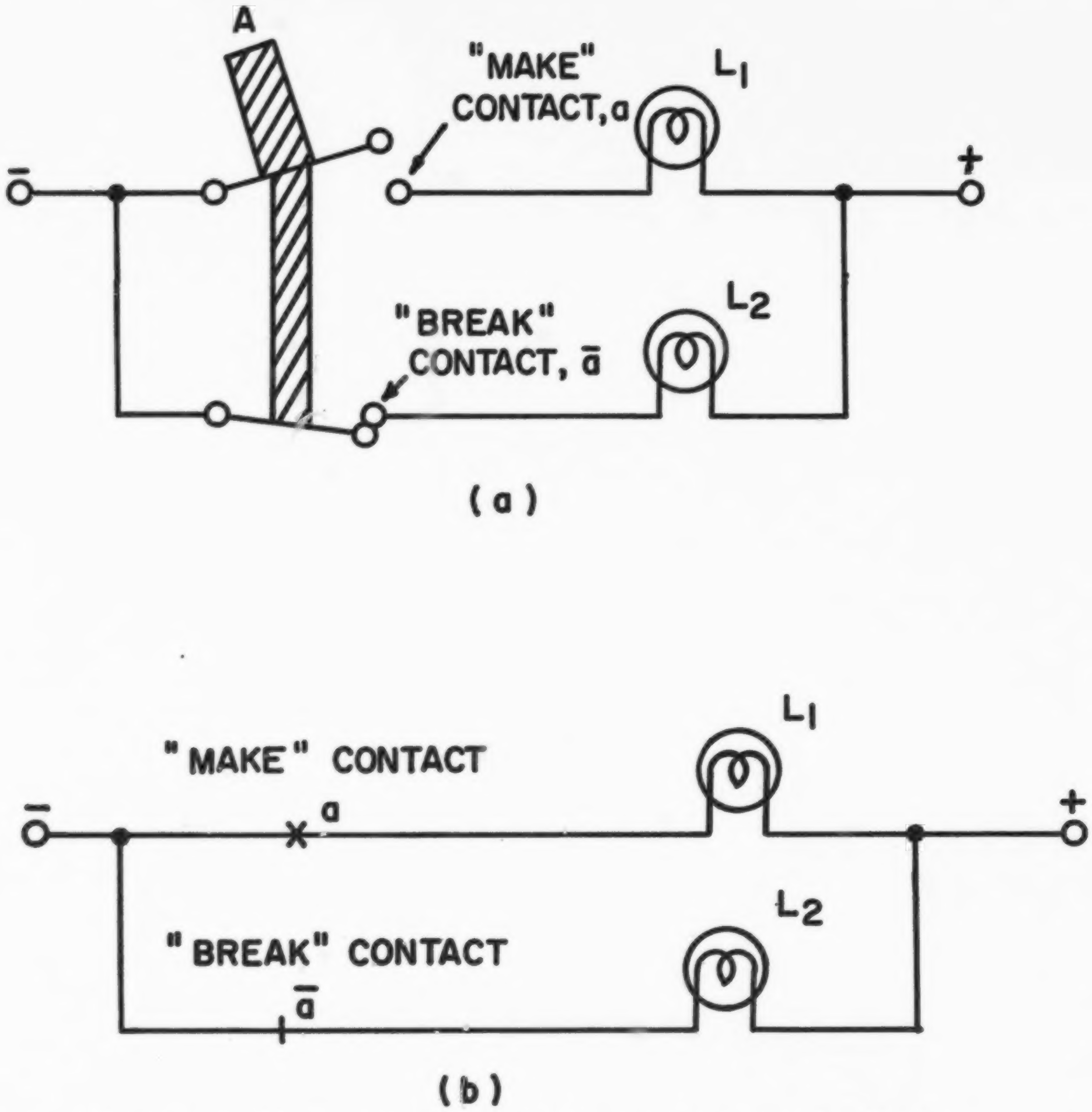
We have seen that "and" and "or" can be represented by series and parallel connections of switch contacts within a circuit. There is a third basic connective of logic – the word "not". How can we represent "not" in circuits having only switches and their contacts? The answer is given by returning once again to see how a handle or lever can control the opening and closing of contacts.

Fig. 7 (a) shows a single switch, A, controlling two different contacts. Whenever the upper one is closed, the lower one is open, and vice versa. Thus these contacts behave in the complementary way which we associate with "not". When the upper contact is closed the lamp L_1 is lighted, and the lower contact is open and the lamp L_2 is not lighted. On the other hand, when the upper contact is open and the lamp L_1 is not lighted, the lower contact is closed and the lamp L_2 is lighted.

It has been seen that a single handle or lever can be used to control one or more contacts in various places in the electric circuit. Now it is also seen that a single switch handle or lever can be made to close one group of switch contacts, while simultaneously opening the contacts of a complementary group.

The entire circuit of Fig. 7(a) is sometimes used in photographic darkrooms to warn people coming to the door that film is being developed inside and that the door cannot be opened without ruining the film. Needed in this situation is a red warning lamp outside the door that goes on when the lamps in the darkroom go off. In this instance, the red lamp would be L_2 and the darkroom lamp, L_1.

Logic circuits may involve a multiplicity of switch contacts, some of which work in a fashion opposite to some of the others. In order to simplify their circuit diagrams we use the following conventions. We speak of a contact as being of either the "make" or the "break" type. In Fig. 7(a) the upper contact is the "make" contact and the lower contact is the "break" contact. The diagram shows the switch in the released state: in that state the make contact is open and the break contact is closed. Depressing the lever in the diagram puts the switch into its operated state; in that state the make contact is closed and the break contact is open. If the switch is labelled with a capital letter ("A") the make contacts



(a)

(b)

Fig. 7. Two complimentary types of contacts, controlled by the same switch.

associated with that switch are labelled with the corresponding lower case letter ("a") and the break contacts are labeled with that letter with a bar over it ("$\bar{a}$"). The make contact on circuit diagrams [see Fig. 7(b)] is denoted by a cross; the break contact is denoted by a bar. (It helps to remember that both "bar" and "break" begin with the letter "b".)

In Fig. 8 there are several circuits, each containing several contacts of both make and break varieties which are controlled by a single switch, S. With the switch in its released state (and later with the switch in its operated state) indicate which of the circuits have a completed path between their left and right terminals and for which circuits this path is interrupted.

Useful circuits using "break" contacts.

There is a common circuit which requires both make and break contacts.

It is the circuit used to turn a light off and on by either of two switches. This is exactly the arrangement needed to control a single light from either end

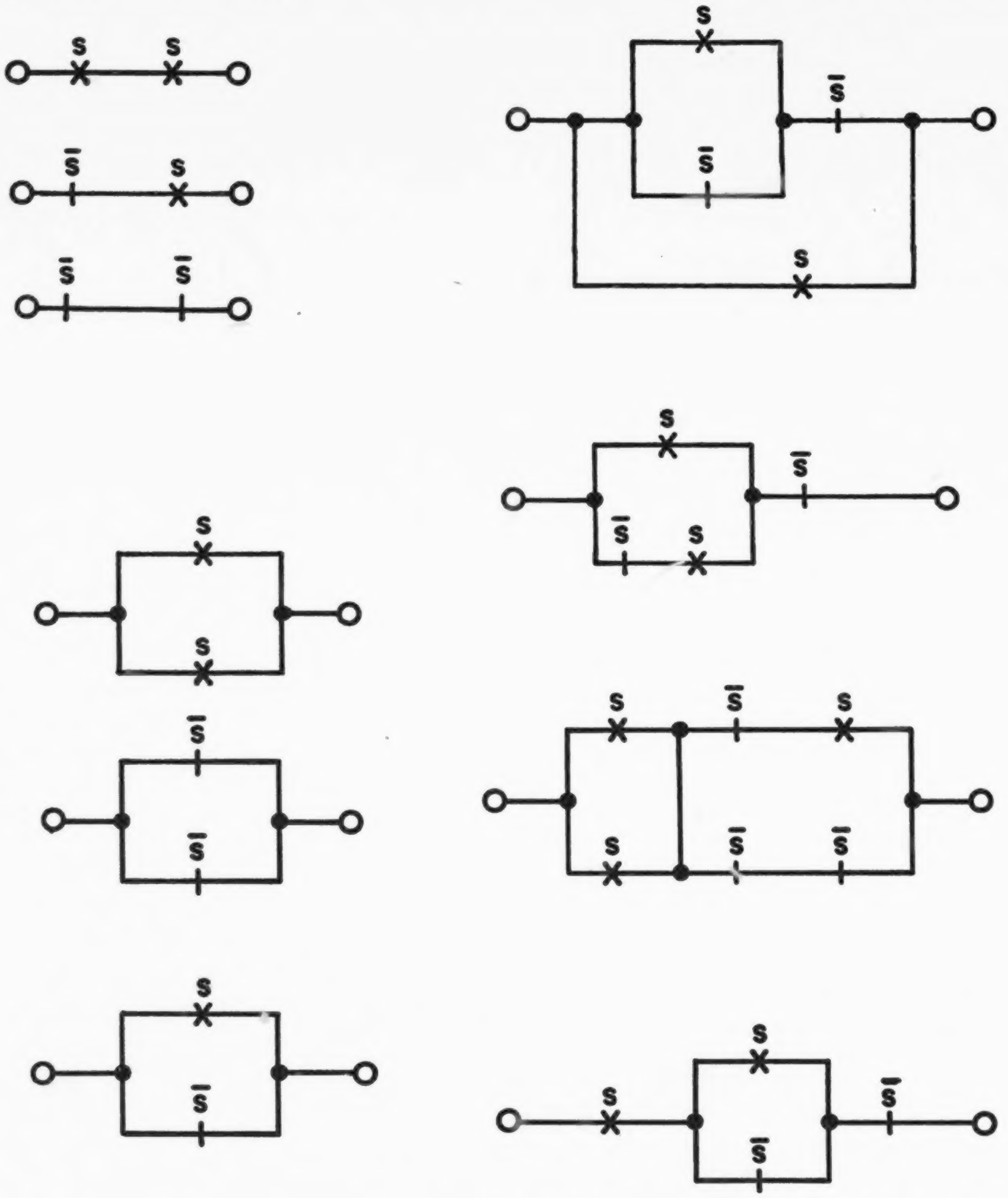


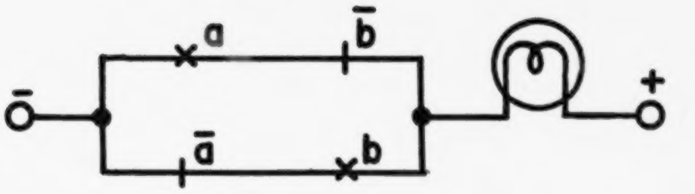
Fig. 8 Circuits with contacts all controlled by the same switch.

of a hall or set of stairs. Changing the position of the switch where you are always changes the condition of the light, regardless of the position of the other switch.

The circuit given in Fig. 9(a) shows how contacts on the switches A and B can be used to control the hall light. The contact network consists of two circuits in parallel and each of these consists of two contacts in series. There will be a path through the contact network when either

(i) switch A is operated and switch B is not operated (released), or
(ii) switch A is not operated (released) and switch B is operated.

Because there is a path when "A or B (but not both)" are operated, the circuit is also referred to as the "exclusive-or" circuit.



(a) The Circuit

Switch A	Switch B	Path (Light)
released	released	open (off)
released	operated	closed (on)
operated	released	closed (on)
operated	operated	open (off)

(b) A table which shows how the hall light is controlled.

a	b	path
0	0	0
0	1	1
1	0	1
1	1	0

(c) Table of combinations.

Fig. 9 The two-variable odd-parity (or "exclusive-or") circuit.

The tables of Fig. 9 (b and c) list the properties of the circuit in terms of the operation of the switches and, equivalently, in terms of the states of the make contacts on the two switches. Note that if a person stands at switch A he can turn the hall light off or on no matter whether switch B is operated or released. The same thing can be done from the opposite end of the hall by using switch B.

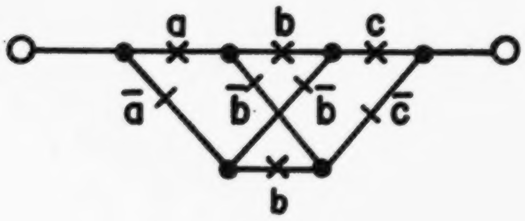
The control circuit we have described is closed whenever just one of the switches is operated, and is open whenever exactly none or two of the switches is operated. It is possible to generalize this idea and design circuits, having three or more switches, which are closed whenever any odd number of switches are operated and open whenever any even number of switches are operated. A circuit of this kind is called an "odd-parity" circuit. Two of them are shown in Fig. 10.

The three variable odd-parity circuit will be particularly important when we discuss the "adding" circuits necessary for the design of computers. The reader should analyze its operation carefully by deriving the appropriate table of combinations and by verifying each entry with the corresponding state of the circuit. The table of combinations is given in Fig. 10(b).

Two additional exercises in analysis are given in Fig. 11. These circuits are closed whenever a majority of their controlling switches are closed. The first of these uses a break contact in a three variable majority circuit although we already know that it is not necessary to use anything but make contacts (see Fig. 5). Its table of combinations is also given in the figure.

5. A MODEL FOR A RIVER-CROSSING PUZZLE

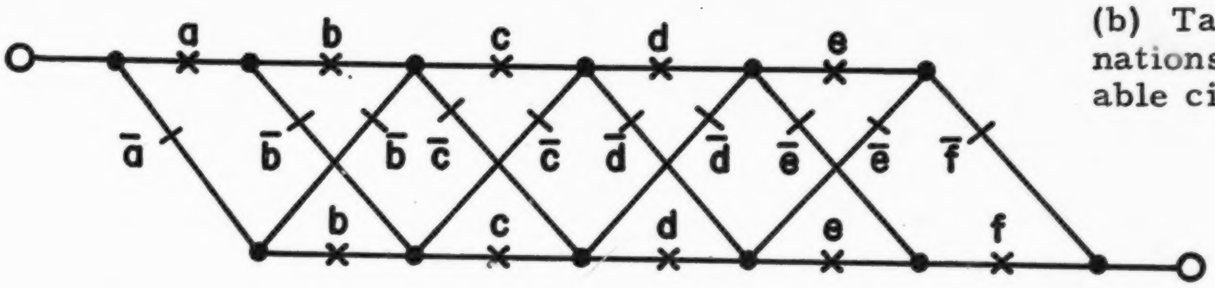
A logic circuit using contacts is ultimately nothing more than a representation for the equivalent logical word statement. One type of application in which


a
b
c
$\bar{a}$
$\bar{b}$
$\bar{b}$
$\bar{c}$
b

(a)
A three-variable odd-parity circuit.

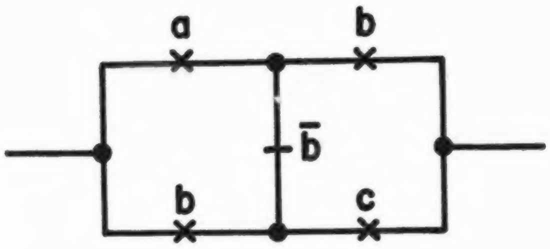
a	b	c	path
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

(b) Table of combinations for three-variable circuit.


a
b
c
d
e
$\bar{a}$
$\bar{b}$
$\bar{b}$
$\bar{c}$
$\bar{c}$
$\bar{d}$
$\bar{d}$
$\bar{e}$
$\bar{e}$
$\bar{f}$
b
c
d
e
f

(c) A six-variable odd-parity circuit.

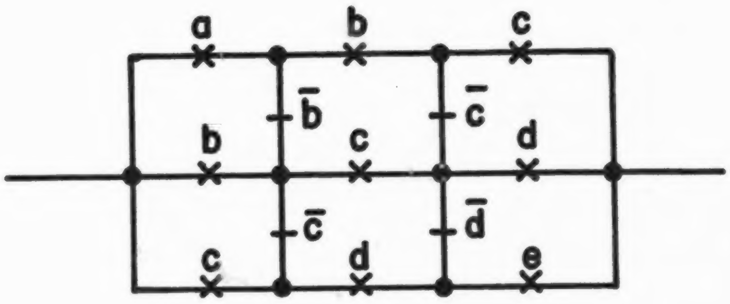
Fig. 10 Three- and six-variable odd-parity circuits.



(a) A three-variable majority circuit.

a	b	c	path
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(b) Table of combinations for the three-variable circuit.


a
b
c
$\bar{b}$
$\bar{c}$
b
c
d
$\bar{c}$
$\bar{d}$
c
d
e

(c) A five-variable majority circuit.

Fig. 11 Two more majority circuits.

this correspondence is quite direct is the representation of the rules for certain kinds of logical puzzles. A traditional problem is the following:

> A boatman must carry a wolf, a goat, and a cabbage across a river in a boat which is so small that he can carry, at most, one of them with him in it at a time. Moreover, whenever the wolf and goat are together, he must also be present to keep the goat from being eaten. Neither can he leave the goat with the cabbage. How can he carry all of them from the south bank of the river to the north bank?

A key initial step for this sort of puzzle is to find a representation for the situation at any given time. Here we let four switches, M (man), W (wolf), G (goat) and C (cabbage) represent the four main characters in the cast. When one of them is on the south shore, the associated switch will be released, and when on the north shore the switch will be operated. The puzzle then becomes one of how, starting with all of the switches released, to operate all of them without violating the rules which have been set down.

The circuit designed later in this section does not itself solve the problem but it will serve as a model of the situation so that various tentative approaches to the problem can be tested without actually getting a wolf, goat, cabbage, boat and a conveniently wide river. The circuit is used to turn on a warning light when one of the conditions of the problem has been violated.

One condition was that on either river bank the wolf should never be with the goat without the man present. Thus, when M is operated (man is on the north shore) and W and G are both released (wolf and goat are on the south shore) the warning light should be on. It should also be on when M is released, and W and G both operated. The other condition was that the goat and cabbage should never be together without the man present. Thus, when M is operated and G and C are both released, or when M is released and G and C are both operated the warning light should be on.

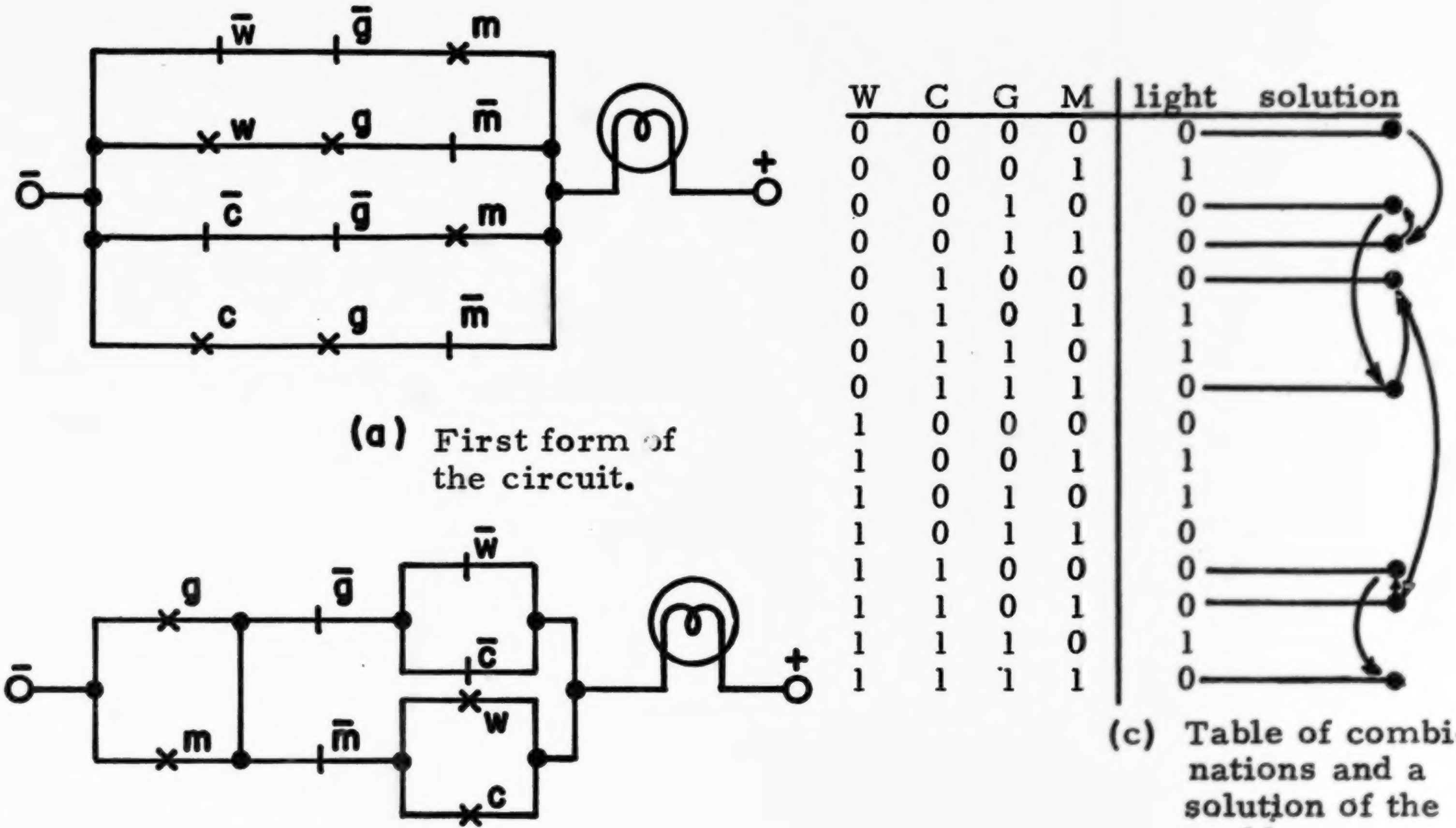
The two preceding pairs of conditions (four in all) lead to a contact network with four corresponding sets of contacts in series, with all of these to be placed in parallel with each other. These sets of contacts are

(1) i: $\overline{w}$, $\overline{g}$, m
(1) ii: w, g $\overline{m}$
(2) i: $\overline{c}$, $\overline{g}$, m
(2) ii: c, g, $\overline{m}$

The corresponding circuit is given in Fig. 12(a). A simplified circuit which is equivalent to that one is given in Fig. 12(b). The reader should verify that these are equivalent by examining each of the sixteen possible combinations in which the four switches can be operated or released.

The rules of the puzzle tell us that the boatman can travel alone between the banks of the river or can transport, at most, one of the others with him. If he goes alone from the north shore to the south shore, this is represented on our model by moving the switch M from the operated position to the released position. If he takes the wolf to the north shore from the south shore, this is represented by operating both of the switches M and W.

The puzzle now becomes one of starting with all switches released and of operating or releasing the switch M (and at most one of the other switches) so that eventually all of the switches are operated and so that in the process, the warning light does not go on. Fig. 12(c) shows one possible sequence of switch



(a) First form of the circuit.

(b) Alternate form of the circuit.

W	C	G	M	light	solution
0	0	0	0	0	●
0	0	0	1	1	
0	0	1	0	0	●
0	0	1	1	0	●
0	1	0	0	0	●
0	1	0	1	1	
0	1	1	0	1	
0	1	1	1	0	●
1	0	0	0	0	
1	0	0	1	1	
1	0	1	0	1	
1	0	1	1	0	
1	1	0	0	0	●
1	1	0	1	0	●
1	1	1	0	1	
1	1	1	1	0	●

(c) Table of combinations and a solution of the problem.

Fig. 12 Circuit models for the "river-crossing" problem.

states which solves the problem. That solution is:

(1) The man crosses to the north shore with the goat.
(2) He goes back to the south shore alone.
(3) The man crosses to the north shore with the cabbage.
(4) He goes back to the south shore with the goat.
(5) The man crosses to the north shore with the wolf.
(6) He goes back to the south shore alone.
(7) The man crosses to the north shore with the goat.

There is another slightly different solution with the same number of steps; can you find it?

6. LOGICAL THOUGHT

What is logical thinking? There is no satisfying answer, for we don't know exactly what mental processes a person uses to draw conclusions from facts at hand. However, we do know that some people, when confronted with problems and puzzles, can obtain answers which seem consistent with conditions of the case.

Centuries ago, mathematicians founded a branch of mathematics which they called logic, to aid thinking of this kind. Originally, these people hoped that all real world problems could be settled by applying logic. Today we know that this feat is impossible. Nevertheless, even today, the ability to draw conclusions which are consistent with facts or theories (sometimes called postulates, as in geometry) is considered a mark of intelligence. Often on intelligence tests, there are questions such as: "Decide if the conclusion 'All zryks sneeze continuously' can be deduced from the postulates 'All three-headed zryks sneeze continuously' and 'All zryks have three heads'." After the initial shock of such a question has worn off, we recognize that whether or not the premises are reasonable, possible or foolish is not at issue. The question seeks to determine if the person being tested can decide what conclusions can legitimately be deduced from even the most unlikely or mysterious premises. This type of "nonsense" question is purposely designed so that experience with the subject of the question, or lack of it, will neither hinder nor help in deducing a legitimate conclusion. Not everyone will agree that questions of this sort really do test intelligence, but many people seem to think so.

Regardless of this issue, however, there is a correct answer to such questions; an answer which can be derived by applying a set of rules. Furthermore, these rules can be stated using the logical terms "and", "or", and "not" which were incorporated into the switching circuits described earlier in this chapter. It is thus possible to deduce logical conclusions from a complex series of facts with the help of switching circuits, assuming that the facts can be represented as circuit elements. This ability to deduce logical conclusions rapidly and accurately is vital in a variety of complex situations. Such situations exist in anti-missile defense, in automatic observation of hospital patients, and in the operation of petroleum refineries, for example.

And

Basically, logic concerns itself with the truth or falsity of compound statements, such as, "The wind is from the northeast and it is raining." There are two component statements in this composite statement, the two being joined by the connective "and". The logical truth of this statement does not depend upon how accurately we measure wind direction or whether we actually observe the rain. Logic specifies only that if both statements are true the compound statement is logically true; otherwise the compound statement is logically false. There is a correspondence here between the "truth-value" (truth or falsity) of the compound statement and the state of the series "and" circuit. The state of that circuit (open or closed) corresponds to the truth value of the entire or compound statement, while the states of the contacts correspond to the truth values of the individual parts or components of the compound statements. Thus, only if both contacts are closed is the circuit closed. The truth-table of Fig. 13 indicates how the logical truth of an "and" statement depends upon the truth of the individual components of the statements. These are labelled A and B for convenience. Note that if "1" were substituted for true and "0" for false, this table would be identical to that for the "and" circuit shown in Fig. 2(c). Note that the logical analysis of the "and" statement depends upon the logical truth of its component statements and not upon the sense or meaning of the compound statements. The truth table of Fig. 13 holds for the statement for "A and B", where the components of the compound statements are represented by letters, as well as for the earlier statement about wind and rain. In other

A	B	A and B
false	false	false
false	true	false
true	false	false
true	true	true

Fig. 13 A truth table for the and connective.

words, logic deals with the logical truth of the relationship between statements and not with the subject matter of the statements themselves, just as the "and" circuit deals with the state of switch contacts and their series connection, not with the size and shape of the contacts, or for that matter any other irrelevant fact about the contacts. To emphasize this point, logicians often use "nonsense" statements, such as "gorgs can play the cello" and "zryk feathers make fine pillows". We use this technique in the next few pages.

Or

Connecting statements to produce a compound statement often involves the use of the word "or". Thus the statement, "Zryk feathers make fine pillows or armidillos shun hyenas" consists of two simpler statements connected by "or". To deduce the logical truth or falsity of the compound statement, we must be aware of the truth or falsity of the individual parts. We must also be careful to interpret properly the exact meaning of the connective "or". The "or" connective can be used in two different senses, designated as the "inclusive or" and the "exclusive or".

In the statement "John will marry either Jane or Linda" we have an example of the "exclusive or". "John will marry Linda" or "John will marry Jane", but in the United States, John obviously will not be permitted to marry both Jane and Linda (at the same time) even though the selection of either is permitted. In the "exclusive or" the acceptance of one choice automatically eliminates the remaining choice.

The word "or" sometimes reflects the possibility that while the selection of one choice or the other may be acceptable, the selection of both choices remains a possibility. This is the "inclusive or". Thus the statement "Payment for the item in six installments can be made by check or money order" uses the or connective as an "inclusive or". Here either checks or money orders or combination of both are acceptable choices.

Normally when a logician uses the connective "or" he uses it in an "inclusive or" sense. If he wishes the "or" to be interpreted as an "exclusive or" he will usually specify this in some fashion such as "A or B but not both". Otherwise his use of the term would mean "A or B or both". In this text, "or" will be assumed to be the "inclusive or", unless otherwise stated.

Returning once again to our statement (which we abbreviate "A or B") we show the associated truth table in Fig. 14. This table should be compared with the table of combinations of Fig. 3(c). Once again we see that a simple contact network can represent a logical statement.

A	B	A or B
false	false	false
false	true	true
true	false	true
true	true	true

Fig. 14 A truth table for the or connective.

NOT

A fundamental rule in logic is that the only allowable truth-values for statements are "true" or "false". (This corresponds to contacts and networks of contacts having two possible states: "closed" or "open".) Words like "maybe" and "perhaps" are useful in everyday conversation but they are not useful in making the definite statements necessary for logical thought.

A system of logic based on statements that can be only true or false is not complete without "complementary statements". A "complementary statement" is one whose truth value is opposite to the original statement. For example, the statement complementary to "Francis smokes cigars" is "Francis does not smoke cigars". The statement complementary to "Gorgs cannot play the cello" is "Gorgs can play the cello."

It is always possible to express the complementary statement by using "not" but often we have other English words which we interpret the same way. "The play was good"; "the play was bad" (not good). "The integer is odd"; "the integer is even" (not odd). "That man is honest"; "that man is dishonest" (not honest).

A word of warning is appropriate, however. We may be used to thinking that certain pairs of words have complementary meanings when they do not. (Consider and criticize the following pairs of words as complementary pairs. Full; empty. White; black. Blonde; brunette. Fast; slow. Positive; negative. Smooth; rough. Heads; tails. Walk; run. Friend; enemy.)

The only safe way to express a complementary idea is to use the word "not". Full; not-full. Empty; not-empty. Fast; not-fast. Walk; not-walk. And so forth.

7. AN EXAMPLE OF A PROBLEM IN LOGICAL THOUGHT

The logical connectives "and", "or", and "not" each have a corresponding switching circuit. Let us now see how this fact may aid logical thinking. Take the following situation:

> Agent 070 has been assigned to watch a house in which a suspected foreign agent, known as Mr. Jones, lives. Agent 070, however, has several other assignments. He can drive by Jones' house only several times a day and observe whether Jones' garage door is open or closed. The "file" on Jones contains the following: Jones is a man of strong habits: whenever his car is in his garage, the garage door is closed. Also, if Jones is at home, then the car is in the garage.

Suppose Agent 070 drives by the house and observes the garage door closed. Can he properly conclude from this observation that Jones is at home?

The knowledge of Jones' habits can be put into two critical logical statements. They are joined by "and" into one compound statement which specifies Jones' behavior:

1. If the car is in the garage, then the garage door is closed, and
2. If Jones is at home, then the car is in the garage.

These statements are not phrased entirely in the logical terms, "and", "or", and "not" which we have studied so far. However, they can be reduced to that form.

Note that these statements take the form "If......., then.......". These have the appearance of logic statements, so let us analyze them as such. Remember that logic is concerned only with the truth or falsity of a statement in terms of its component statements; that is, logic deals with the truth value of statement (1) above in terms of the truth value of the component statements, "the car is in the garage" and "the garage door is closed". In discussing the logic of "If....., then.......", it is useful to think of the entire statement as a promise; thus, "if you study regularly, the teacher will give you a passing grade" is promise of a reward based on a condition. The logic of this relation is summarized in the truth table of Fig. 15(a). In discussing this table, let us abbreviate the statement to: "If A, then B". Now, clearly if A is true (the condition satisfied) and B is true (the reward given), then the overall statement is also true (promise fulfilled); if A is true (condition satisfied) and B is false, (the reward withheld), the statement is contradicted (promise defaulted) and so is false.

Note that if the condition is not fulfilled, then the statement, "If.........., then........" promises nothing. Thus whether the reward is bestowed or not, the promise can be taken as fulfilled. In line with this thinking, when A is false, (condition not fulfilled), the composite statement is taken as true regardless of whether B is true or false.

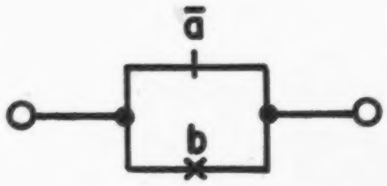
Now we consider a switching circuit to deal with these facts. This circuit must have the same truth table as that of Fig. 15(a). Note that the "If........., then........." column shows all "trues" except when A is true and B is false. Note, too, that the "or" truth table (Fig. 14) shows all "trues" except when A and B are both false. A change of A to "not A" will make these two tables the same as indicated in Fig. 15(b). Thus the statements, "If A, then B" and "not A or B" are logically equivalent. The importance of this result is that a statement used in logical arguments (If______, then______) can be represented by "not" and "or" only. So also a "not______ or______" circuit represents "If....., then.....". Thus, the contact circuit for "if A, then B" is a simple parallel connection of an "$\bar{a}$" contact and a "b" contact as shown in Fig. 15(c).

A	B	If A then B
false	false	true
false	true	true
true	false	false
true	true	true

(a) The truth table

A	B	Not-A	Not-A or B
false	false	true	true
false	true	true	true
true	false	false	false
true	true	false	true

(b) Showing the equivalence of "not-A or B" and "If A, then B".



(c) The contact circuit.

Fig. 15 The if-----then----- connective.

Let us now return to 070 and Jones, and rephrase the logical statements using the equivalences above. First, let us set up the following abbreviations:

A - The car is in the garage
B - The garage door is closed
C - Jones is at home

The logic statements are then

1. If A, then B and 2. If C, then A

Using the logical equivalent for "if___, then ____"; the statements become

1. Not-A or B
   and
2. Not-C or A

The contact network for this compound statement is shown in Fig. 16 (a).

This circuit is a model of the logical situation at Jones' house just as the "river-crossing" circuit of Fig. 12(a) and (b) is a model of the situation there. Neither of these circuits gives the answer to its problem directly, but each circuit can be manipulated by a person to find the answer. The circuit of Fig. 16(a) can, therefore, help in deciding whether or not Jones is at home when 070 observes that the garage door is closed. Note that the problem as posed says that the statement about Jones' habits is true. In the network model of this problem, then, we are interested only in those switch states for which the network is closed.

Recall that the original problem was to decide if Jones is necessarily at home when 070 observes that the garage door is closed. In the network this is equivalent to asking: "If switch B is operated, must switch C be operated in order to close the network?"

Consulting the network again, we see in Fig. 16(b)(where switch B is operated) that the network is closed if either A is operated or C is released or both. Thus, there is a way to close the circuit without operating C (A operated,

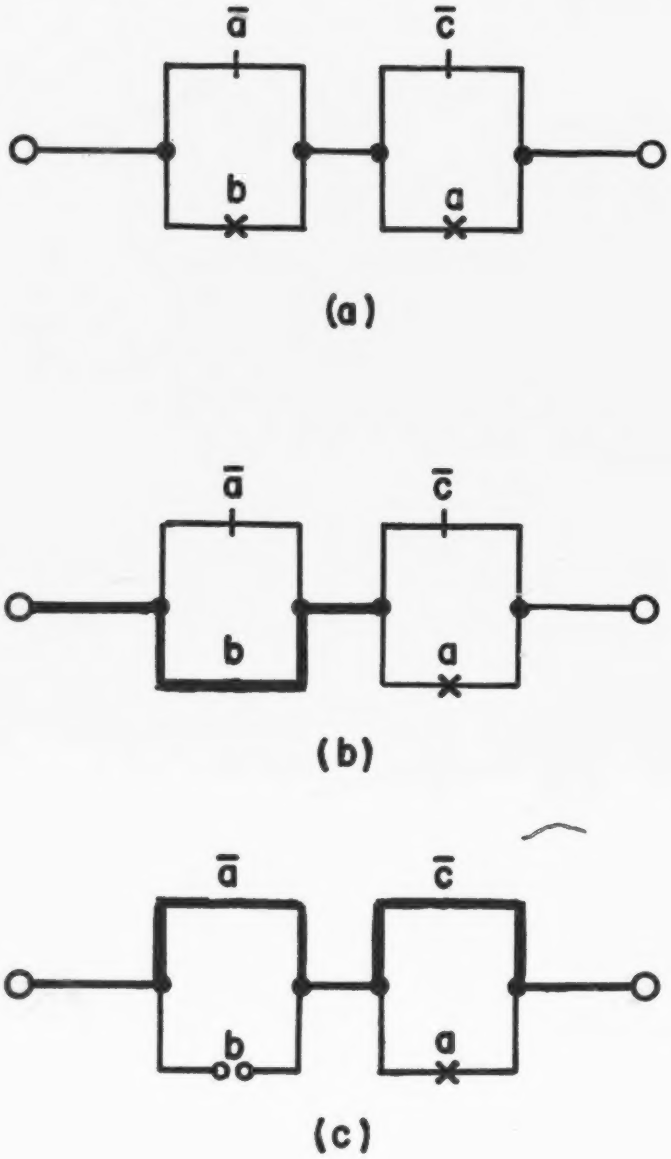


Fig. 16 A Logic Circuit for Agent 070.

C released). Also, the circuit can be closed when C is operated (by operating A). In other words, the conclusion is that if the car is in the garage, 070 does not have enough information to decide whether Jones is at home or not.

This indeterminate situation arises because the condition of the A switch is not determined by the conditions of the problem. That is, 070 does not know whether the car is in the garage or not (he only knows that the garage door is down, not whether the car is or is not in the garage.)

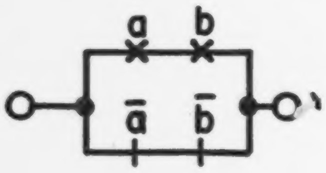
If 070 observes the garage door is open, can he conclude that Jones is at home? In the network model, B is released and the circuit of Fig. 16(c) is applicable. As can be seen there, this circuit is closed only when $\bar{c}$ is closed (C is released), and $\bar{a}$ is closed (A is released). For $\bar{c}$ open (C operated) there is no setting of the A switch which gives a closed path). This situation allows us to conclude that "statement C is false" is the logical answer. Thus Jones is not at home.

This same method can be used to solve much more complex problems in logic. Sometimes in such problems, another logical connective is used. It takes the form "A if and only if B". This statement, too, can be reduced to a form containing only "not" and "or". Let us take an example. Consider this statement:

"Jones is at home if and only if the car is in the garage." This statement says that Jones and the car are always at home or always away at the same time. So the compound statement is true only if both component statements are true or if both are false. That is, the statement is false if the car is at home and Jones is away or vice versa. This situation is summarized in the truth table of Fig. 17(a).

P	Q	P if and only if Q
false	false	true
false	true	false
true	false	false
true	true	true

(a) The truth table



(b) The contact network

Fig. 17 Truth table and corresponding contact network for the if and only if connective.

A slightly different form of the compound statement will aid in finding the contact network for "A if and only if B". The logically equivalent statement is "(A and B) or (not-A and not-B)": (Jones is at home and the car is in the garage) or (Jones is not at home and the car is not in the garage). The corresponding contact circuit is therefore the one shown in Fig. 17(b).

8. CONCLUSION

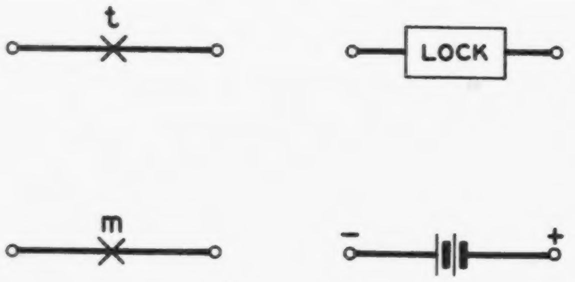
This chapter shows how electric circuits made up of on-off elements (contacts controlled by switches) are put together to represent the basic connectives of logic: "and", "or", and "not". Furthermore, these basic contact circuits can be combined to represent, or model, logical situations such as the river crossing problem, or to perform logical operations, such as in the majority circuit. Logic circuits are at the root of automatic aids to human thought and action. Thus, any problem or operational requirement (as in the hall light problem) which can be stated in logical form can be met by a logic circuit.

So in this sense, logic circuits can be an aid to thought. There are many other aids to human thought, but none relieve us of the obligation to think. For example, reducing a real problem to logical terms may be a mental challenge as great as or greater than the logical manipulations themselves. Furthermore a mistake in this reduction can easily result in an inappropriate answer. Nevertheless, aids to thought put us a step ahead. We return to these ideas later in this course as we talk about computer problem-solving and the use of models.

Meanwhile, we have progressed a significant step toward understanding digital computers. We have seen that switch-controlled contacts properly organized into circuits can perform useful tasks. We see in the next chapter how these circuits can in turn be assembled into larger units to form the basic sub-units of a computer.

PROBLEMS

2-1 A door lock is to be operable only when time switch T and manual switch M are both activated. Draw the circuit from the components shown below

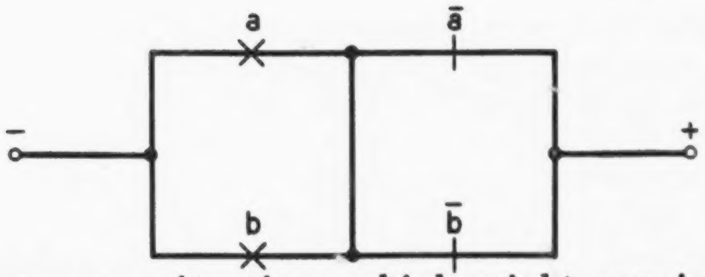


2-2 A single house electric bell is to be operated when either the front or rear door push buttons are operated. Draw the wiring diagram.

2-3 Review the seat ejection problem in Section 2. Complete the truth table for the circuit in Fig. 4.

A	B	CANOPY CHARGE	SEAT EJECTION
0	0		
0	1		
1	0		
1	1		

2-4 The figure below shows a circuit which is to be analyzed.
(a) Construct and complete a truth table for the network.
(b) Compare the truth table with that of Problem 3.
(c) Which of the following is a correct description of the circuit?
and; or; odd-parity; even-parity.



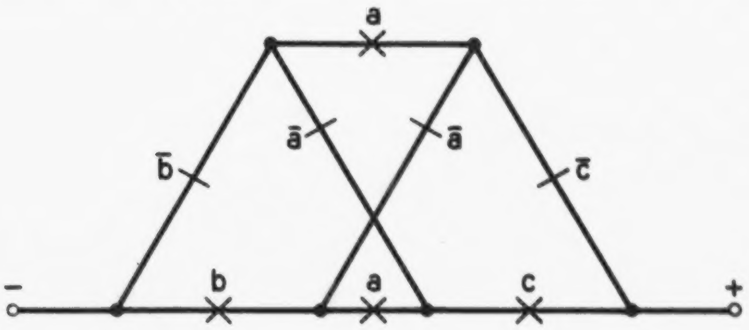
2-5 Describe one or more situations which might require the operation of three contacts in series.

2-6 Describe one or more situations which might require the operation of three contacts in parallel.

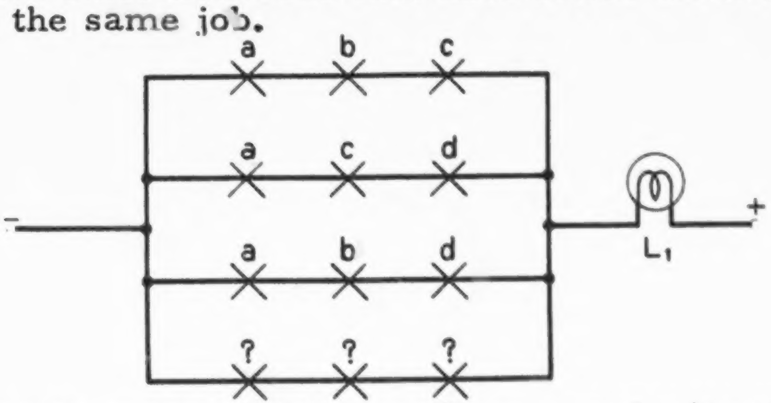
2-7 Construct and complete a truth table for
(a) Fig. 9(a)
(b) Fig. 10(a)
(c) Fig. 10(c) (optional)

2-8 Define odd-parity as used in contact network analysis. Define even-parity as used in contact network analysis.

2-9 Construct and complete a truth table for the network shown below.



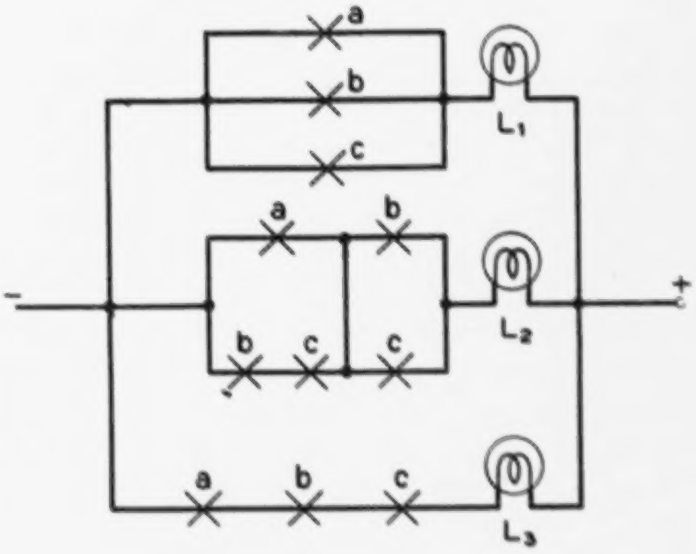
2-10 A board of trustees for the Last National Bank consists of four voting members. All loans must be approved by at least three of the board members before it is accepted. The members wish to vote in secret but wish to know if any three or more members voted yes. Below you will find a contact network for this "at least 3 out of 4" vote problem.
(a) What three contacts should be placed in the bottom branch of this network so that the network is completely specified?
(b) Are any other branches in parallel necessary? If so, why?
(c) Draw two other networks that have fewer contacts and which will do the same job.



2-11 Describe one or more situations where a majority circuit would be appropriate.

2-12 (a) Complete the truth table for the contact network shown below.
(b) How many switches must be operated in order to light only L_1?
(c) How many switches must be operated in order to light only L_1 and L_2?
(d) How many switches must be operated in order to light all three lamps?
(e) Does the order in which the switches are operated determine which lamps will be lighted?
(f) Can you think of a real-life situation in which this circuit might be used?

NETWORK



TRUTH TABLE

a	b	c	L₁	L₂	L₃
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

2-13 Districts I, II and III combined to form a regional school with each district having two members on the Board of Education. Action by the board requires a majority vote by district. A negative vote by one representative of a district acts as a veto on a positive vote by the other representative. Design a circuit which will permit secret voting by individuals. Use a lighted lamp to indicate affirmative action by the board.

Chapter A-3

BINARY NUMBERS AND LOGIC CIRCUITS

1. INTRODUCTION

In Chapter A-2, binary variables (1's and 0's) were used to represent the truth or falsity of logical statements. Furthermore, we found that contact networks can be used to represent such statements. Two such networks, developed in Chapter A-2, were the majority circuit and the odd-parity circuit. In this chapter we see how these can be used to construct a larger circuit which adds numbers. Our adder, however, will work for numbers only if they are expressed as a string of "0's" and "1's", for in this form there can be a direct relation between the individual digits of the numbers and the switch contacts. Thus, even though decimal numbers may be more familiar, binary numbers are more convenient for logic circuits and for computers.

2. THE DECIMAL AND THE BINARY NUMBER SYSTEMS

In the decimal system, a number is expressed as an ordered sequence of digits, such as 85283. Each digit can have one of ten values, 0 through 9. The position of each digit in the sequence determines its value in units, tens, hundreds, etc. The rightmost number gives the units and is called the least significant digit*, while the leftmost number is the most significant digit and in our example represents the number of ten thousands. The values of the digits in a decimal number are all powers of ten. For instance, the value of the leftmost 8 in the number 85283 is $10^4 = 10 \cdot 10 \cdot 10 \cdot 10 = 10,000$. Remembering that $10^3 = 1,000$, $10^2 = 100$, $10^1 = 10$, and $10^0 = 1$, the number 85283 represents

$$\underline{8} \cdot 10^4 + \underline{5} \cdot 10^3 + \underline{2} \cdot 10^2 + \underline{8} \cdot 10^1 + \underline{3} \cdot 10^0 = 8 \cdot 10000 + 5 \cdot 1000 + 2 \cdot 100 + 8 \cdot 10 + 3 \cdot 1$$

A similar system is used for binary numbers. A binary number is expressed as an ordered sequence of binary (two-valued) digits, such as 10111. The value of each of these digits is a power of two, and the digits are ordered so that the least significant digit is at the right and the most significant at the left, as before. Therefore, the binary number 10111 has the decimal value

$$\underline{1} \cdot 2^4 + \underline{0} \cdot 2^3 + \underline{1} \cdot 2^2 + \underline{1} \cdot 2^1 + \underline{1} \cdot 2^0$$

---

*This positional notation, which we now take for granted, did not always exist. In more ancient systems, such as the Roman (in which the number 1984 would be written MCMLXXXIV), computation with numbers was so difficult that only a learned scholar could handle even the simplest problems in arithmetic. We use the Hindu system, which was developed from the earliest positional notation we know of -- that of the Babylonians and Sumerians.

A significant advantage of a positional notation is that after learning only a few rules which apply to the numbers represented by single digits, one can use these same rules to add, subtract, multiply and do other computations on numbers of any size whatsoever.

$= 1 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 1 \cdot 1$

$= 23$*

Any number in the binary system can be converted to a decimal number by this technique. The list of numbers below, written in both the binary and decimal systems, are examples for practice.

Binary representation	Decimal representation
$1 \cdot 2^4 + 1 \cdot 2^2 + 1 \cdot 2^0 = 10101$	$2 \cdot 10^1 + 1 \cdot 10^0 = 21$
111001	57
1001101	77
1110111	119
1111111	127
11111000000	1984
1010	10
1100100	100
1111101000	1000
11110100001001000000	1000000

Some preliminary comments must be made before we can determine how to convert a number from the decimal notation to the binary notation. First, recall that to multiply a decimal number by ten, you must shift the number one place to the left and add a zero. (Note that 852830 is ten times 85283.) Similarly, a number in the binary system is multiplied by two by shifting it one position to the left and putting a "0" to the right of the shifted number. Thus, 101110 is twice 10111.

If a number is written in the binary system, it is easy to determine whether it is even or odd by looking at the rightmost digit. If that digit is "0", the number is exactly twice the value of some other integer, and consequently the number is even. If the rightmost digit is "1", the number is not twice another integer and therefore the number must be odd. In that case, the number can be made even (and therefore divisible by two) by subtracting 1 from it.

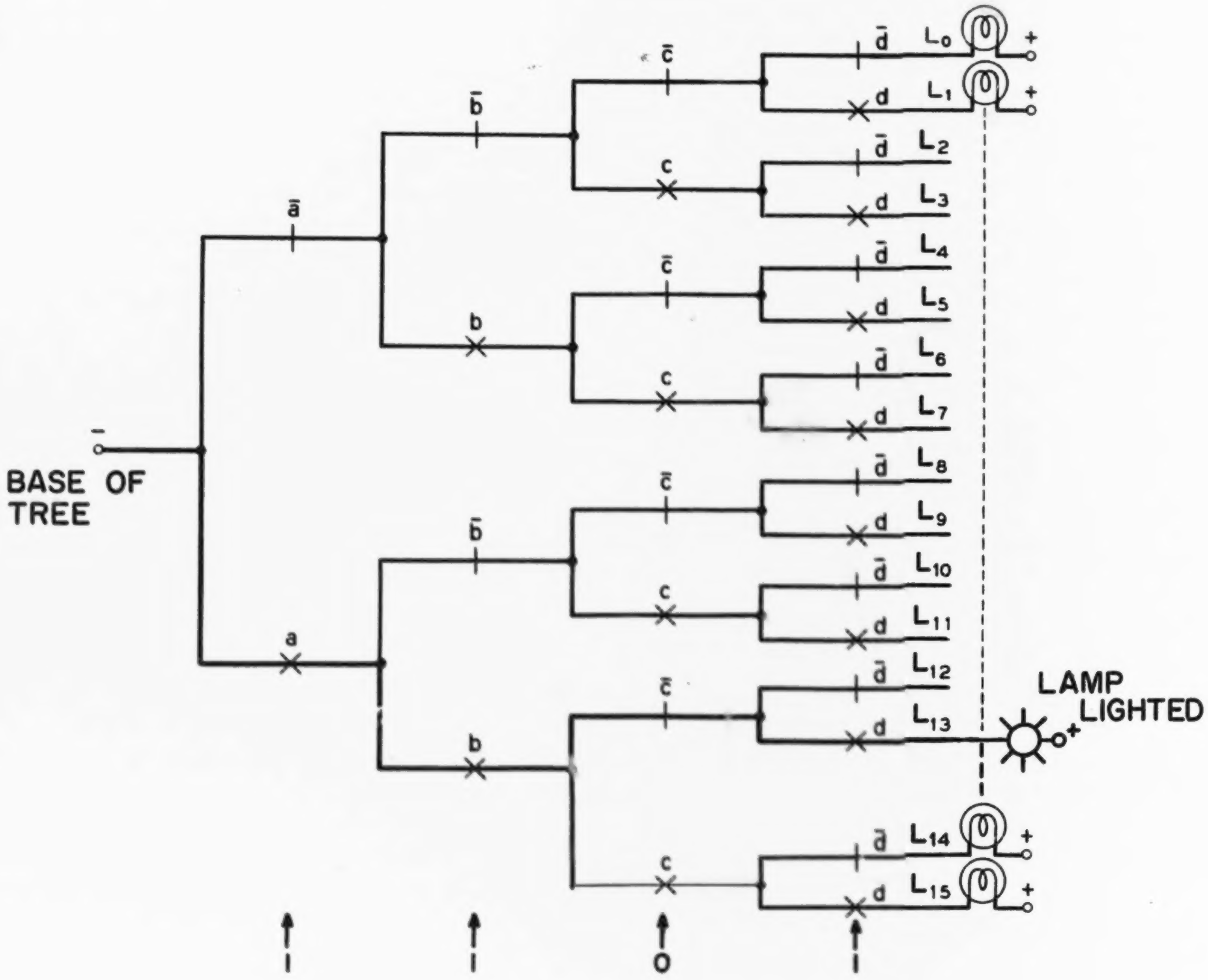
These facts can be exploited for converting decimal numbers to the equivalent binary numbers. The procedure determines the digits in the binary number, one at a time, starting with the rightmost, or least significant, digit and working to the left. For example, conversion of the decimal number 117 is illustrated below:

117 is odd.	Thus the rightmost or least significant digit is 1.
(117-1)/2 = 58 is even.	Thus the next digit is 0.
58/2 = 29 is odd.	Thus the next digit is 1.
(29-1)/2 = 14 is even.	Thus the next digit is 0.
14/2 = 7 is odd.	Thus the next digit is 1.
(7-1)/2 = 3 is odd.	Thus the next digit is 1.
(3-1)/2 = 1 is odd.	Thus the next digit is 1.
(1-1)/2 = 0. Finish.	

---

*The smallest possible integer base for a positional notation is "2". Any integer could be used as a number base. From time to time advocates of a base-12 or "duodecimal" system appear. They claim that this base would have many advantages over the decimal system because 12 is divisible exactly by four smaller integers -- 2, 3, 4 and 6 -- rather than only by the two -- 2 and 5 -- which are possible in the decimal system. The duodecimal system would require twelve different symbols; for example, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, α, and β.

Thus, Decimal 117 = Binary 1110101. Note that the basic procedure consists of successive divisions by 2, except that the number is always made even (by subtraction of 1, if necessary) before the next division.



BASE OF TREE
−
ā
a
b̄
b
c̄
c
d̄
d
L0
L1
L2
L3
L4
L5
L6
L7
L8
L9
L10
L11
L12
L13
L14
L15
+
LAMP LIGHTED
1
1
0
1

Fig. 1 A selection "tree" circuit set to 1101.

3. A "TREE" CIRCUIT

A circuit which converts binary to decimal numbers is shown in Fig. 1. Known as a selection tree circuit, it will play an important role in our computer later. Imagine that we wish to turn on one of sixteen different lights (numbered L_0 through L_{15}) by operating four switches (A, B, C, and D). The state of the switches represents the binary number and the lighted lamp the corresponding decimal number. For instance, if switches A, B, C and D are set to 1, 1, 0, and 1, respectively, lamp #13 is turned on since the contacts "a", "b", "c" and "d" are closed. When these contacts are closed, the only path completed from the "base" of the tree to a lamp is the one which goes to lamp #13.

Tree circuits are commonly used in computers to connect any one part to another selected part so that information in the form of binary digits may be transferred between them. In Chapters A-4 and A-5 we investigate in more detail how this is done.

4. THE "SIGN-MAGNITUDE" NUMBER REPRESENTATION

Very often negative numbers must be represented in a computer. We have chosen the "sign-magnitude" method, which is a common one. Here the leftmost digit of a binary number is used to specify the sign of the number. This digit is called the sign-bit. If the sign bit is "0" the number is positive, and if the sign bit is "1" the number is negative. (The magnitude, or "size", of a number is always positive.) Therefore, for example,

0000000000101 represents +5, and
1000000000101 represents -5.

In this example, there are 13 bits in each number. Since the sign occupies the first bit, there are only twelve bits available to represent the magnitude. Thus, numbers as large as $2^{12}-1 = 4095$ can be represented. The representation for +4095 is 0111111111111 and that for -4095 is 1111111111111.

S = Sum
X = First number to be added
Y = Second number to be added

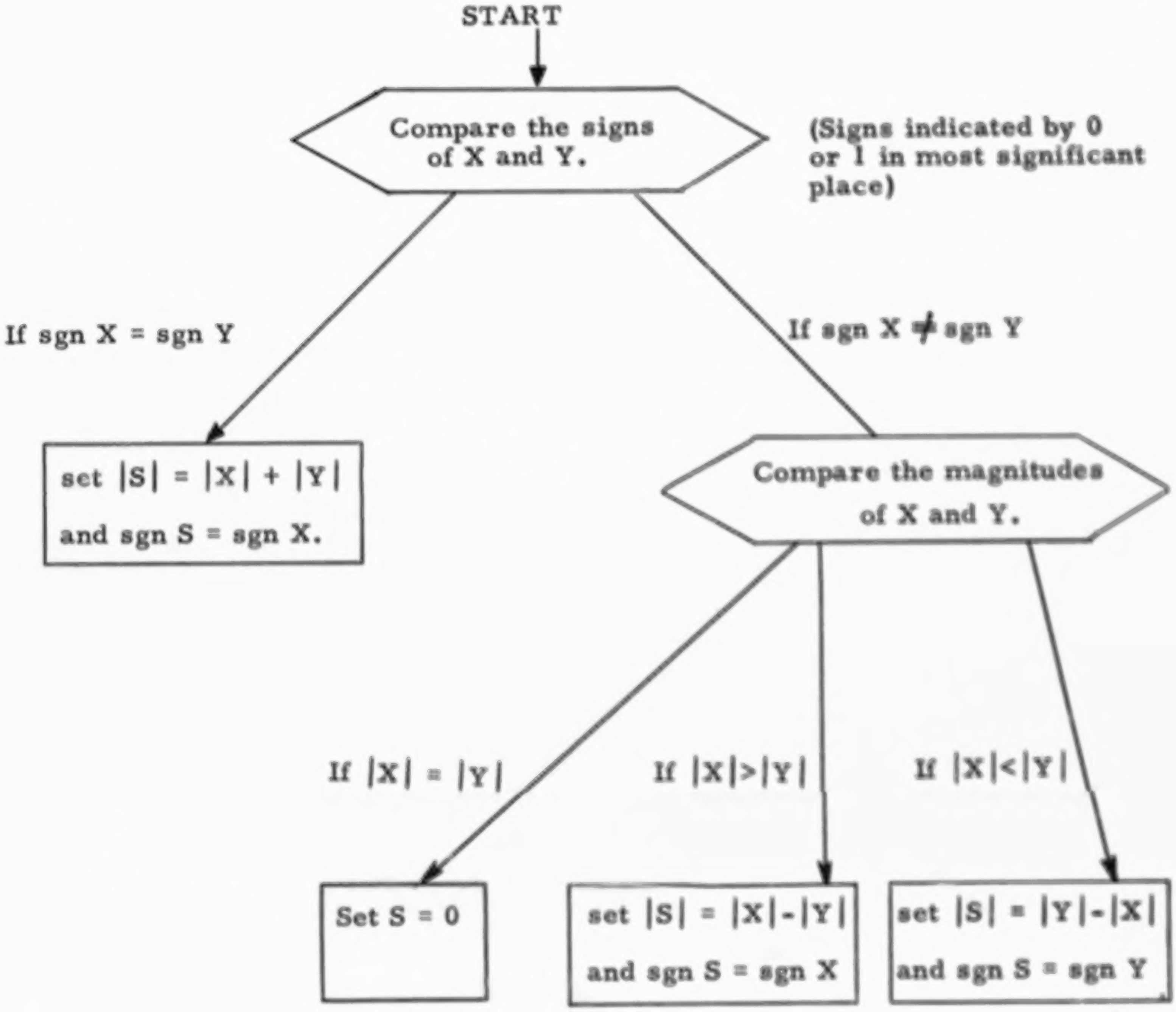

START
Compare the signs of X and Y.
(Signs indicated by 0 or 1 in most significant place)
If sgn X = sgn Y
If sgn X ≠ sgn Y
set |S| = |X| + |Y| and sgn S = sgn X.
Compare the magnitudes of X and Y.
If |X| = |Y|
If |X|>|Y|
If |X|<|Y|
Set S = 0
set |S| = |X|-|Y| and sgn S = sgn X
set |S| = |Y|-|X| and sgn S = sgn Y

Fig. 2 A flow chart showing how to compute S = X + Y

D = Difference
X = First number
Y = Second number
(i.e., D = X - Y)

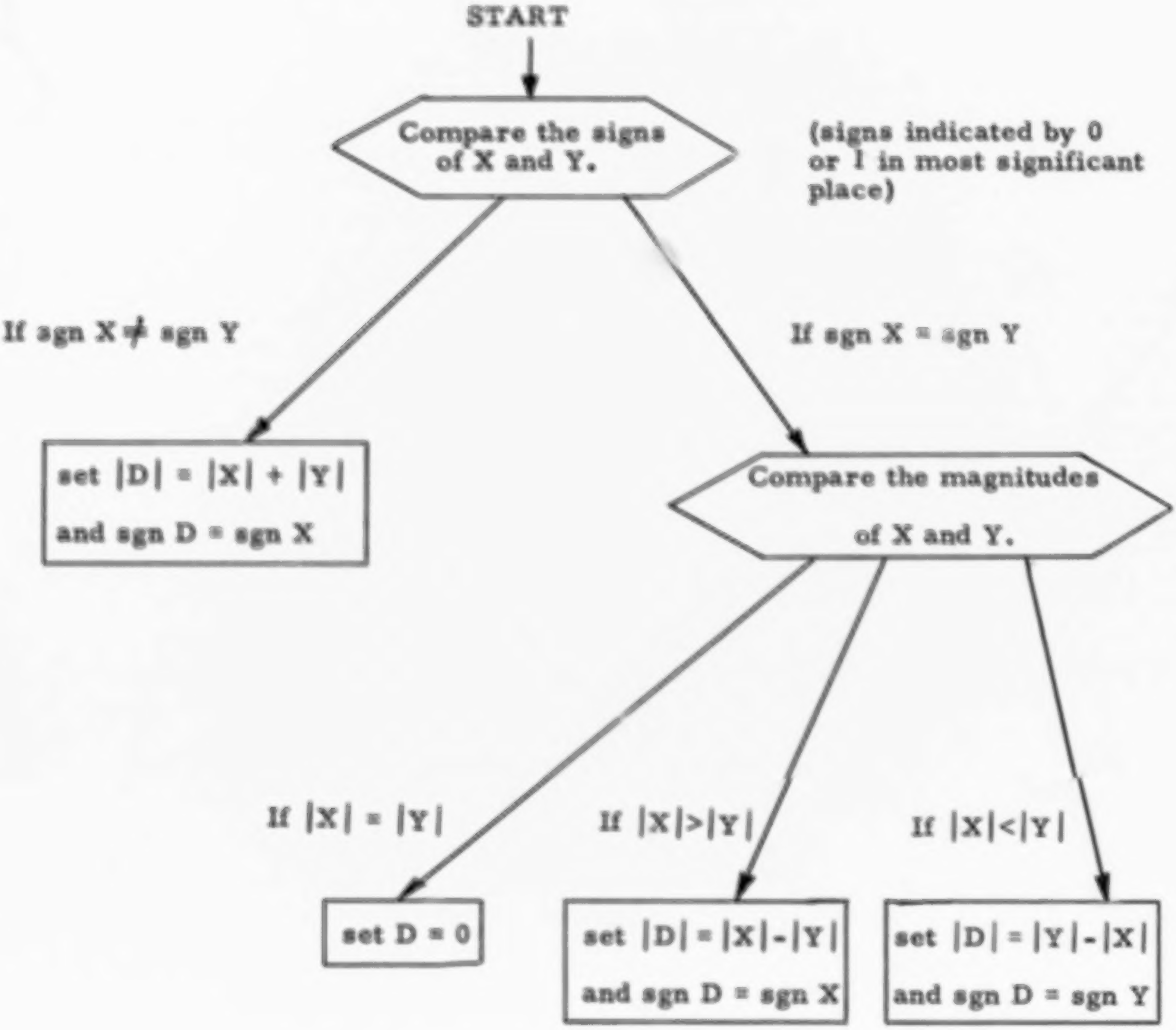

START
Compare the signs of X and Y.
(signs indicated by 0 or 1 in most significant place)
If sgn X ≠ sgn Y
If sgn X = sgn Y
set |D| = |X| + |Y| and sgn D = sgn X
Compare the magnitudes of X and Y.
If |X| = |Y|
If |X|>|Y|
If |X|<|Y|
set D = 0
set |D| = |X|-|Y| and sgn D = sgn X
set |D| = |Y|-|X| and sgn D = sgn Y

Fig. 3 A flow chart showing how to compute D = X - Y

It is simpler to design adding circuits if they only need to add positive numbers. It is simpler, also, to design subtracting circuits for subtracting a smaller positive number from a larger positive number. Thus the add and subtract circuits of our computer operate only on the magnitudes of the pairs of numbers, X and Y. This requires that to compute the sum, S, or difference, D, of numbers which can be either positive or negative, the signs and magnitudes of these numbers must be treated separately. For example, in subtracting "-5" from "+3" (3-(-5) = 8), there are two parts to be determined: namely, the magnitude of the difference which is "3 plus 5"; and the sign of the difference which is positive. In general, the details of the decisions which must be made are shown in Fig. 2 and 3 as flow charts. In these "flow charts" we have represented the magnitude of a number N by "$|N|$" and the sign of N by "sgn N".

The decision procedures shown in the flow charts are based upon comparisons of the signs and of the magnitudes of the numbers X and Y. In these charts the result of a comparison determines which of several things should be done next; the arrows show the choices which are available. The two-variable "odd-parity" circuit (Fig. 9(a), Ch. A-2) may be used to compare the sign bits of the two numbers. We have, as yet, no specific procedures or circuits for comparing the magnitudes of two binary numbers. Such a circuit is discussed in the next section.

5. A CIRCUIT WHICH COMPARES TWO POSITIVE INTEGERS

If two positive numbers are expressed in the binary system, they may be compared to determine which is the larger by the rules shown in the flow chart in Fig 4(a). For this scheme to produce correct results, each number should contain the same number of digits. Any difference in length can be eliminated by placing one or more "0's" to the left of the shorter number. In a computer, the magnitudes of numbers are ordinarily represented this way. Now, the flow chart says: Starting with the leftmost position, compare the digits in the corresponding positions of the two numbers. If these digits have the same value, compare the next pair of digits to the right. Continue to do this until an unequal pair is found or until there are no more digits. If an unequal pair is found, the number which contains a "1" at this position is larger than the number which has the "0". If the values of the digits are the same in each position, the numbers are equal. This procedure is illustrated in Fig. 5.

Problem: Which is the larger number?

$|X| = 0\ 0\ 1\ 1\ 0$

$|Y| = 0\ 1\ 0\ 1\ 1$

Steps according to Fig. 4 (a)

Step 1. Compare the leftmost pair of digits
Result: Equal
Step 2. Compare the next pair of digits
Result: digit of $|Y| >$ digit of $|X|$
Conclusion: The magnitude of Y is greater than the magnitude of X.

Fig. 5

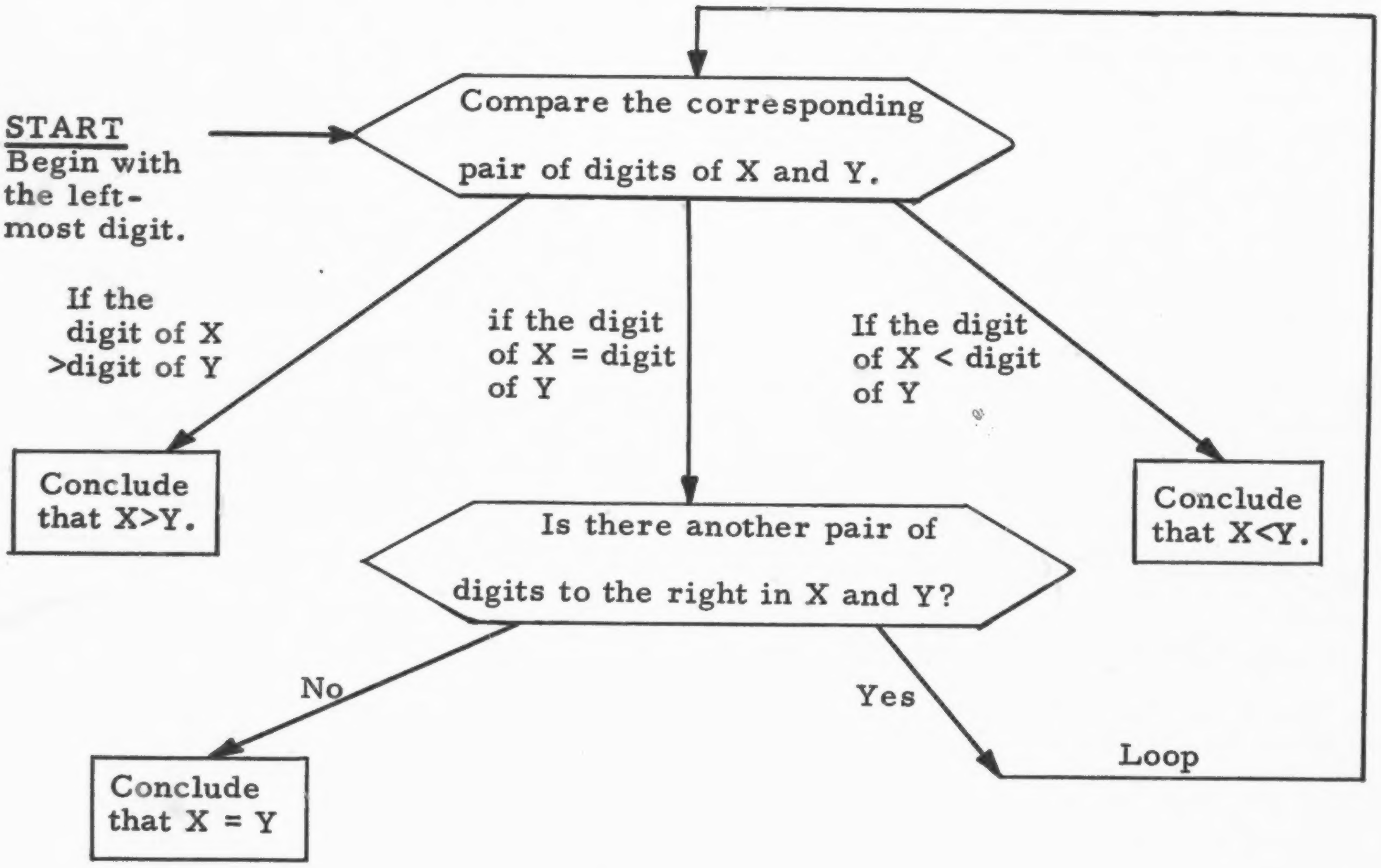

START
Begin with the left-most digit.
Compare the corresponding pair of digits of X and Y.
If the digit of X >digit of Y
if the digit of X = digit of Y
If the digit of X < digit of Y
Conclude that X>Y.
Conclude that X<Y.
Is there another pair of digits to the right in X and Y?
No
Yes
Loop
Conclude that X = Y

Fig. 4(a) Comparing two positive binary numbers

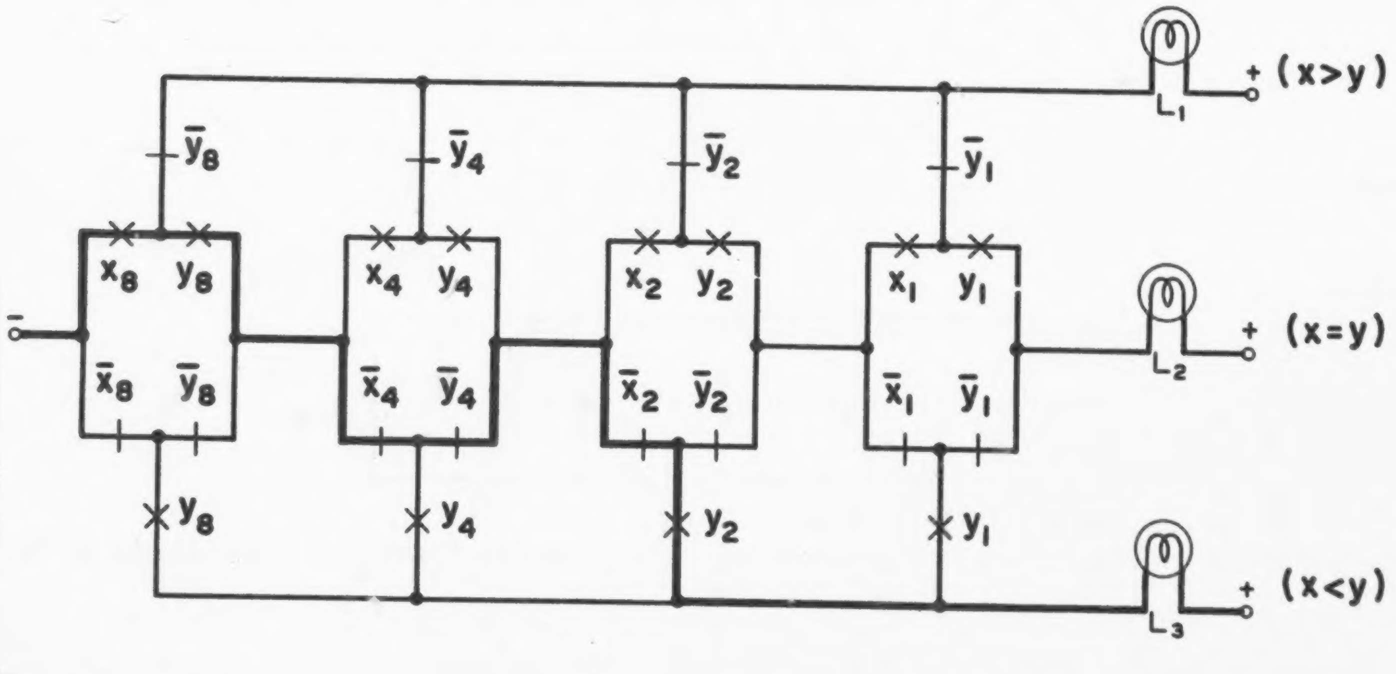

$\bar{y}_8$
$\bar{y}_4$
$\bar{y}_2$
$\bar{y}_1$
x_8 y_8
$\bar{x}_8$ $\bar{y}_8$
x_4 y_4
$\bar{x}_4$ $\bar{y}_4$
x_2 y_2
$\bar{x}_2$ $\bar{y}_2$
x_1 y_1
$\bar{x}_1$ $\bar{y}_1$
y_8
y_4
y_2
y_1
L_1
L_2
L_3
+ (x>y)
+ (x=y)
+ (x<y)
−

Fig. 4 (b) The comparison circuit

The flow chart procedure applies equally well to numbers expressed in any base. Some procedure of this kind is followed in comparing two decimal numbers. The comparison always proceeds from left to right. The "loop" in the flow chart dramatizes the fact that there is a simple basic step which may be repeated over and over again. This step is merely the comparison of the corresponding digits. It may be expected, therefore, that the comparison circuit will have one basic contact network repeated many times. The circuit in Fig. 4(b) has this property. It has four repeated "stages" but it could be extended to as many stages as there are positions in the numbers to be compared.

In the figure we have assigned the contacts on a set of four switches X_8, X_4, X_2, and X_1 to represent the digits of the number X and those on switches Y_8, Y_4, Y_2, and Y_1 to represent the digits of Y. (The subscripts indicate the values of the digits.) For example, take specific values of X and Y:

X = 1001 (nine) is represented by $X_8 = 1$, $X_4 = 0$, $X_2 = 0$, $X_1 = 1$ and

Y = 1010 (ten) is represented by $Y_8 = 1$, $Y_4 = 0$, $Y_2 = 1$, $Y_1 = 0$.

The operated switches are X_8, X_1, Y_8, and Y_2. The others are released. Under these conditions the heavily lined path is completed to lamp L_3, thereby indicating that $X < Y$. In general, a path will be completed to L_1 if X is greater than Y, to L_2 if X is equal to Y, and to L_3 if X is less than Y.

6. ADDING OF TWO BINARY NUMBERS

Every general purpose digital computer has an arithmetic unit, the heart of which is an adder capable of adding two binary numbers. The logic circuit for doing this is so fundamental that it is considered as a basic building block of large computers.

We add two numbers in the binary system in much the same way that we add numbers in the decimal system. One number, X, is placed above the other, Y, so that digits of the same value, or weight, are in the same column. We work progressively from the rightmost column towards the left. At each step the pair of corresponding digits of X and Y (the digits in a given column) are added together, along with any "carry" digit from the previous column. The sum of these three digits is computed. In general, this sum may require two digits in its representation: the carry digit and the sum digit. The sum digit is written at the foot of the column in the required sum. The carry digit is placed in the next column to the left, and the procedure is repeated until the sum is complete.

Weights:	64	32	16	8	4	2	1	
Carry digits, C:	1	1	0	1	1	1	(0)	
First number, X:	0	0	1	0	1	1	1	(=23)
Second number, Y:	0	1	1	0	0	1	1	(=51)
Sum, S:	1	0	0	1	0	1	0	(=74)

Fig. 6 (a) An example of binary addition which illustrates the rules.

Arithmetic sum of the digits of C, X and Y	Carry Digit	Sum Digit
zero	0	0
one	0	1
two	1	0
three	1	1

(b) Rules for determining the carry and sum digits for binary addition.

c	x	y	Carry Digit	Sum Digit
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

(c) Table of combinations for the carry and sum digits for binary addition.

Fig. 6 Addition of two binary numbers

As an example of this procedure we have, in Fig. 6 (a), shown how to add the numbers 10111 and 110011, which are equivalent to the decimal numbers 23 and 51. The reader should reconstruct the individual steps required by following the rules given in the above paragraph.

Note that in any column the sum of the carry digit and the digits from the numbers X and Y may have any value between zero and three. The binary representations of these sums are 00, 01, 10, and 11, respectively. In the first two of these sums, the carry digit is 0. In the remaining two the carry digit is "1" which must be transferred to the next column to the left. These facts are summarized in Fig. 6 (b).

The table of combinations given in Fig. 6(c) shows in greater detail all possible combinations of the values for C (carry), X and Y digits, and the resulting values for their sum and their carry digits. From our viewpoint, the most important thing about this table is that we have seen its right-hand two columns before. The carry digit column is the same as that in Fig. 11(b) and the sum column is the same as that in Fig. 10(b), both of Chapter 2. These described the behavior of the three variable majority and odd-parity circuits, respectively.

This indicates that the carry digit in our adder circuit can be generated by the same circuit which the three legislators used to decide if a majority of them had voted "Yes", and that the sum digit can be generated by the same odd-parity circuit needed to control a hall light from each of three positions. The simple rule for determining the sum digit in any column is: If the column contains an odd number of "1's" (odd parity), the sum digit is 1; otherwise the sum digit is 0. For the carry digits: If the column contains two or more "1's" (majority), the carry digit is 1; otherwise it is zero. It is apparent that these two basic rules can be used for columns of indefinitely long binary numbers. Note that there is no carry digit in the rightmost column (that is, C_1 always has the value 0). Thus C_2 is "1" only when both X and Y are 1, and S_1 is 1 when either X or Y (but not both) is 1.

When two sufficiently large binary numbers are added the number of digits in the sum may be one more than the number of digits in either number. This fact will require special steps when an adder circuit is used.

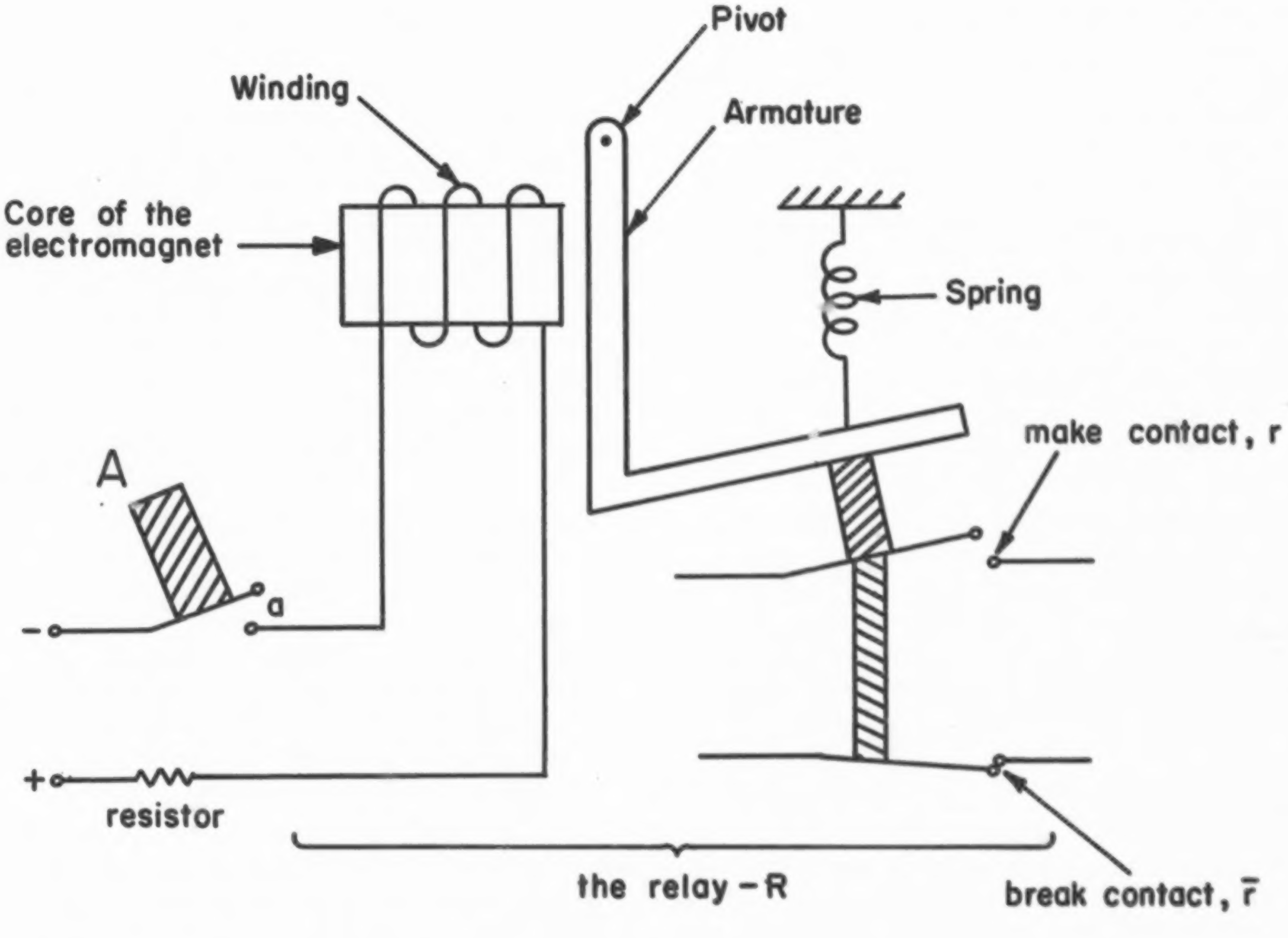

Pivot
Winding
Armature
Core of the electromagnet
Spring
make contact, r
A
a
−
+
resistor
the relay − R
break contact, $\bar{r}$

Fig. 7 Schematic diagram of a relay.

7. HOW TO CONSTRUCT A CIRCUIT WHICH ADDS TWO NUMBERS

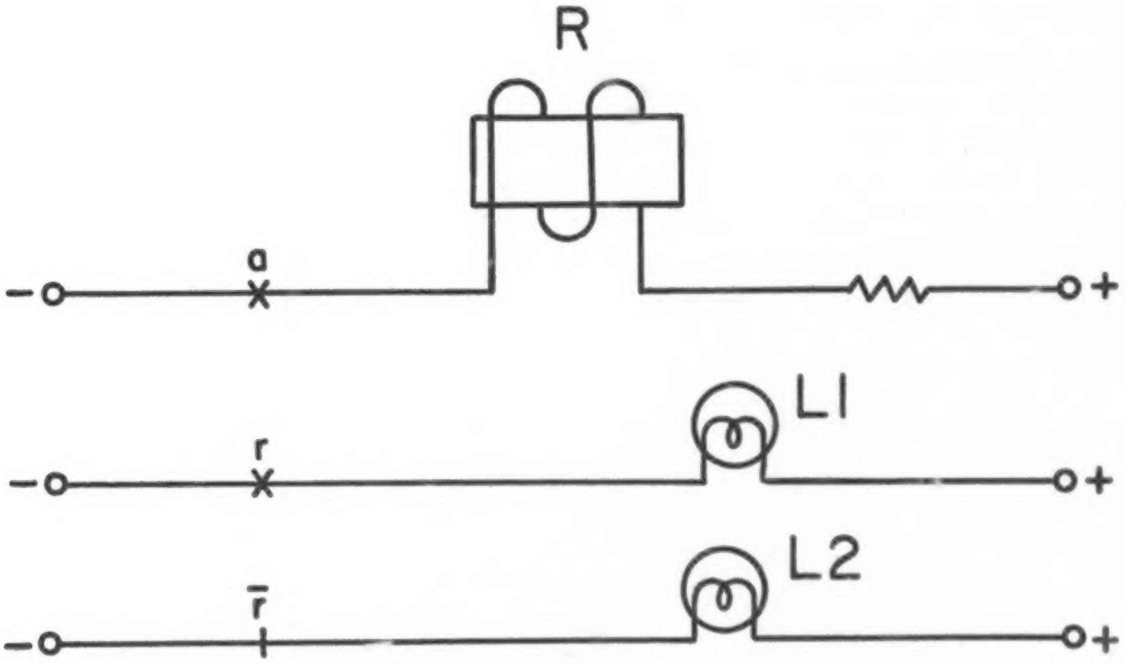
A new element, the relay, will be needed for the adder circuit. A relay (see Fig. 7) is a device which controls make and break contacts through a lever operated and released by an electromagnet, rather than by hand. An electromagnet consists of a coil of wire called the winding wound around a piece of iron or other easily magnetized substance called the core. When a sufficiently large electric current exists in the winding, the electromagnet attracts the armature which is made of magnetic material and is free to move about a pivot. When the armature has been pulled toward the core the relay is said to be operated, and its make contacts are closed, its break contacts are open. When there is no current in the winding a spring pulls the armature away from the core, the relay is then said to be released. In that state the make contacts are open, and break contacts are closed. The physical appearances of these various parts on actual relays may differ in detail from the diagram. Though Fig. 7 shows one make and one break contact, most relays are constructed to control many contacts. The diagram shows how a relay "R" could be used to allow a hand operated switch, A, to control the make and break contacts, r and $\bar{r}$, on the relay. When the "a" contact is closed the relay will operate. (The resistor shown in the winding circuit simply limits the amount of current in the relay winding. If the resistance is too large, there will not be enough current to operate the relay. If the resistance is too small there will be an excess of current and the winding may overheat). Operation of the relay closes the make contacts and opens the break contacts. When the switch A is released, the current in the winding ceases, the pull of the electromagnet on the armature disappears, the spring returns the armature to the original released position, the make contacts open and the break contacts close.

The action of any make contacts ("r") on the relay is the same as the action of the make contact ("a") in series with the relay winding. However, the contact "a" is sometimes replaced by a complex network of contacts. In these cases, the single contact "r" reflects the action of the entire network: when the network is open, "r" is open, and when the network is closed, "r" is closed. Note that the break contact, "$\bar{r}$", behaves oppositely. Thus the relay permits a single network to control many make or break contacts on a single relay. Thus in applications where many copies of a complex contact network would be required, one copy of the entire network suffices. Its action is duplicated by using relay controlled contacts.

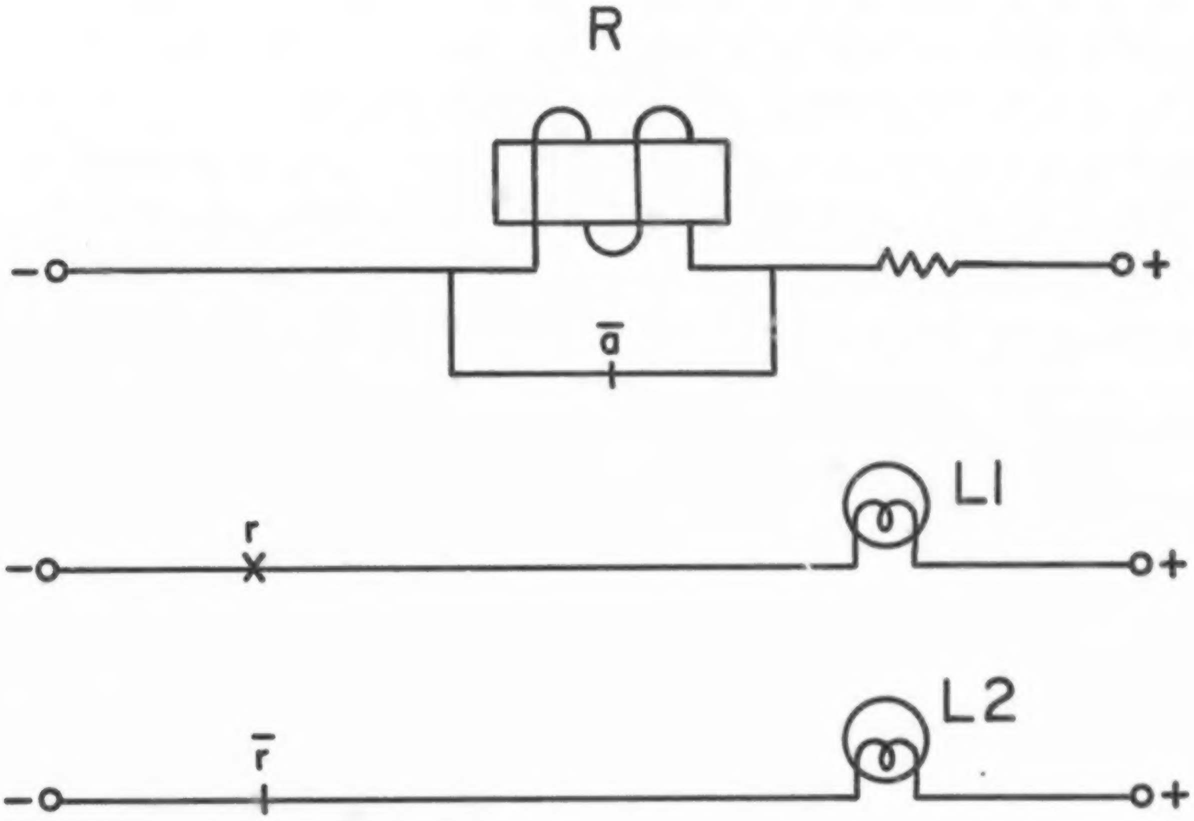
When a relay is shown in a circuit diagram most of the working parts are not included. It is only necessary to indicate how the current through its winding is controlled and how its contacts are used. In Fig. 8 (a) a typical circuit diagram is shown. The contact "a" in series with the relay winding controls the operation of the relay. The relay contacts, r and $\bar{r}$, operated by the relay R, control the lamps, L_1 and L_2. (Compare the circuit with that of Fig. 7 in Chapter 2).

Because the "a" contact is in series with the relay winding this arrangement is called series control. Series control is used in the design of the adder in the next section.

Another way of controlling a relay is shown in Fig. 8 (b). Here a break contact, "$\bar{a}$", has been placed in parallel with the relay winding. When this contact is open (that is, when the switch A is operated) current will exist in the winding and the relay R will become operated. When the contact "$\bar{a}$" is closed, however, the current from the electrical source through the resistor is diverted, or shunted, around the relay winding. This type of control of a relay is called shunt control.


R
a
LI
r
L2
$\bar{r}$
−
+

(a) Series control


R
$\bar{a}$
LI
r
L2
$\bar{r}$
−
+

(b) Shunt control

Fig. 8 Series and shunt control of a relay.

Almost all of the current follows the path through the shunting contact rather than through the winding because the resistance of the contact is so much less than the resistance of the winding. With the current shunted around the winding (that is, with the switch A released) the relay will become released.

The same relay action can be obtained with either series or shunt control. However, as indicated in Fig. 8(a) and 8(b) complementary contacts (make in one case, break in the other) are necessary on the relay winding. Note in the figure the lamp response to the action of switch A is identical in both circuits. Combinations of these two basic types of control are also possible. We examine some examples in the following chapter.

8. THE ADDER AND ITS RELAYS

The logical rules for generating the sum and carry digits in a binary adder have already been presented. Now they must be incorporated into contact networks. The adder diagramed in Fig. 9 adds numbers having three binary digits, but the circuit could be extended to any number of digits. The number X will be entered into the circuit by three switches which we call X_4, X_2, and X_1; switches Y_4, Y_2 and Y_1 provide the second number, Y. (For example to add X = 6 (binary 110) and Y = 7 (binary 111), set X_4, X_2 and X_1 to 1, 1 and 0, and set Y_4, Y_2 and Y_1 to 1, 1 and 1).

Remember that the sum and carry digits in each column of an addition involves the carry digit from the previous column. In the adder, the carry digits are represented by the states of three relays, called C_8, C_4 and C_2. The digits of the sum, S, are to be displayed by the lamps S_8, S_4, S_2 and S_1 -- "0" by an unlighted lamp and "1" by a lighted lamp. (The lamp S_8 is controlled by the left-most carry, C_8. If the sum of the added numbers could always be represented by three digits that lamp would not be necessary).

Returning to the example, the six switches, X_4, X_2, X_1, Y_4, Y_2, and Y_1, are set as noted above. The actions of the contact networks which control the relays and lamps are as follows:

$X_1 = 0$ and $Y_1 = 1$: Thus, $S_1 = 1$ ("odd-parity" of x_1 and y_1);
also, $C_2 = 0$ ("and" of x_1 and y_1).
$C_2 = 0$, $X_2 = 1$ and $Y_2 = 1$: Thus, $S_2 = 0$ ("odd-parity" of c_2, x_2 and y_2);
also, $C_4 = 1$ ("majority" of c_2, x_2 and y_2).
$C_4 = 1$, $X_4 = 1$ and $Y_4 = 1$: Thus, $S_4 = 1$ ("odd-parity" of c_4, x_4 and y_4);
also, $C_8 = 1$ ("majority" of c_4, x_4 and y_4).
$C_8 = 1$: Thus, $S_8 = 1$.

The lamps S_8, S_4, S_2, and S_1 indicate, by their states, that the sum of 6 (binary 110) and 7 (binary 111) is 13 (binary 1101).

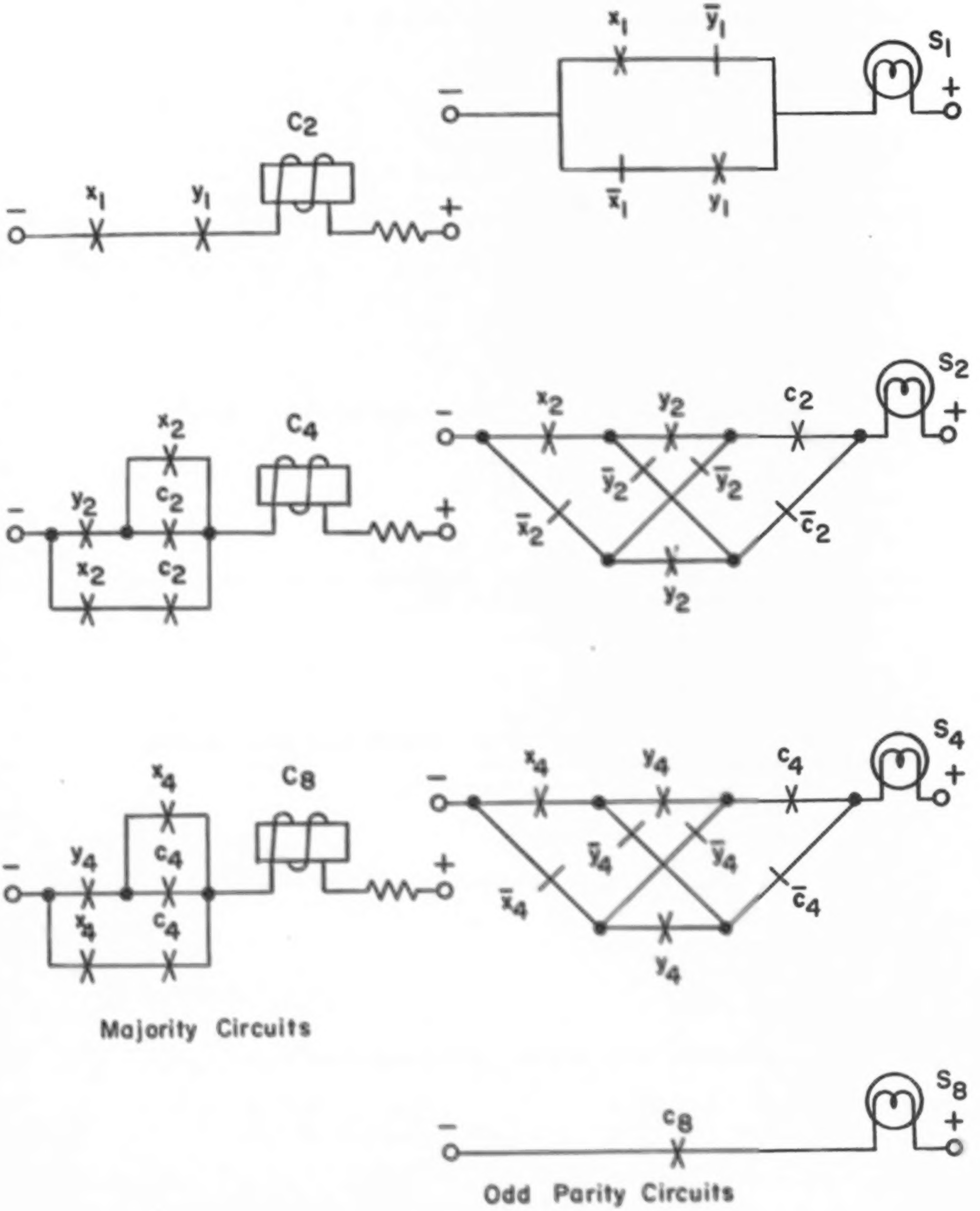

x_1
$\bar{y}_1$
S_1
$\bar{x}_1$
y_1
C_2
x_1
y_1
S_2
x_2
y_2
c_2
$\bar{y}_2$
$\bar{y}_2$
$\bar{x}_2$
$\bar{c}_2$
y_2
C_4
x_2
y_2
c_2
x_2
c_2
S_4
x_4
y_4
c_4
$\bar{y}_4$
$\bar{y}_4$
$\bar{x}_4$
$\bar{c}_4$
y_4
C_8
x_4
y_4
c_4
x_4
c_4
Majority Circuits
S_8
c_8
Odd Parity Circuits

Fig. 9 A three-digit binary adder.

PROBLEMS

3-1 Convert the following binary numbers to decimal form.

11
101
110
1011
1010
10000
110010
11010
1100100
1111101000

3-2 Convert the following decimal numbers to binary form.

1
8
4
2
9
15
16
31
32
27

3-3 Convert the following decimal numbers to binary form. Use the technique illustrated in Section 2.

73
119
237
527
512
256

3-4 Add the following pairs of numbers. Perform the addition in binary arithmetic, and express the answer in binary form.

100 + 11 =
101 + 11 =
1000 + 1000 =

3-5 Perform the following binary addition. Check your work by converting the numbers to decimal form.

1010 + 110010 =
1011 + 11010 =

3-6 Perform the following binary subtractions. Check your work as in Problem 3-5 above.

100 - 11 =
1010 - 101 =
110011 - 11010 =

3-7 Perform the following subtractions, and check your work.

1100100 - 11010 =
11010 - 110100 =

3-8 Refer to Section 5 and Fig. 4

(a) What is the meaning of the subscripts on the switch letters in Fig. 4?
(b) For the following pairs of numbers, which lamp will light? Why?

(1) A = 1010
B = 111
(2) A = 1010
B = 1100
(3) A = 1010
B = 1010

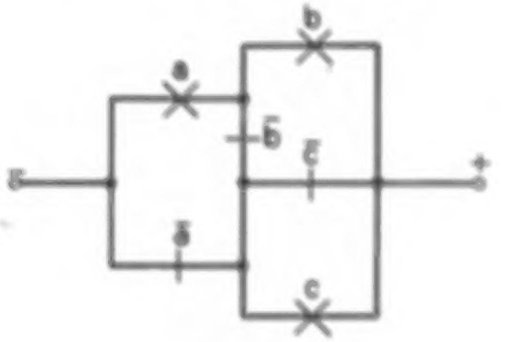
3-9 Explain the function of
(a) the first left-hand section of the binary adder in Fig. 9.
(b) the first right-hand section of Fig. 9.

3-10 Construct and complete [for Fig. 9] a truth table for
(a) the left-hand section of the first stage;
(b) the right-hand section of the first stage.

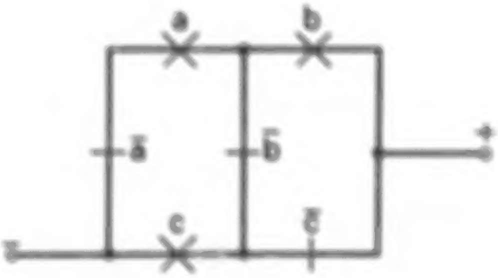
3-11 Name one application of a tree circuit.

3-12 Study Fig. 1. Can you find a closed path between lamps 6 and 8 for any state of the switches? Between any other pair of lamps?

3-13 Complete the truth table for the two networks shown below. For each determine a simplified network which has the same truth table.



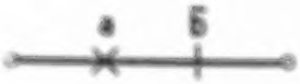
NETWORK SC



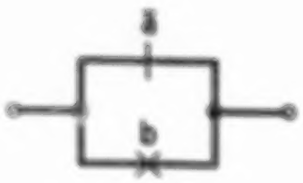
NETWORK OC

A	B	C	SC	OC
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

3-14 A general rule is illustrated in the six networks shown below. Find the rule by completing the truth tables and use it to get a network for column Z. (Hint: Your rule will need the words "series", "parallel" and "not".)

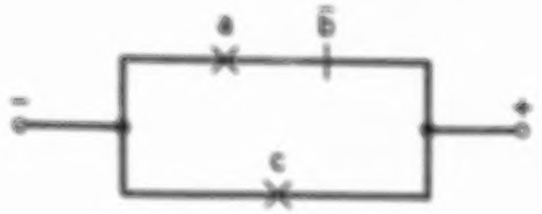


NETWORK S



NETWORK T

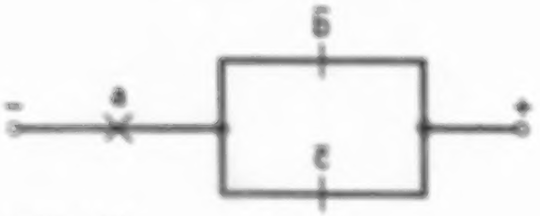
A	B	S	T
0	0		
0	1		
1	0		
1	1		



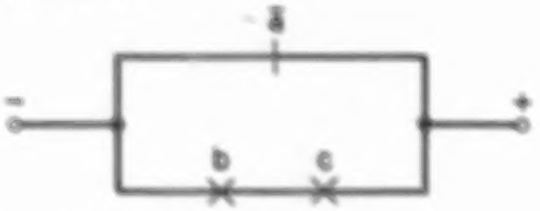
NETWORK Y

A	B	C	Y	Z
0	0	0		1
0	0	1		0
0	1	0		1
0	1	1		0
1	0	0		0
1	0	1		0
1	1	0		1
1	1	1		0

NETWORK Z
(DRAW THIS NETWORK)

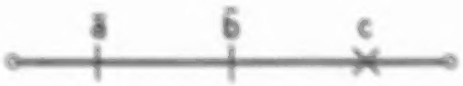


NETWORK W

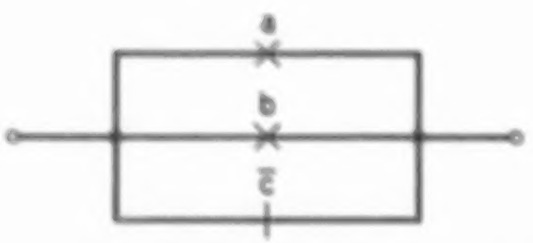


NETWORK X

A	B	C	W	X
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		



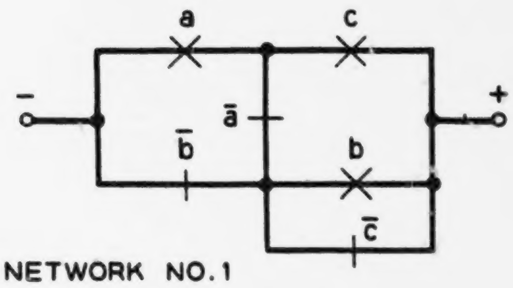
NETWORK U



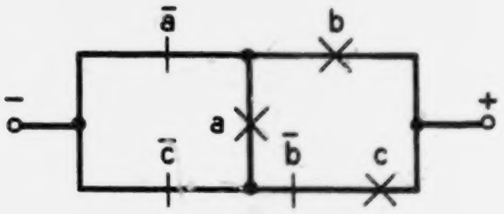
NETWORK V

A	B	C	U	V
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

3-15 Derive a truth table for each of the two networks shown below. For each determine a less complicated network which has the same truth table.



NETWORK NO. 1



NETWORK NO. 2

Chapter A-4

LOGIC CIRCUITS WITH MEMORY

1. INTRODUCTION

One of the most fascinating facts about circuits containing logic elements such as contacts controlled by relays is that they can "remember" something about what has happened to them in the past. This fact is exploited on a large scale in digital computers, where data and the instructions about what to do with these data are stored in the computer's memory. Memory is needed in a less obvious way to accomplish the necessary control within a computer. For instance, as we see in Section 6, even the simple process of counting requires memory in an elementary form.

"Memory" circuits are widely used in everyday life. There are circuits at the telephone exchange which remember the individual digits one by one as they are dialled, until the dialling procedure is completed. When you push the "Up" or "Down" button to call an elevator to your floor, you do not have to continue to press it because there are circuits which remember which button was pushed. At many pedestrian crossings there are traffic lights which can be changed by pushing a button. The circuits for these are often designed so that a pedestrian can push the button and cause the light to change immediately to "Red" for the automobile traffic. If a second pedestrian pushes the same button a fraction of a minute later, the light will not change at once because the circuit has been designed to remember that the button was pushed less than, say, one minute ago. When the one-minute interval is over, however, the circuit remembers that the button has been pushed a second time, and it will again turn the light on.

None of the circuits discussed in the previous chapter on logical design had memory of the kind illustrated above. The lamp controlled by the majority-vote circuit lighted or did not light in accordance with the present positions of its switches and was not affected by what settings these switches might have had in the past. The hall-light, odd-parity circuit was affected only by the present status of its switch positions and not by what happened to these switches previously. The adder circuit delivered its answer independently of the past history of its use. In the following sections we examine circuits that store the evidence of past events.

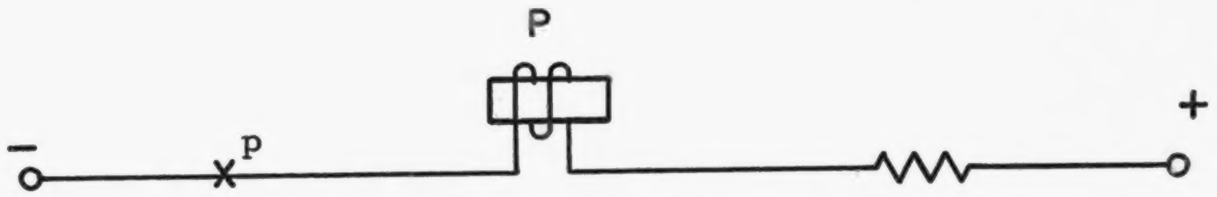
In human beings and in computers the words "memory" and "state" are very closely associated with each other. Out mental state changes as the result of experience and this change usually involves the memory of that experience. Human memory is a complicated and subtle thing to define or to localize. The state of our brain seems to depend upon the chemical and electrical states of its billions of neurons. We have not been able to identify any specific area of the brain as the repository for memories of past events. No surgeon has been able to eradicate the recollection of a predetermined single specific event by cutting into some specific area of the brain.

In our understanding of computers we are in a much more favorable situation because it is we who have controlled how the computer elements are interconnected. We also know exactly how the states of the computer elements are determined and how they can be changed. Though there are a legion of magnetic and electronic devices that can be used for computer memory, in this course relays and the contacts on these relays will be used. The remainder of this chapter is devoted to investigating the use of relays for the simple memory functions required in computers.

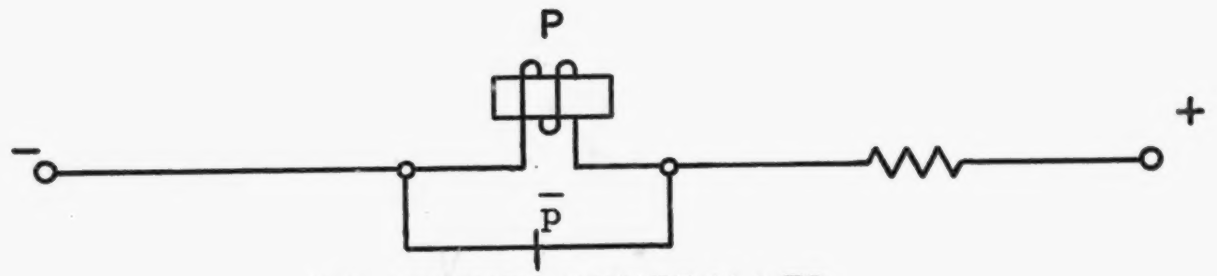
2. MEMORY AND FEEDBACK

Circuits with two stable states

The basis for memory in relay circuits is illustrated by the pair of circuits shown in Fig. 1. In each of these a single contact (the "holding" contact) on a relay controls the current through that same relay. In the series circuit there are two possible states for the circuit. If the make contact "p" is open there is no path for current through the winding. Therefore, the relay will remain released. If, on the other hand, the relay somehow becomes operated, the contact "p" will be closed and a path for current through the winding will be maintained. In either case the holding contact assures that the relay will remain in the state (operated or released) it is set in, and consequently the circuit is stable.



(a) SERIES CONTROLLED



(b) SHUNT CONTROLLED

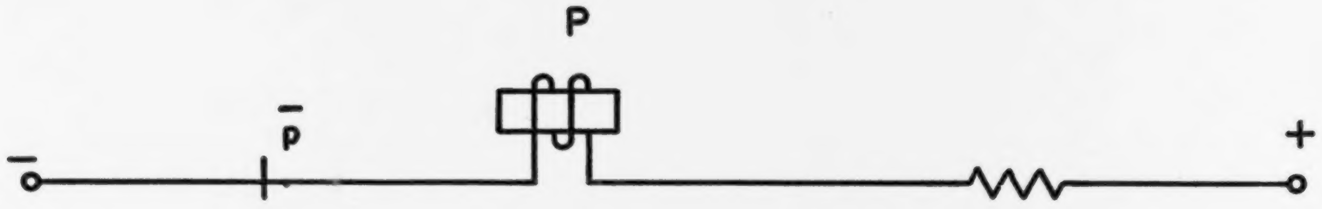
Fig. 1 Relay circuits with two stable states

In the shunt-controlled circuit there are also two possible states, both stable. If the relay is initially in the released state the break contact "$\overline{p}$" will be closed and current will be shunted around the relay winding; this will cause the relay to remain in the released state. If the relay is originally operated the "$\overline{p}$" contact will be open and the relay will continue to be operated.

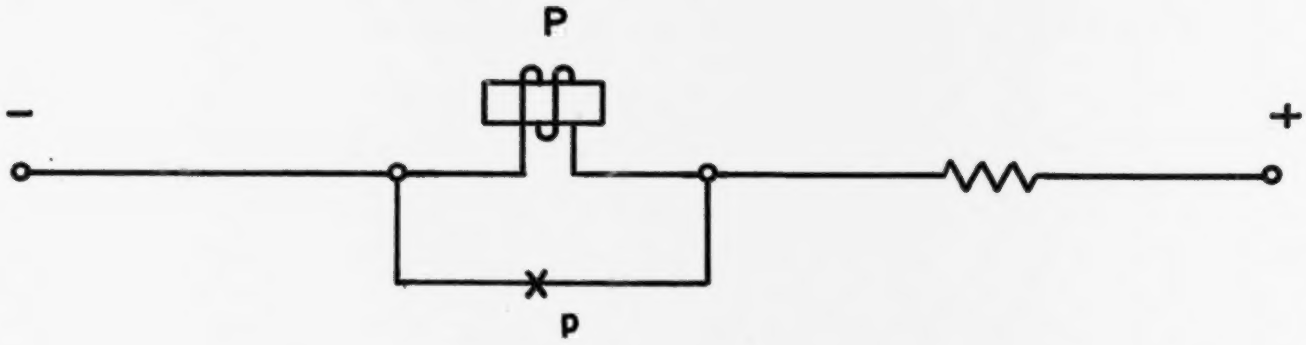
The circuits of Fig. 1 are not very useful but they do illustrate that a single relay, controlled by one of its own contacts, exhibits elementary memory. If the armature of either relay is set to the operated position it will remain there indefinitely. If the armature is pushed to the released position the relay will also remain in that state. The relay "remembers" in which position its armature was last set. In Section 3 we discover methods for setting the state of the relay circuit which do not involve pushing and pulling on the relay armature itself.

Buzzers

A relay may be controlled by its own contacts so that it is unstable rather than stable. In Fig. 2(a), assume that the relay is released. The break contact "$\overline{p}$" will then be closed and there will be current in the relay winding. As soon as the pull on the armature is great enough the relay will operate and the contact "$\overline{p}$" will open. With this contact open the current in the winding will be reduced to zero and the relay will release. This cycle of events will occur again and again (with most relays, between about five and fifty times a second) and the relay will act as a buzzer.



(a) SERIES CONTROLLED



(b) SHUNT CONTROLLED

Fig. 2 Relay circuits with two unstable states.

Another buzzer circuit [Fig. 2(b)] uses a make contact to shunt-control a relay. Again neither the operated state nor the released state of the relay is stable and the relay will buzz.

The distinction between energizing and operating a relay

The simple circuits shown in Figs. 1 and 2 illustrate that a relay controlled by one of its own contacts may have memory or may buzz. Which action occurs depends upon the type of contact and its connection to the relay armature. In such circuits, it is important to distinguish between a relay being operated and being energized.

A relay is operated or released depending upon the position of its armature and therefore depending upon the states of the contacts coupled to the armature. A relay is energized if there is a path for current through the relay winding; otherwise the relay is de-energized. For example, in the buzzer circuits of Fig. 2, an operated relay may be de-energized, and a released relay may be energized. Both of these conditions are unstable because a de-energized relay will release and an energized relay will operate. In either case, it takes some time for the release or operate action to take place. Usually these "release" and "operate" times are fractions of a second. Circuits with holding contacts (Fig. 1) illustrate that the released state of a relay is stable if the relay is also de-energized and that the operated state is stable if the relay is energized at the same time.

The necessary distinctions are further emphasized by letting lower-case letters represent the state of the make contacts on a relay (they reflect whether the relay is operated or released) and the corresponding capital letters represent the state of the relay winding (energized or not) as reflected by the state of the network which would be used for series control of the relay. That is,

$p = 0$ means that the relay is released
$p = 1$ means that the relay is operated
$P = 0$ means that the relay (winding) is de-energized
$P = 1$ means that the relay (winding) is energized

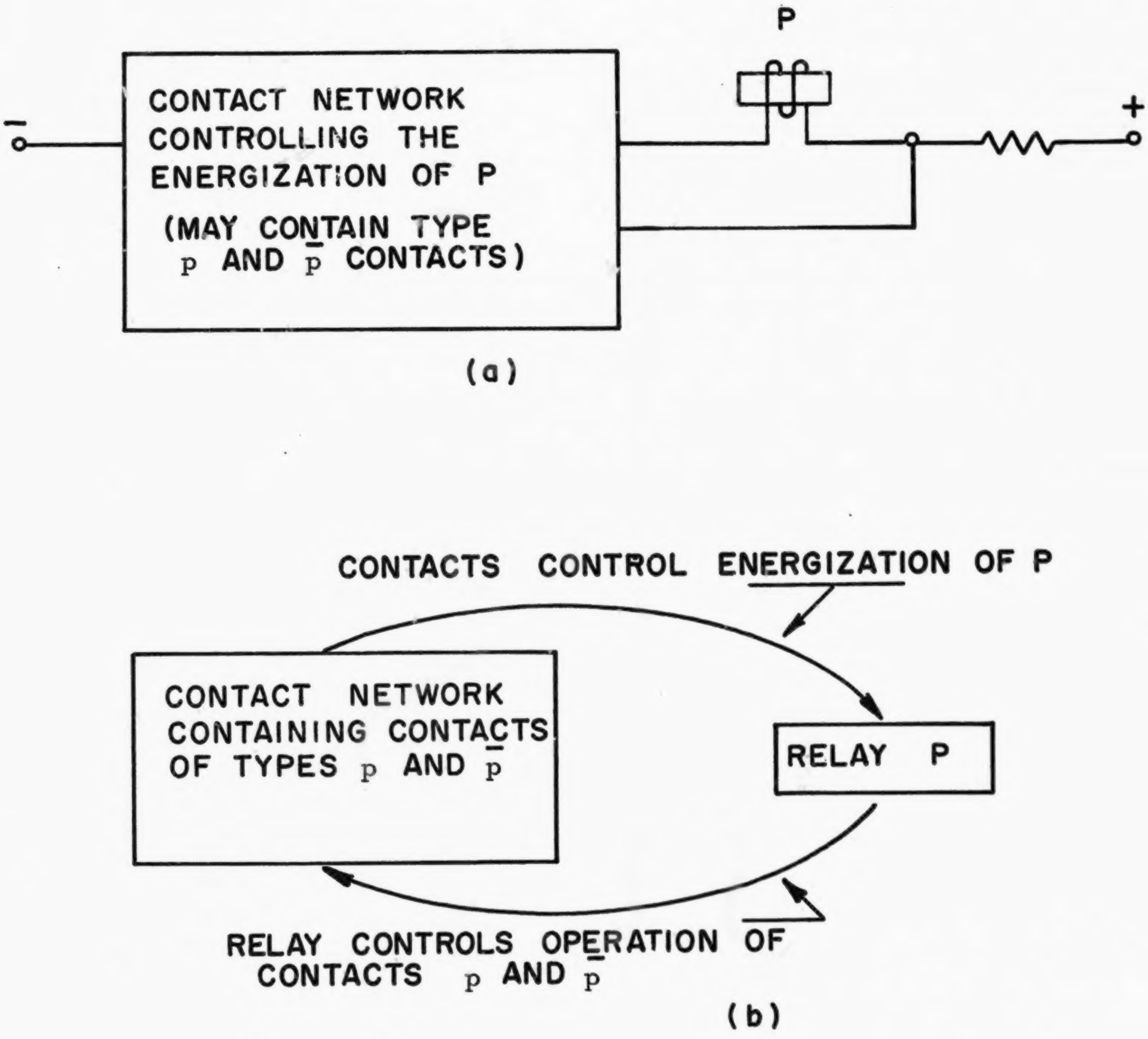
For the stable circuits in Fig. 1, $P = p$, implying that when the relay is energized, the relay contact is closed. For the unstable circuits in Fig. 2, $P = \overline{p}$.

Note that in the circuits of Figs. 1 and 2, the relay contacts control the state of the relay winding, and vice versa. This situation is an example of feedback. A contact network controlling current through a relay winding is suggested in Fig. 3(a). The flow of influence is illustrated in Fig. 3(b). The state of the relay winding is "fedback", influencing its contacts which in turn influence the state of the winding (energized or not). The ideas of feedback and stability are taken up in greater detail later in the course.

3. CIRCUITS HAVING STATES WHICH CAN BE CHANGED

Memory circuits which are more useful

The preceding discussion indicates that the state of a memory circuit can be changed by opening or closing contacts in the controlling network. If these contacts are not all on the memory relay then the memory state can be controlled by a switch whose state is not affected by the memory relay. In terms of Fig. 3(b), such a switch is outside the feedback loop, though its contacts would appear


P
−
+
CONTACT NETWORK CONTROLLING THE ENERGIZATION OF P
(MAY CONTAIN TYPE p AND $\bar{p}$ CONTACTS)
CONTACTS CONTROL ENERGIZATION OF P
CONTACT NETWORK CONTAINING CONTACTS OF TYPES p AND $\bar{p}$
RELAY P
RELAY CONTROLS OPERATION OF CONTACTS p AND $\bar{p}$

(a)

(b)

Fig. 3 Schematic diagrams demonstrating feedback in relay circuits.

in the box marked "contact network". The six circuits in Fig. 4 are completely equivalent with regard to how the state of the relay (operated or released) depends upon the opening and closing of the contacts on switches A and B, which are outside the feedback loop. Imagine that switches A and B are to be operated as "push buttons". That is, each switch is operated only while it is being pushed. When the pressure is removed the switch is released again.

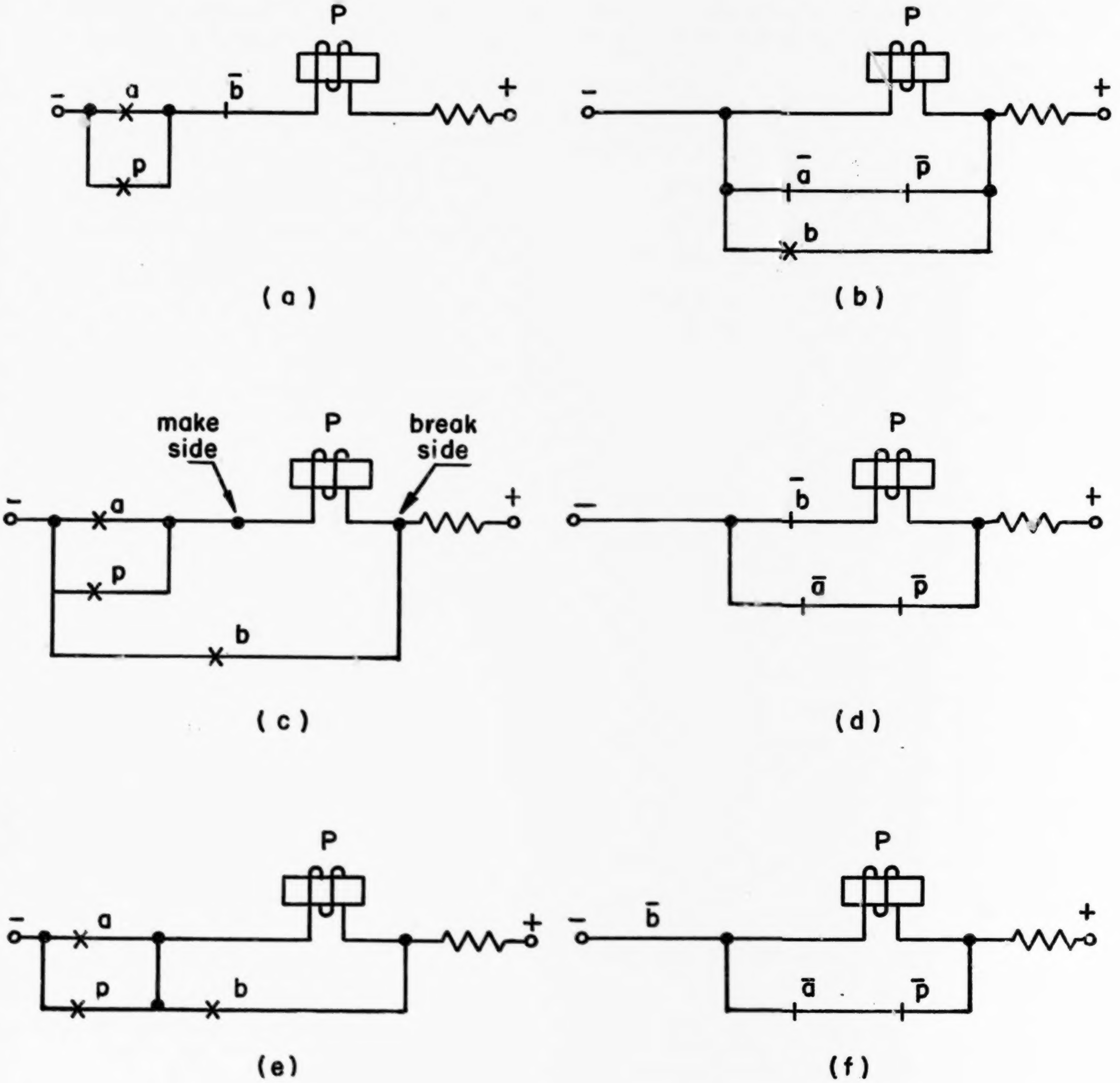

P
a
$\bar{b}$
p
+
−
(a)
P
$\bar{a}$
$\bar{p}$
b
+
−
(b)
make side
P
break side
a
p
b
+
−
(c)
P
$\bar{b}$
$\bar{a}$
$\bar{p}$
+
−
(d)
P
a
p
b
+
−
(e)
P
$\bar{b}$
$\bar{a}$
$\bar{p}$
+
−
(f)

Fig. 4 Six equivalent circuits with two-state memories.

With neither switch A nor switch B operated, the "a" and "b" contacts are open and the "$\bar{a}$" and "$\bar{b}$" contacts are closed. The effect in the circuits (a), (c) and (e) of the figure is to place a "p" contact in series with the relay winding; in those of parts (b), (d) and (f) the effect is to place a "$\bar{p}$" contact in parallel with the winding. In all six cases the resulting circuit is stable and the relay can be in either its operated or its released state. (Recall Fig. 1).

We describe in detail what happens only to the circuit in (c), which uses a combination of series and shunt control. (The states of the other five circuits depend upon the states of the A and B switches in exactly the same way as does the state of the (c) circuit.) If switch A is now momentarily operated the relay P will be energized (regardless of its state of operation) and eventually it will become operated. That is, its "p" contact will close. If switch A is released the relay will remain operated because "p" is now closed.

To change the relay from the operated to the released state, operate the B switch momentarily. The closing of the "b" contact provides a shunt around the relay winding and the relay will be de-energized and will eventually release. That is, the "p" contact will open. If switch B is released the relay will remain released because "p" is now open.

The momentary operation of the switch A allows us to "store a 1" in the relay circuit. The left hand side of the relay is called the "make" side. The momentary operation of switch B allows us to "store a 0" in the circuit. Thus the righthand side is known as the "break" side. As long as neither switch is operated the relay will continue to remain in the state in which it was last placed.

The simple circuits in Fig. 4 are also some of the most useful. The one shown in part (c) is used as the basis of our computer memory. Also this circuit can be used to remember that an elevator has been called by a person wishing to use it. The "elevator button" pushed by the prospective passenger is switch A. The relay P could be used to control a light showing that the button has been pushed. Switch B is the one operated automatically when the elevator arrives at the calling floor. Switch B remains operated until the elevator leaves that floor. Switch B is usually operated by the movement of the elevator itself, and it is customarily put inside the elevator shaft where passengers cannot see it.

The same circuit may be used as a burglar alarm. Switch A is placed where a burglar will unwittingly step on it (for instance, under the mat just inside the front door of the store we wish to protect). A bell, which rings whenever the relay is operated, is added to the circuit (it should be connected in parallel with the winding of the relay). Switch B is used to turn off the alarm bell after the burglar has been captured and the store owner has grown tired of the sound. (Sometimes B is called a "reset" switch since it restores the circuit to its "alert" condition.)

Analysis by tables of combinations

A properly compiled table of combinations can indicate clearly how the energizing of the relay P depends upon the states of its contacts and of switches A and B. Consider the series-controlled circuit in Fig. 4. As usual, make a list of all possible states of contacts a, b, and p as in Fig. 5. For each set of states, it is a straightforward matter to determine if the circuit is open or closed, and so if the relay is energized or not. If the relay is energized, enter a "1" in a new column labeled P; otherwise enter a "0" in it.

a	b	p	P	
0	0	0	0	stable
0	0	1	1	stable
0	1	0	0	stable
0	1	1	0	unstable
1	0	0	1	unstable
1	0	1	1	stable
1	1	0	0	stable
1	1	1	0	unstable

Fig. 5 A table of combinations for the circuits of Fig. 4.

Remember that when the values of "p" and of "P" are the same, the circuit is in a stable state. Whenever "p" and "P" have complementary values, the circuit is unstable. For example, the fourth row of the table tells us that if a = 0, b = 1, and p = 1 the contact network which controls the relay is open (P = 0). But if the network is open the relay is de-energized and the relay, which is presently operated (p = 1), will become released (p = 0). The arrow in the table indicates that the circuit will go to the state a = 0, b = 1 and p = 0. This state is stable. The circuit will remain in this state until both switch A and switch B are changed.

For example, start with the circuit in the stable state shown in the top row of the table, a = 0, b = 0 and p = 0. Operate the switch A; the circuit temporarily goes into state a = 1, b = 0 and p = 0; this state is unstable. The action of the relay itself will cause the next state to be a = 1, b = 0 and p = 1; this state is stable. If switch A is released again the stable state a = 0, b = 0 and p = 1 will result. The momentary operation of the switch A has stored a "1" in the circuit. To change the state of contacts on the relay it was first necessary to put the circuit into an unstable state; from the unstable state the action of the circuit itself caused the relay to change its state of operation.

Analysis of feedback logic circuits by the use of tables of combinations becomes more and more complex as the number of relays increases. However, the principles are the same as those illustrated by our simple circuit. We have occasion to analyze only one such complex circuit.

4. AN ADDRESSABLE MEMORY

The simple circuit of Fig. 4(c) could store a "0" or a "1". When the switch A (connected to the make side) was operated and released a "1" was stored (that is, the resulting stable circuit state had p = 1). When switch B (connected to the break side) was operated and released a "0" was stored (that is, a stable state in which p = 0 resulted). That circuit has been redrawn in a slightly different form in Fig. 6(a). A light bulb has been added in parallel with the relay winding so that we can sense or read the state of the relay by looking at the corresponding state of the bulb.

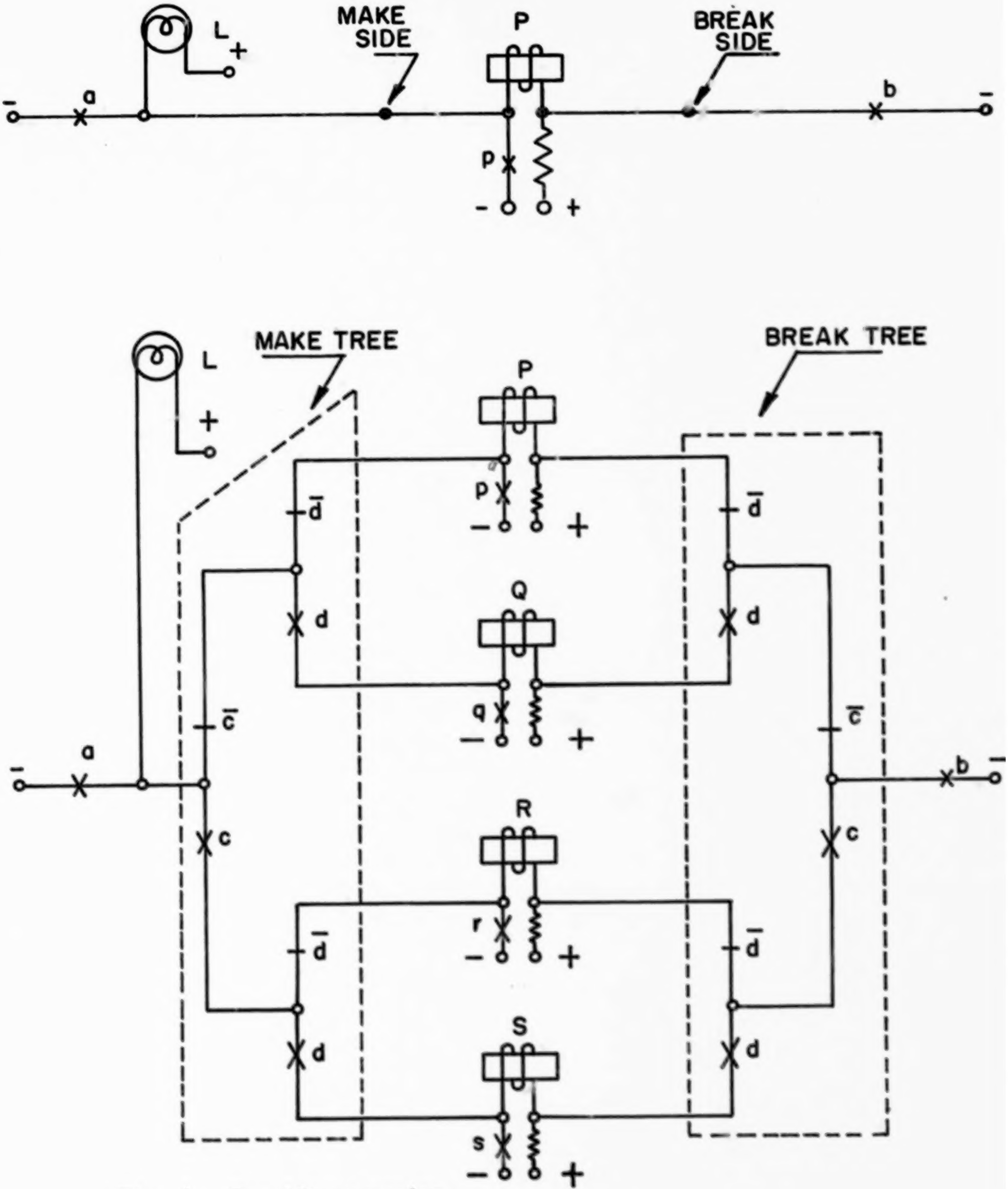

MAKE SIDE
P
BREAK SIDE
L
+
a
b
p
MAKE TREE
BREAK TREE
P
Q
R
S
p
q
r
s
$\bar{d}$
d
$\bar{c}$
c
+
−

Fig. 6 One-bit memories.
(a) A single one-bit memory cell.
(b) Four addressable one-bit memory cells.

In a computer we want to be able to store and read binary digits (commonly called bits) in any one of many different memory circuits. In order to do so we must be able (1) to select the appropriate circuit, and then (2) to set the selected circuit to the desired "0" or "1" state. The arrangement of Fig. 6(b) shows four separate memory circuits, P, Q, R, and S, each capable of storing a single bit, and two selection trees similar to those discussed in Section 3 of Chapter 2. By setting switches C and D appropriately the make tree selects the make side of a specific relay while the break tree simultaneously selects the break side of the same relay. After this selection has taken place the actual setting of the selected relay is accomplished by connecting either the root of the make tree to minus (by operating A, thereby storing a "1") or the root of the break tree to minus (by operating B, thereby storing a "0"). Clearly A and B should not both be operated simultaneously. Since a unique path is set up between the root of the make (or of the break) tree and the chosen relay, none of the other relays are connected to minus (unless through their holding contacts). Thus their states are not changed by the operation of A or B. To record a "0" or "1" bit in another cell, the two step process is repeated by selecting the desired relay with tree switches C and D, and setting the correct state with input switch A or B.

By using selection trees constructed from more switches we may address a larger number of memory relays. Nine switches would allow us to choose among $2^9 = 512$ different relays. The illustrative computer of the next chapter has this number of memory locations.

5. SHIFTING AND SHIFT-REGISTERS

Multiplication in the binary number system

It is useful for a computer to be able to shift the digits in a binary number to the left or to the right. Shifting is used in multiplying two numbers. For example, two binary numbers are shown in Fig. 7. The decimal number 27 ($= 1\cdot 16 + 1\cdot 8 + 0\cdot 4 + 1\cdot 2 + 1\cdot 1$) is the same as the binary number 11011. The decimal number 23 ($= 1\cdot 16 + 0\cdot 8 + 1\cdot 4 + 1\cdot 2 + 1\cdot 1$) is equivalent to the binary number 10111. Figure 7(a) shows how the clerical work required for multiplying these two numbers might be written. There are five digits in N. Thus, five numbers, each of which is a shifted version of M multiplied by an appropriate digit of N, must be added together to form the product. Remember that in binary multiplication:

$$1 \times 1 = 1$$
$$0 \times 1 = 0$$
$$1 \times 0 = 0$$
$$0 \times 0 = 0$$

So in multiplying binary numbers, only shifted versions of one number and rows of all zeros will be required. This fact is illustrated in Fig. 7(a), where four shifted versions of M (= 11011) and one version of 00000 are added to form the product.

Another method of multiplication, one used in computers, is known as "partial sums". This method is illustrated in Fig. 7(b). The same five numbers are added together, but one at a time rather than all five simultaneously as in Fig. 7(a). Thus there are four "partial sums", each the result of adding the previous partial sum to a shifted version of M, multiplied by one digit of N.

Shifting binary numbers in a computer is accomplished using a circuit with memory called a shift register. This circuit stores a binary number, such as 11011, and on command can shift it so as to create, for example, 110110.

Weights:	256	128	64	32	16	8	4	2	1	
M = 27:					1	1	0	1	1	
N = 23:					1	0	1	1	1	
					1	1	0	1	1	= 1·1·M
				1	1	0	1	1	(0)	= 1·2·M
			1	1	0	1	1	(0	0)	= 1·4·M
		0	0	0	0	0	(0	0	0)	= 0·8·M
	1	1	0	1	1	(0	0	0	0)	= 1·16·M

(Add the five numbers above to obtain the desired product.)

(Sum equals 23·M)

(a) How to form the desired product.

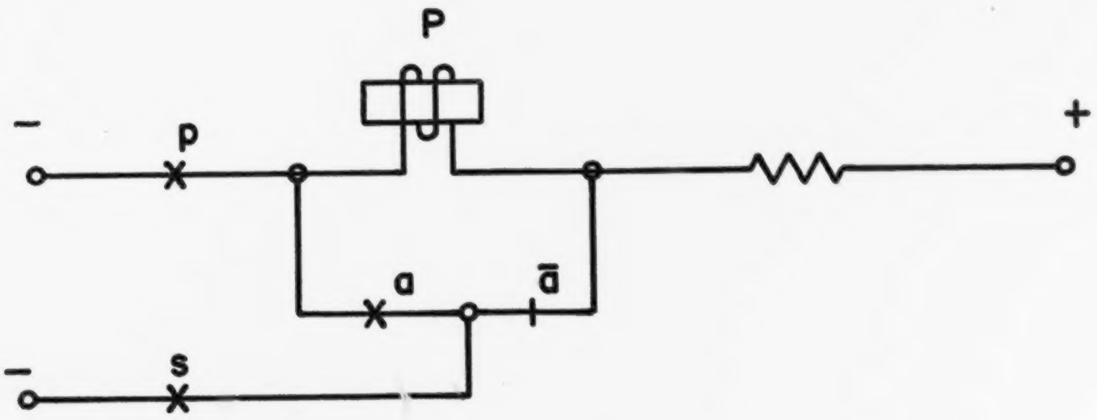
Weights:	512	256	128	64	32	16	8	4	2	1	
Initial partial sum:						0	0	0	0	0	
					+	1	1	0	1	1	= 1·1·M
First partial sum:						1	1	0	1	1	
				+	1	1	0	1	1	(0)	= 1·2·M
Second partial sum:				1	0	1	0	0	0	1	
			+	1	1	0	1	1	(0	0)	= 1·4·M
Third partial sum:			1	0	1	1	1	1	0	1	
		+	0	0	0	0	0	(0	0	0)	= 0·8·M
Fourth partial sum:			1	0	1	1	1	1	0	1	
	+	1	1	0	1	1	(0	0	0	0)	= 1·16·M
Final sum:	1	0	0	1	1	0	1	1	0	1	= 621

(b) The building up of the product by partial sums.

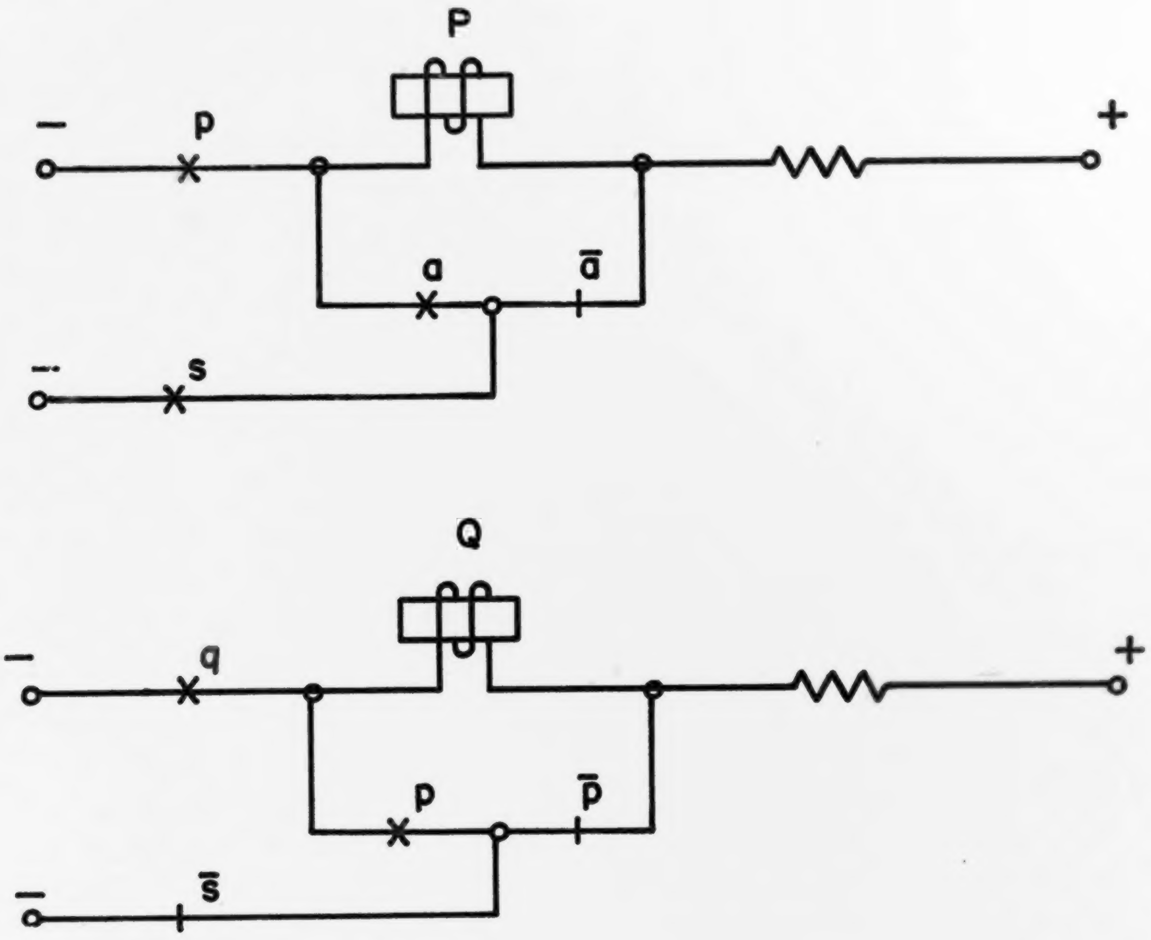
Fig. 7 The use of shifted binary numbers in forming products.

A single shift-register stage

Shift registers are built from nearly identical smaller circuits. The basic circuit for shift-register is shown in Fig. 8(a). This circuit can store one bit. When switch S is operated the negative terminal of the electrical source will be connected to the left terminal of the relay if a = 1, and it will be connected to the right terminal of the relay if a = 0 (if $\overline{a} = 1$). Therefore, when S is operated, the state (operated or released) of the relay will be the same as the state (operated or released) of the switch A.



(a) A Simple Memory Circuit.



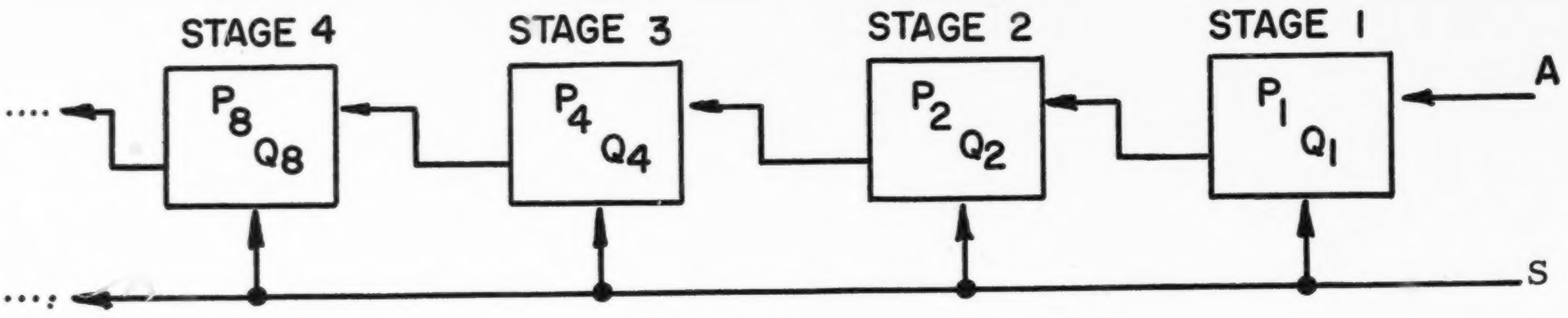
(b) A single shift-register stage.

Fig. 8 Basic circuits for the shift-register.

When the switch S is released the contacts "a" and "$\bar{a}$" are effectively in series. Regardless of the state of switch A, therefore, no path can be completed through both of these contacts. Thus, when switch S is released the relay is controlled by its own holding contact, p. The value of the bit stored in the relay depends, therefore, upon the previous history of the circuit.

A bit of information is stored in the circuit as follows: with the switch S released, set the state of switch A to the value we want to store (1 or 0). The relay will, for the time being, remain in its original state. The state of switch A can be stored by operating and then releasing switch S. After switch S is released, switch A can no longer affect the state of P.

The basic action of shifting a stored bit from one relay to another requires a second circuit almost the same as the one discussed above [see Fig. 8(b)]. Consider what happens when S is released. The relay P will be operated or released, depending upon the value of the bit stored there. The state of Q will be the same as the state of P because, with the contact "$\bar{s}$" closed, the state of the "p" and "$\bar{p}$" contacts determine the state of Q. With the circuit in this state we may set the switch A to either state without affecting the states of either P or Q.



(a) A cascade connection of four shift-register stages.

	Q_{256}	Q_{128}	Q_{64}	Q_{32}	Q_{16}	Q_8	Q_4	Q_2	Q_1	A
Initially:	0	0	0	0	0	0	0	0	0	1
After 1st shift:	0	0	0	0	0	0	0	0	1	1
After 2nd shift:	0	0	0	0	0	0	0	1	1	0
After 3rd shift:	0	0	0	0	0	0	1	1	0	1
After 4th shift:	0	0	0	0	0	1	1	0	1	1
After 5th shift:	0	0	0	0	1	1	0	1	1	0
After 6th shift:	0	0	0	1	1	0	1	1	0	0
After 7th shift:	0	0	1	1	0	1	1	0	0	0
After 8th shift:	0	1	1	0	1	1	0	0	0	0
After 9th shift:	1	1	0	1	1	0	0	0	0	0

(b) An example of the shifting action.

Fig. 9 Illustrating the action of a shift-register.

Set A to some new value to be stored on P and then shifted to Q. With A in its "new" state, operate S. In the upper half of the circuit this causes P to become operated in the same way that A is operated. With S operated, however, the contact "$\overline{s}$" in the lower half of the circuit will be open and the relay Q will retain its former state. Now release S once again. The value stored in the upper half of the circuit in the relay P will be passed on to the relay Q. After switch S is released, the state of A can be changed once again without influencing the state of P. The original state of operation of the A switch has been shifted through the P relay to the Q relay.

A complete shift-register

A shift-register can be made by cascading several copies of the circuit in Fig. 8(b). The circuits are connected as suggested by the diagram in Fig. 9(a). Each of the pairs of relays in a stage of the shift-register has a subscript which denotes the significance of the binary digit stored in that stage. The shifting signal, S, actuates all of the stages. Each time it changes from "0" to "1" and back to "0" the bit value stored in each stage is shifted one stage to the left. The example in part (b) of the figure shows how the number 11011 is inserted into the register one bit at a time by properly setting the switch A for each shift. Further shifting moves that binary number still further to the left so that it can be in proper location for being added into a partial sum (see Fig. 7).

The more detailed circuit diagram of Fig. 10 shows four stages of the shift-register. Note especially that the "q" contacts from one stage are used in the next stage in the same way that the "a" contacts are used in the first stage.

6. CIRCUITS THAT COUNT

The analysis of a single counter stage

Counting is a fundamental operation in a computer. A counting circuit, usually called a counter, is the means for carrying out this function. A counter furnishes as its output a set of binary digits representing a number. This number can be increased by one by an input signal to the counter. The binary digits at the counter output can be used to set the address of a memory cell, using a selector tree, just as the switches C and D did in Fig. 6(b). It is also useful to be able to set the counter to a desired number on certain occasions. All these features can be incorporated into a single counter.

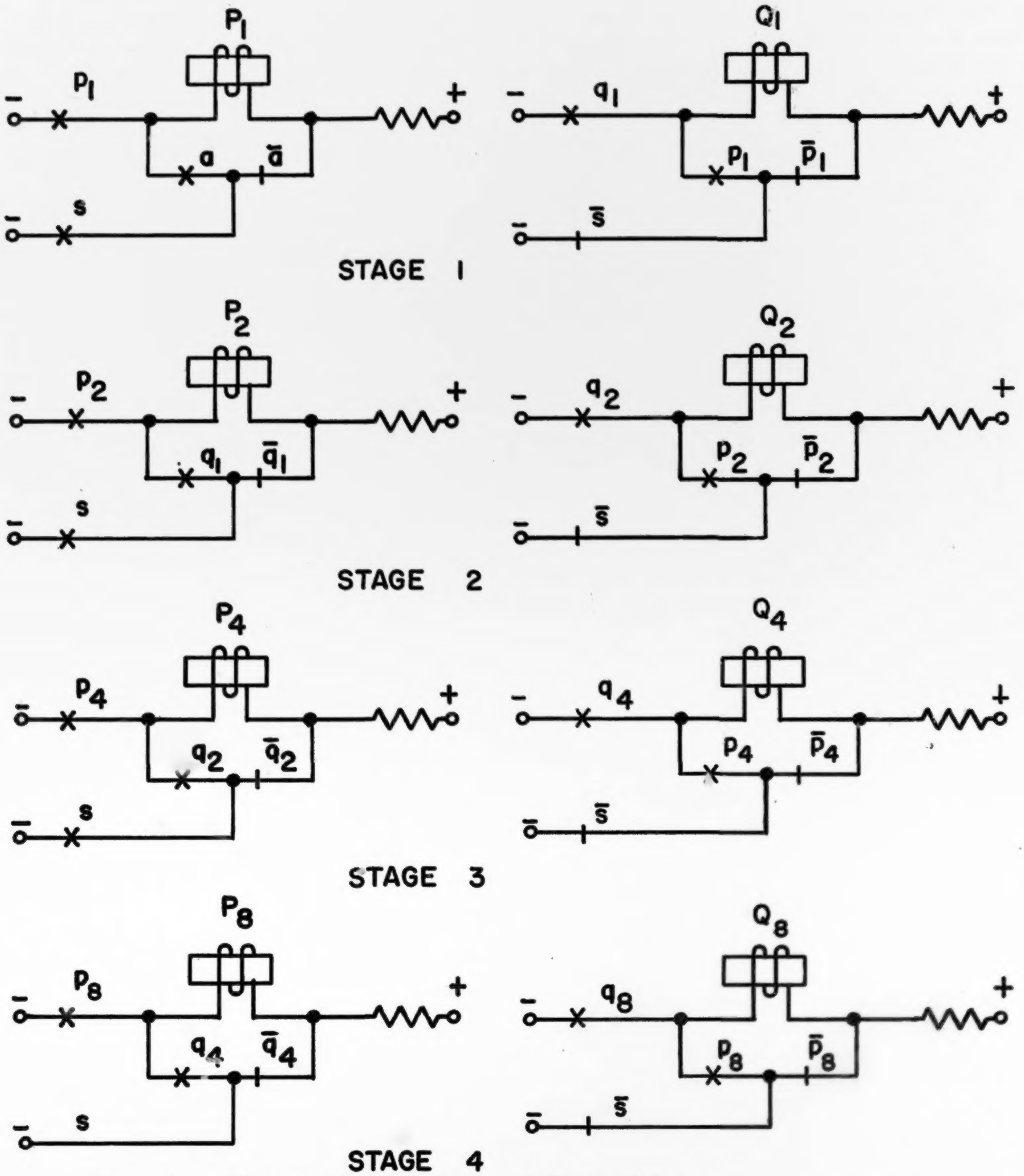
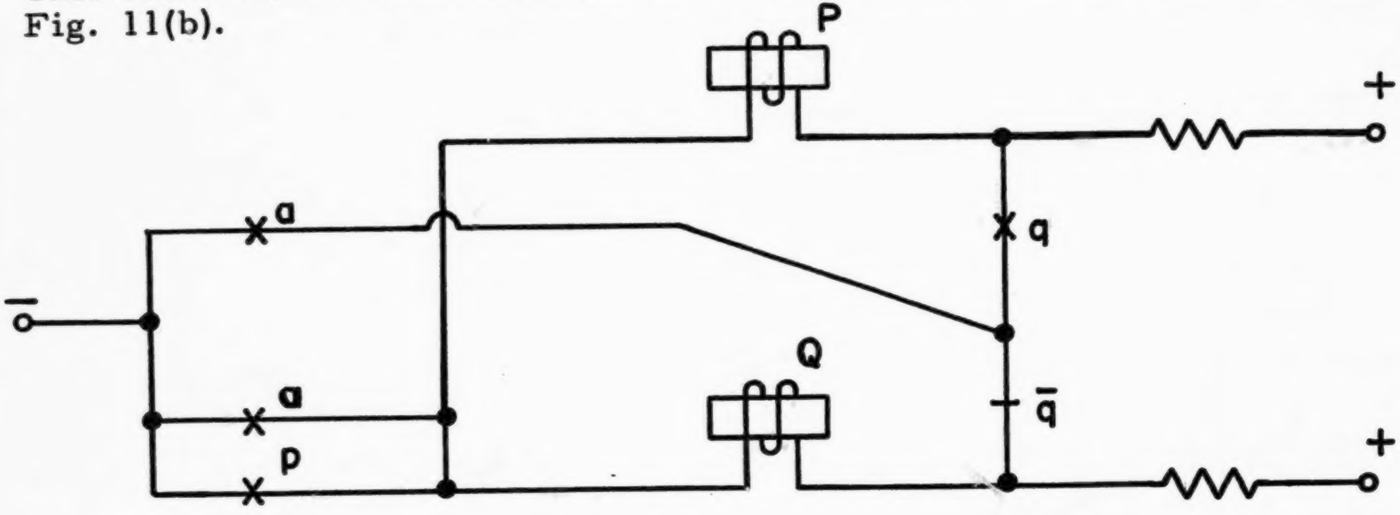

P1
p1
a
ā
s
Q1
q1
p1
p̄1
s̄
STAGE 1
P2
p2
q1
q̄1
s
Q2
q2
p2
p̄2
s̄
STAGE 2
P4
p4
q2
q̄2
s
Q4
q4
p4
p̄4
s̄
STAGE 3
P8
p8
q4
q̄4
s
Q8
q8
p8
p̄8
s̄
STAGE 4
-
+

Fig. 10 Circuit diagram for a shift-register.

One stage of the counter is shown in Fig. 11(a). (The counter itself will be made by cascading several similar stages, just as in the shift-register.) The only contacts which can be controlled from outside the circuit are the two "a" contacts. We can see from the circuit diagram that the relay P will be energized if and only if "a" or "p" is closed, and if "a" and "q" are not closed. This information is listed in the "P" column of the table of combinations in Fig. 11(b).



(a) The circuit diagram.

a	p	q	P	Q	
0	0	0	0	0	stable
0	0	1	0	0	Q is unstable
0	1	0	1	1	Q is unstable
0	1	1	1	1	stable
1	0	0	1	0	P is unstable
1	0	1	0	1	stable
1	1	0	1	0	stable
1	1	1	0	1	P is unstable

(b) The table of combinations.

Step:	(0)	1	2	3	4	5	6	7	8	9	10	11 ...
a:	0	1	0	1	0	1	0	1	0	1	0	1
p:	0	1	1	0	0	1	1	0	0	1	1	0
q:	0	0	1	1	0	0	1	1	0	0	1	1

(c) A summary of the action of one stage of the counter.

Fig. 11 The analysis of one stage of a "counting" circuit.

The conditions for energizing the Q relay can be stated similarly. The circuit indicates that "Q is energized if and only if "a" and "q" are closed ($\bar{q}$ is open), or if "a" is open and "p" is closed". This information is listed in the "Q" column of the table of combinations.

Remember that if a relay is energized and operated or if it is de-energized and released it is in a stable state. Otherwise the relay is in an unstable state. The table of combinations shows that four of the eight circuit states are stable; that is, for these states P = p and Q = q. One or the other of the relays is unstable for each of the other four states. The arrows indicate which stable state follows each of these unstable states.

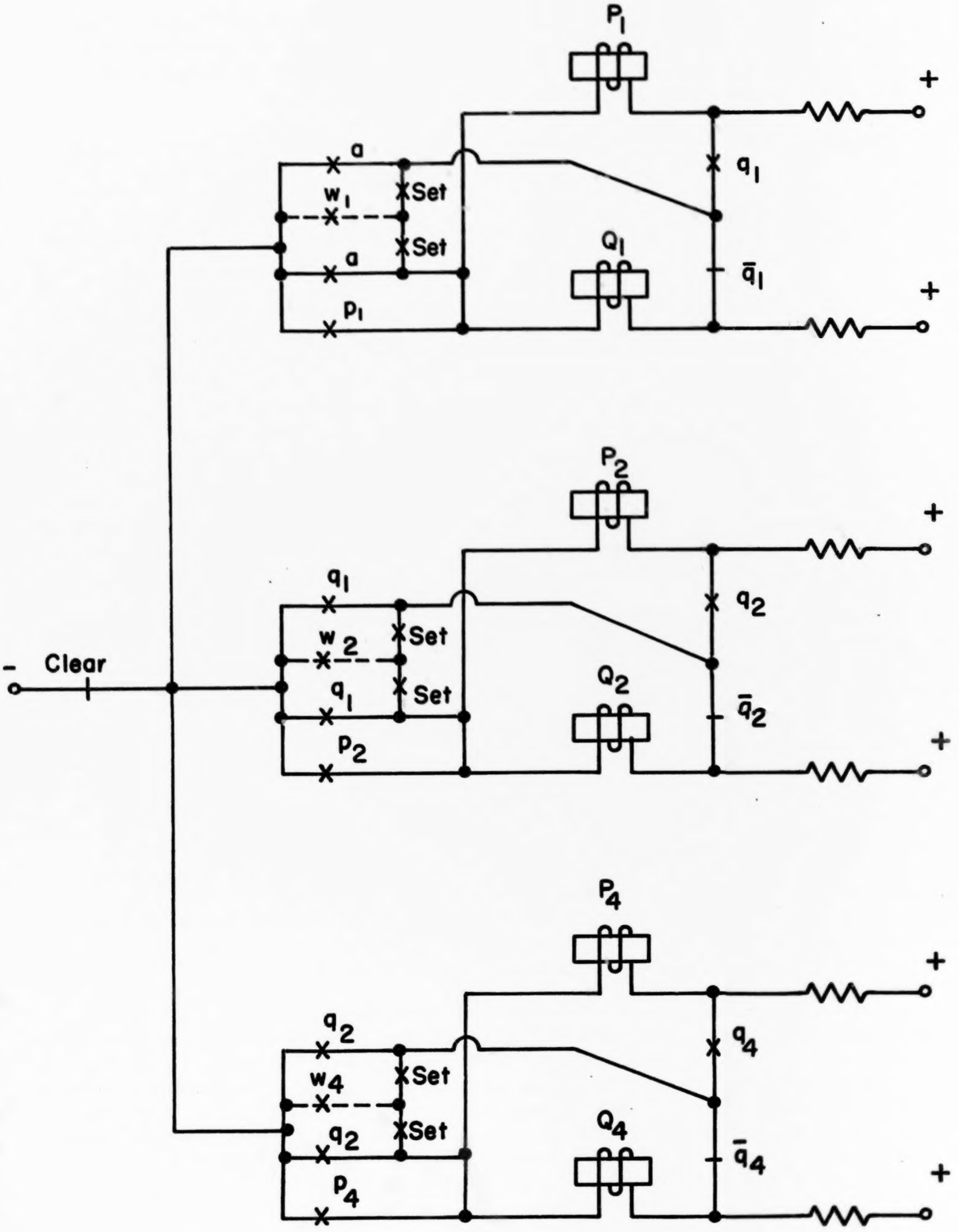
The table of combinations specifies what happens when the switch A is alternately operated and released. Assume that the switch and both relays are released; that is, that the state of the circuit is "000" (a = 0, p = 0 and q = 0). When A is operated the state becomes "100", an unstable state, which leads to the stable state "110". When A is now released the state becomes "010", which leads to "011". If switch A is operated a second time the state becomes "111" and this leads the circuit to the stable state "101". Another release of A leads the circuit through "001" to "000", the initial state. Further opening and closing of the "a" contacts causes a cyclic repetition of these four stable states to occur. The reader should review the cycle of actions by tracing paths through the various contacts in the circuit itself to see how the relays are energized and operated, de-energized and released.

A summary of the actions of relays P and Q is shown in Fig. 11(c). For brevity only the cyclic sequence of the four stable states is given. Notice that it requires two openings and closings of the "a" contacts to cause one opening and closing of "q".

A multistage counter

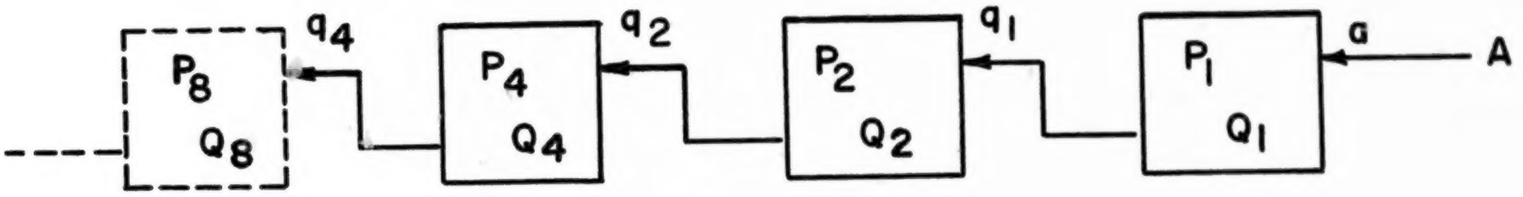
Any number of the circuit stages shown in Fig. 11(a) can be cascaded to form a counter. The circuit of Fig. 12(a) has three stages interconnected. (The subscripts indicate the "weight" of the binary digits generated at each stage. Ignore the dotted part of the circuit for now.) Notice that the opening and closing of the "q_1" contacts in the second stage influence that stage in the same way that the opening and closing of the "a" contacts did in the first stage. In the next stage the "q_2" contacts serve similar purposes. Part (b) of the figure shows schematically how one stage influences the next.

When the several stages of a counter are interconnected two openings and closings of the "q" contact from one stage cause the "q" contacts in the next stage to open and close just once. This fact is summarized in Fig. 12(c). Finally, notice that the values q_4, q_2 and q_1 are a sequence of three-digit binary numbers which starts at 000 and progresses through 001, 010, 011, 100, 101, 110 and 111. The cycle of actions is then repeated. In effect, the circuit "counts" how many times the A switch has been operated. The q relays provide the counter output. A counter with M stages can count from 0 through 2^M-1.


P1
+
a
q1
w1
Set
Set
Q1
a
q̄1
+
p1
P2
+
q1
q2
w2
Set
Clear
-
q1
Set
Q2
q̄2
p2
+
P4
+
q2
q4
w4
Set
q2
Set
Q4
q̄4
p4
+

(a) The circuit diagram

Fig. 12 A three-stage resettable counter



(b) Illustrating the influence of one stage of the counter upon the next

NUMBER OF 'a' PULSES	0		1		2		3		4		5		6		7		8		9
a	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
q_1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
q_2	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0
q_4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0

(c) Summary of the actions of a three-stage counter circuit.

Fig. 12 A three-stage resettable counter

Clearing and resetting the counter

When the "clear" and "set" switches, which control the corresponding contacts (shown dotted) in the circuit of Fig. A-12(a) are released the circuit operates as a three-stage counter as just described. The "clear" switch sets the count to zero when it is operated and then released, for the break contact it controls is in series with the rest of the circuit. After the clear switch is released it is possible, by closing the make contacts controlled by the "set" switch, to set the count directly to any number. If, for instance, the count is to be set to "5" (represented by $q_4 = 1$, $q_2 = 0$ and $q_1 = 1$) the "w" contacts are set to $w_4 = 1$, $w_2 = 0$ and $w_1 = 1$. Next, the "set" contacts are closed and reopened. This action produces the same effect as if the "q_2" and "a" contacts were closed and reopened. In other words, in the resulting state of the circuit, the count in the first stage will be "1" ($p_1 = q_1 = 1$), in the second stage "0" ($p_2 = q_2 = 0$) and in the third stage "1" ($p_4 = q_4 = 1$).

A counter can be used to control a "tree" circuit. The diagram in Fig. 13 demonstrates how the "q" contacts from three stages of a counter can control a tree. When the switch A is operated and released the count is increased by one and the completed path in the tree is connected from the "base" of the tree to the terminal of the right which has the next greater number.

Since the switch A is operated and released twice for each operation of Q_1, contacts from Q_2, Q_1 and A could be used in the tree instead of those from Q_4, Q_2 and Q_1, respectively. [Exactly this is done in the circuit of Fig. 14(b).]

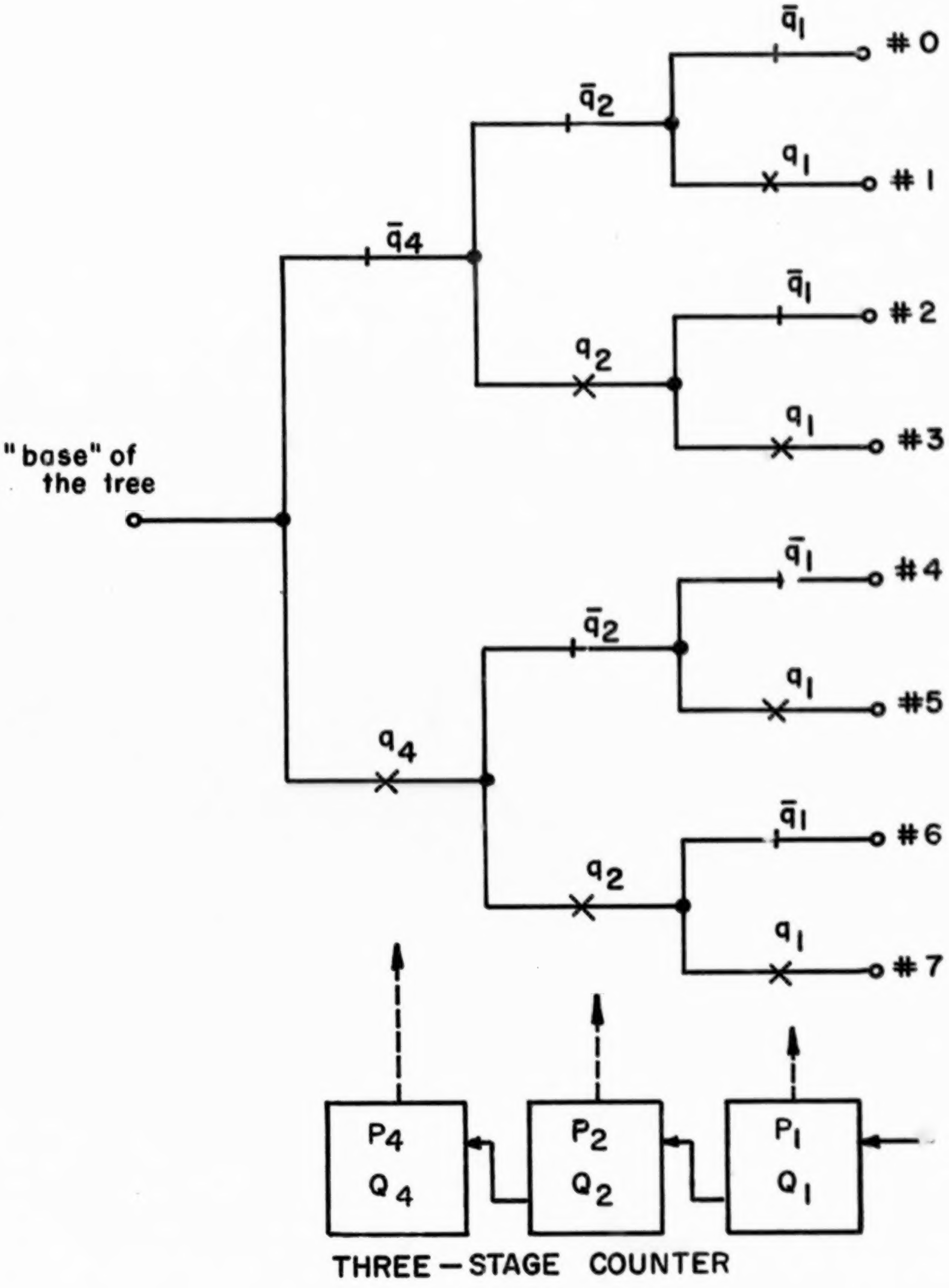

"base" of the tree
$\bar{q}_4$
q_4
$\bar{q}_2$
q_2
$\bar{q}_2$
q_2
$\bar{q}_1$
q_1
$\bar{q}_1$
q_1
$\bar{q}_1$
q_1
$\bar{q}_1$
q_1
#0
#1
#2
#3
#4
#5
#6
#7
P_4 Q_4
P_2 Q_2
P_1 Q_1
THREE — STAGE COUNTER

Fig. 13 A counter-controlled tree.

In that case only one change in the state of A--rather than two-- is necessary to advance the closed path to the next terminal.

Trees controlled in this way are used in addressable memories [see Fig. 6(b)], and permit examination of memory locations one after another. To "jump" to any particular memory location, clear and reset the counter to the appropriate number. It is in just such a way that the memories of computers are "addressed". Another use of counter-controlled trees is examined in the next section.

7. SYNTHESIS OF AN AUTOMATIC MORSE CODE TRANSMITTER

A computer is "programmed" by ordering it to perform, in sequence, a number of elementary operations, such as "add", "subtract", "store the result", "get a new number from the input", "send a number to the output"--and so forth. In order to carry out even one of these simple operations, a timed sequence of connections must be made among the various circuits of the computer. Each sequence may consist of several steps. The circuit which "translates" each elementary operation into the corresponding sequence of actions is called the "operation decoder". The particular operation desired at any time is specified by a binary number or "code". If there are eight possible operations, the states of three switches are sufficient to specify one of the eight. The crucial point is that a binary number, in the form of switch settings, can lead to a sequence of actions. Each different setting specifies a different sequence. A circuit which is very close to the decoder in its principle of operation is one which gives a sequence of "dots" and "dashes" (on lamps) corresponding to the Morse code for eight English letters. The table in Fig. 14(a) shows the sequence of dots and dashes corresponding to the letters A through H. Settings of switches X, Y, and Z which correspond to these letters are given in the column labelled "binary code".

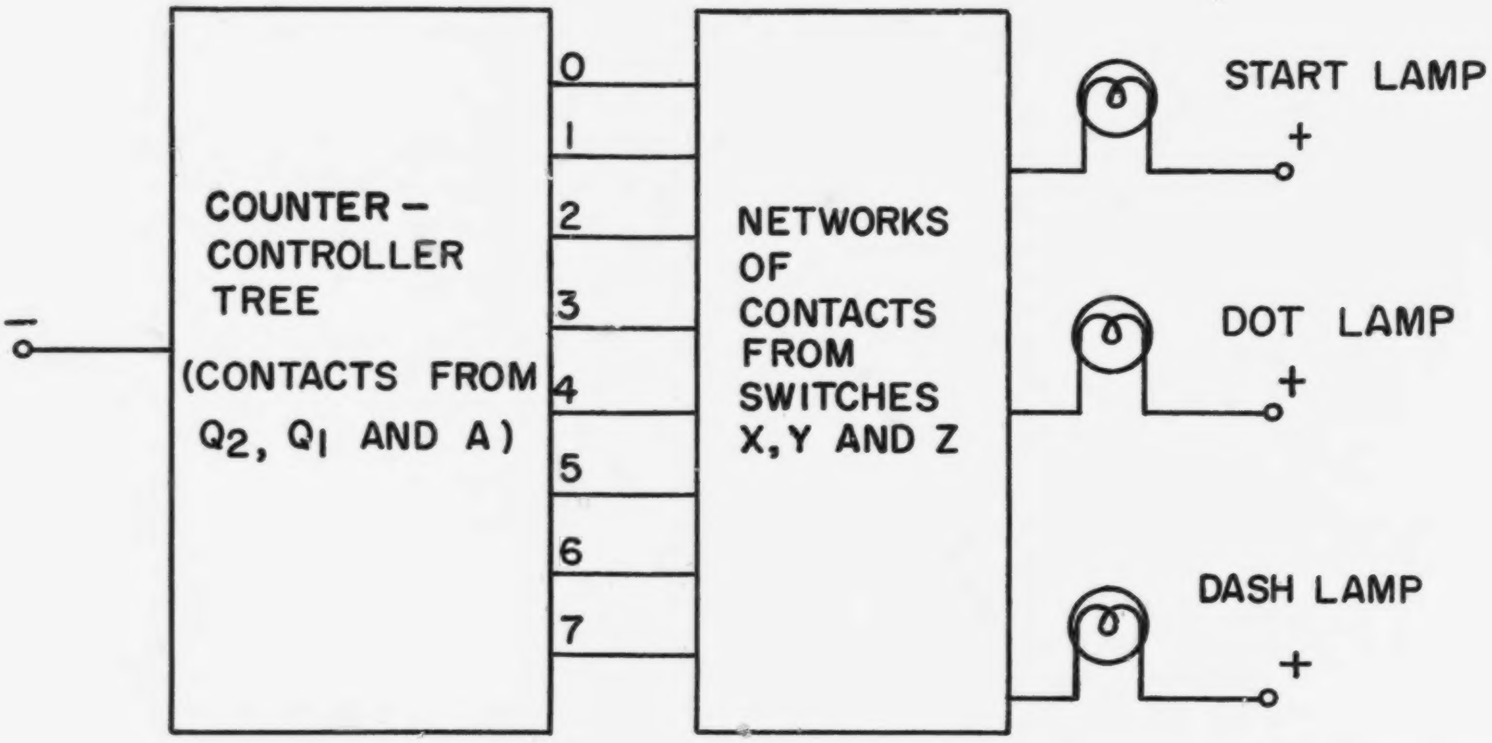
Assume that there is available a counter-controlled tree built from contacts on relays Q_2 and Q_1 and switch A of the counter circuit discussed previously. During eight successive time intervals, operate and release switch A to connect the eight tree terminals successively to the tree base. This action is indicated roughly in the left part of Fig. 14(b). The Morse code output is to be produced on the lamps at the right, as the tree is cycled through its eight connections. The table in Fig. 14(a) indicates which positions are to light "dot" and "dash" lamps. Another lamp, called the "start" lamp, is to be lighted in interval #0 to indicate the beginning of the Morse code representation for each letter.

The complete Morse code transmitter requires a group of contact networks from the switches X, Y and Z to be placed between the tree terminals and the three lamps. The specifications of these networks can be determined from the table. For instance, the table tells us that in time interval #1 there is to be a connection to the "dot" lamp when both "x" and "z" have the value "1". A simple series circuit of the two corresponding make contacts, placed between terminal #1 of the tree and the "dot" lamp, will do the job.

The reader should verify that the contact networks shown in Fig. 15 do yield the appropriate light sequence as given in the table. Consider one of the more complicated networks, that for the "dot" lamp in interval #7. There are four combinations of the values of X, Y and Z for which the lamp should be lighted; and correspondingly for which there should be a connection between

	binary code			Morse code output							
				time interval							
letter	X	Y	Z	0	1	2	3	4	5	6	7
A	0	0	0				●		—		
B	0	0	1		—		●		●		●
C	0	1	0		—		●		—		●
D	0	1	1		—		●		●		
E	1	0	0				●				
F	1	0	1		●		●		—		●
G	1	1	0		—		—		●		
H	1	1	1		●		●		●		●

(a) A table showing Morse code for the letters A through H.


−
COUNTER-
CONTROLLER
TREE
(CONTACTS FROM
Q_2, Q_1 AND A)
0
1
2
3
4
5
6
7
NETWORKS
OF
CONTACTS
FROM
SWITCHES
X, Y AND Z
START LAMP
+
DOT LAMP
+
DASH LAMP
+

(b) Block diagram of the circuit.

Fig. 14 Specifications for a Morse code transmitter.

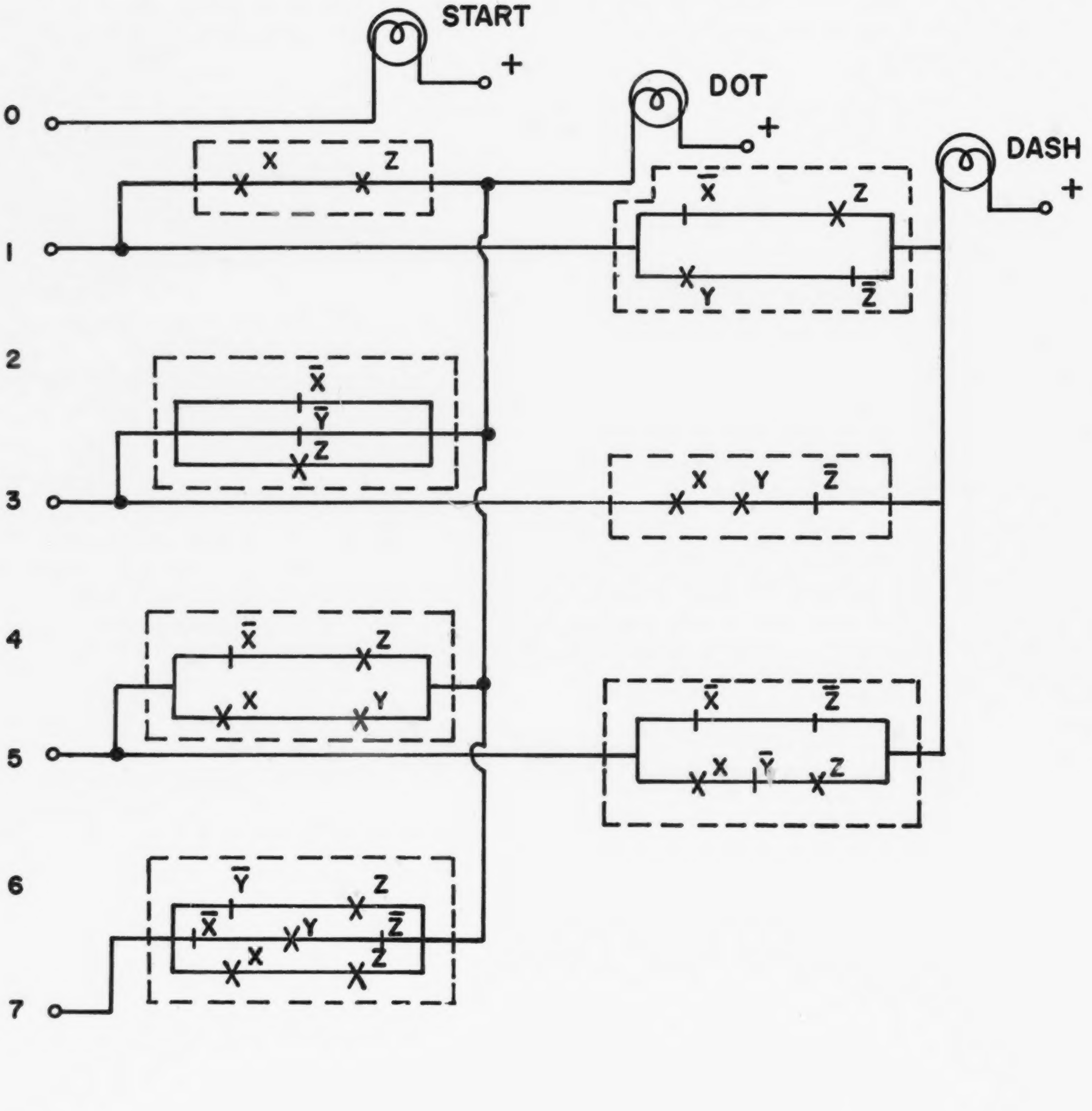

START
+
DOT
+
DASH
+
0
1
2
3
4
5
6
7
X
Z
$\bar{X}$
Z
Y
$\bar{Z}$
$\bar{X}$
$\bar{Y}$
Z
X
Y
$\bar{Z}$
$\bar{X}$
Z
X
Y
$\bar{X}$
$\bar{Z}$
X
$\bar{Y}$
Z
$\bar{Y}$
Z
$\bar{X}$
Y
$\bar{Z}$
X
Z

Fig. 15 The contact networks for the Morse code transmitter.

terminal #7 and the dot lamp. These are

X, Y, Z: 001, 010, 101 and 111.

Notice that the corresponding network should be closed whenever Y = 0 and Z = 1, regardless of the value of X; also, that it should be closed when X = 1 and Z = 1, regardless of the value of Y. (The case X = 1, Y = 0 and Z = 1 is covered by *both* statements; each statement is valid nevertheless.) Finally, when X = 0, Y = 1 and Z = 0 the network is to be closed. Summarizing, the network should be closed between the #7 terminal of the tree network and the "dot" lamp whenever

Y=0 *and* Z=1, *or* X=1 *and* Z=1, *or* X=0 *and* Y=1 *and* Z=0.

The network, then should consist of three parallel branches, each with a series of contacts as follows: (1) $\bar{y}$, z; (2) x, z; and (3) $\bar{x}$, y, $\bar{z}$.

8. SUMMARY

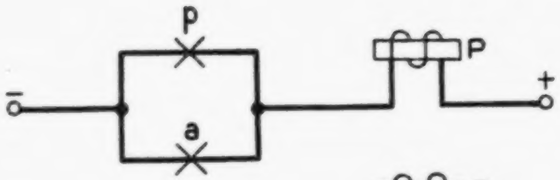
There are a variety of logic circuits with memory which are important in designing computers. These include the shift register, which is useful for shifting binary numbers to determine products. The binary counter can "count" to 2^M-1 with only M circuit stages, each identical to the others. Operation decoding circuits can be designed to translate binary numbers specifying basic computer operations into the sequence of actions necessary for carrying out the individual steps of those operations.

The single most critical logical circuit with memory is also the simplest [Fig. 6(a)]. When sufficient numbers of them are brought together, they can store an indefinitely large number of bits. These can be changed when necessary and can be grouped so that individual groups can be read by a single address number. The addressable memory is the heart of the *stored-program* computer which is investigated in the next chapter.

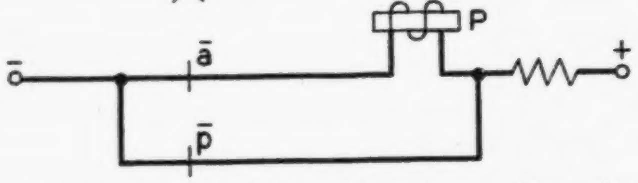
PROBLEMS

4-1 In the two circuits below, discuss what happens when, starting with the condition that relay P is released, the switch A is operated. What happens when, starting with condition that P is operated, A is released?

(a)

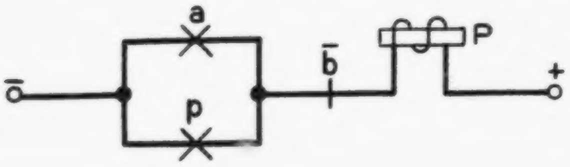


(b)

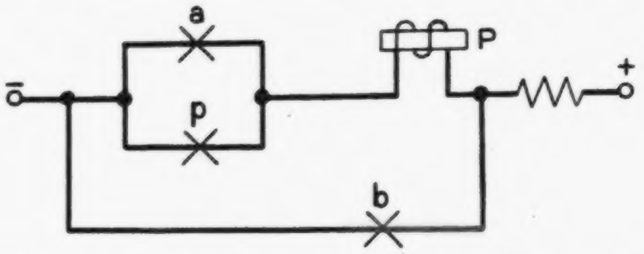


4-2 Analyze the operation of the following two circuits; that is, describe step-by-step, starting with all switches and the relay unoperated, what happens when
(a) switch A is operated and then switch B is operated;
(b) switch B is operated and then switch A is operated.

(a)

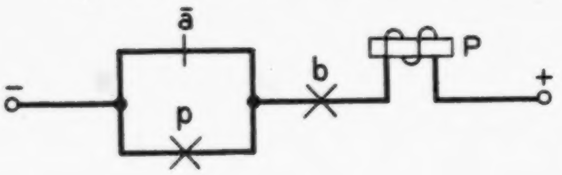


(b)

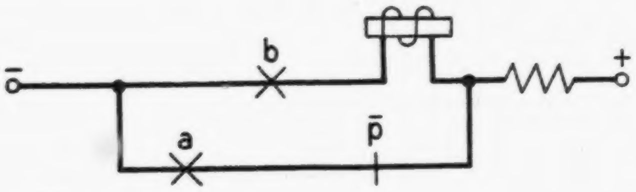


4-3 In each of the three circuits below analyze what happens, starting with both switches and the relay unoperated, when
(a) switch A is operated and then released, then switch B is operated and then released;
(b) switch B is operated and then released, then switch A is operated and then released.

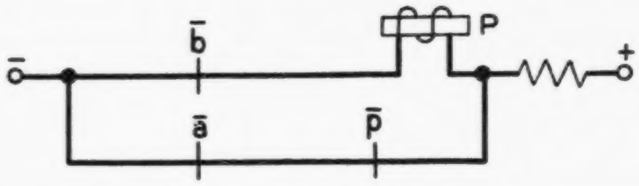
(a)



(b)

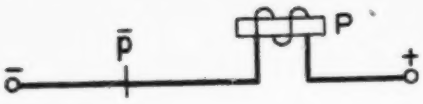


(c)

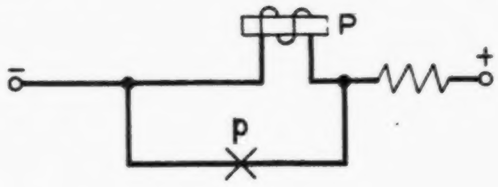


4-4 Explain the operation of the two circuits below.

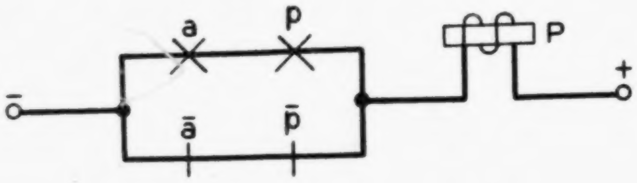
(a)



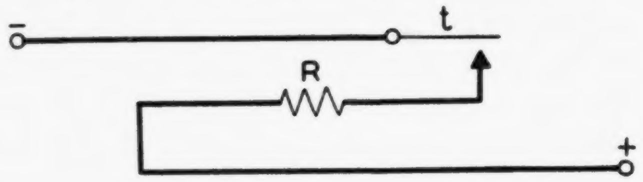
(b)



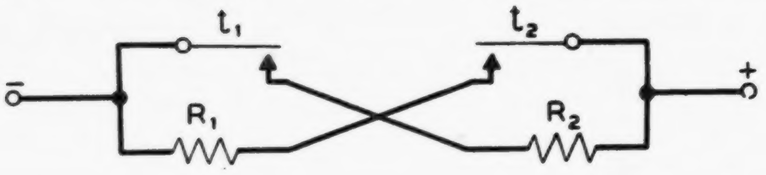
4-5 In the circuit below, switch A has been released for a long time, and then it is operated. What are the possible resulting states of operation of the relay P? Explain.



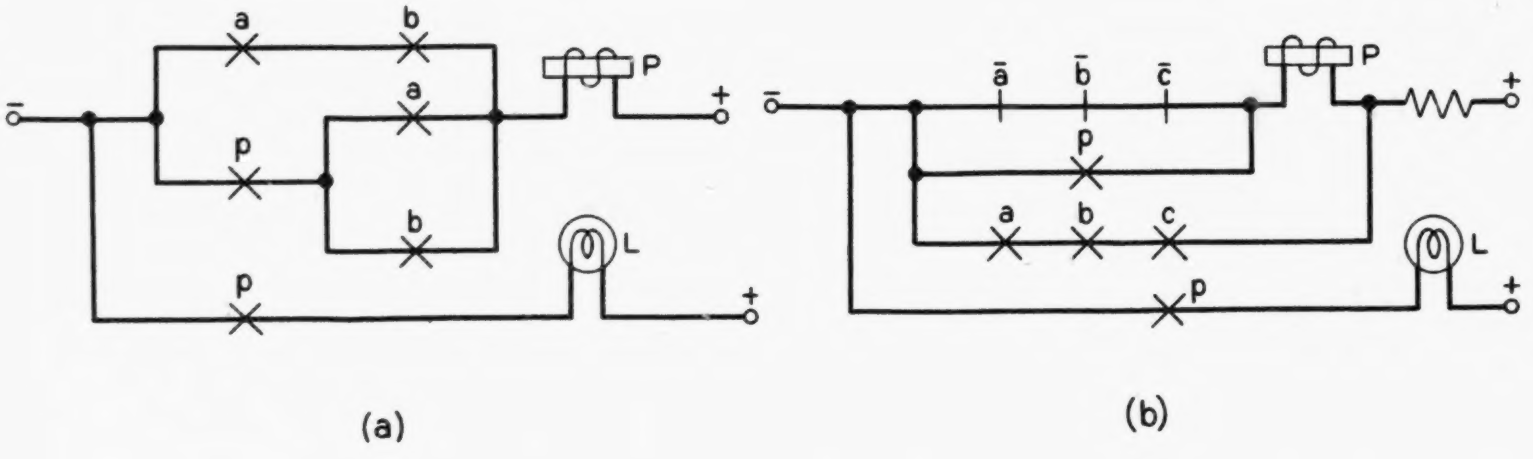
4-6 A bimetallic strip controls a contact, t, so that when the temperature is high the contact is open and when the temperature is low the contact is closed. The contact is placed in series with a resistor, R, the heat from which can raise the temperature of the bimetallic strip. Explain what happens when the circuit is in operation over an extended period of time. Would you call this a stable or an unstable circuit? Why?



4-7 Two bimetallic strips control two contacts t_1 and t_2 (when the temperature at a strip is high the contact is open; when the temperature is low the contact is closed). The resistor R_1 is placed next to contact t_1 so that when current flows in R_1 the temperature at t_1 is raised. How would you expect the circuit to behave over an extended period of time? Would you call this circuit stable or unstable? Discuss.

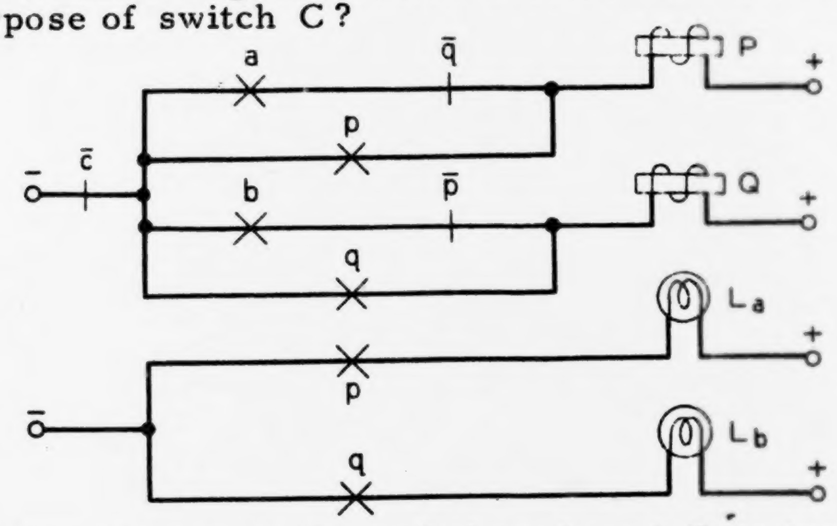


4-8 Discuss what conditions are necessary to turn the lamp L on and off in the following two circuits.

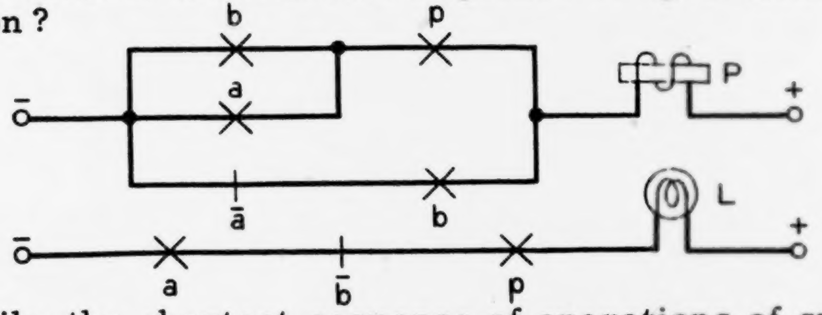


(a) (b)

4-9 The following circuit is used to determine which of two contestants in a T.V. quiz show operates his switch first. Discuss the operation of this circuit assuming that switch C is released. What is the probable purpose of switch C?

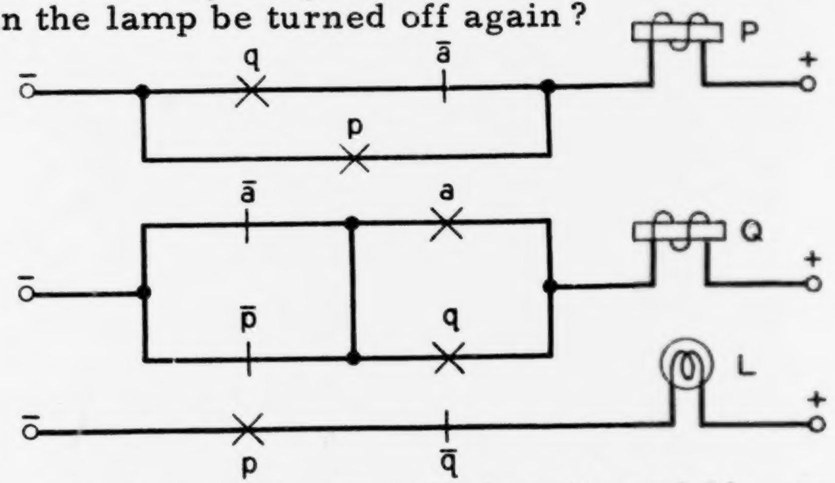


4-10 Discuss how to operate the switches A and B to turn the lamp L on and off in the circuit below. If the relay and switches are initially unoperated what is the shortest sequence of operations which will turn the lamp on?



4-11 (a) Describe the shortest sequence of operations of switch A which will cause the lamp L to be turned on. Assume that all switches and relays are initially unoperated.

(b) How can the lamp be turned off again?



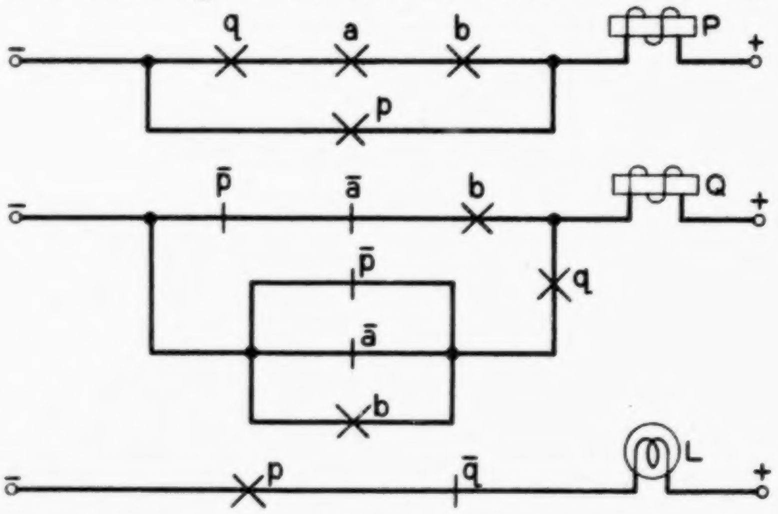
4-12 Refer to Fig. 6 of the text. Recall that 0's and 1's can be stored by operating and releasing switches A and B, and that the states of operation of C and D determine the address at which data are stored or sensed. For all parts of this problem assume that all switches and relays are initially unoperated.

(a) What and where is information stored when the switches D and A are operated (in that order) and switch A is released?

(b) What and where is information stored when C is operated, A is operated and released, and B is operated and released (in that order)? What sequence of operations is necessary to

(c) store a 1 in the relay S?
(d) store a 0 in the relay R?

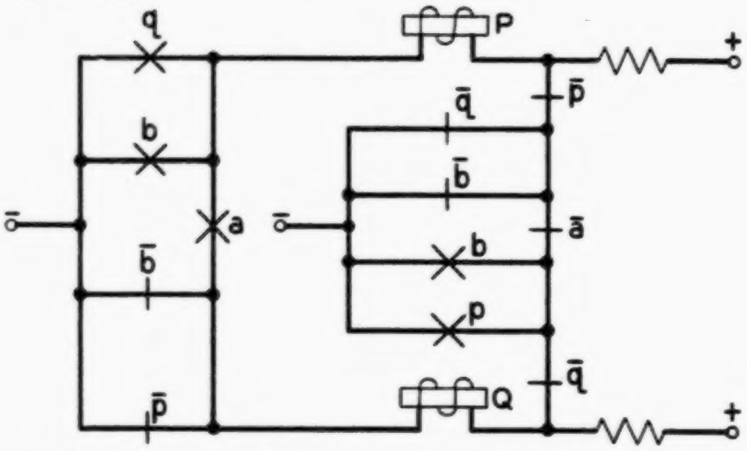
4-13 Describe the sequence of operations that are necessary to turn on the light in the following circuit. Give the reason for each operation.


q
a
b
P
+
p
p̄
ā
b
Q
+
p̄
q
ā
b
p
q̄
L
+

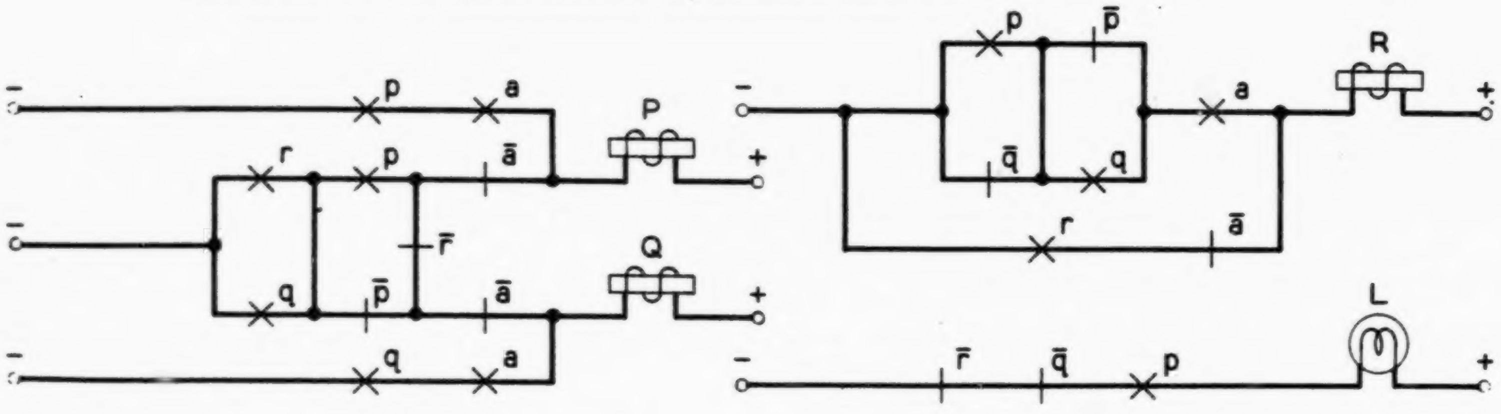
4-14 The switches in the following circuit are operated in the following sequence:

	1	2	3	4	5	6	7	• • • •	13	14
a:	0	0	1	1	0	0	1	• • • •	0	0
b:	0	1	1	0	0	1	1	• • • •	0	1

Describe how the relays P and Q operate and release, and how the number of times they operate relates to the number of operations of switches A and B.


q
P
+
p̄
q̄
b
b̄
a
ā
b
b̄
p
q̄
p̄
Q
+

4-15 Describe, step-by-step, what happens in the following circuit when switch A is alternately operated and released.


p
a
P
r
p
ā
+
r̄
q
p̄
ā
Q
+
q
a
p
p̄
R
a
+
q̄
q
r
ā
L
r̄
q̄
p
+

Chapter A 5

THE ORGANIZATION OF A COMPUTER

1. INTRODUCTION

The basic elements of a general-purpose digital computer are the logic circuits of Chapters A-3 and A-4. This chapter shows how such circuits can be connected together and controlled to perform a large variety of tasks, that is, to form a general-purpose computer. In performing any one of a vast number of tasks, such a computer differs from its component logic circuits, each of which is designed for one specific task. In fact, a general-purpose computer can perform any task which can be specified in detail. The function of this chapter is to explore the basis upon which such a computer is organized to achieve this objective. Chapter A-6 explains how one goes about specifying tasks in detail; i. e., how one "programs" the computer to execute a particular sequence of simple steps.

The basic quantity processed by mechanical devices such as an electricity generating plant or an internal combustion engine is energy, the basic quantity processed in a computer is information. "Data" is often used as a synonym for "information", and the terms "data processing" and "computing" can be used interchangeably. As we saw in previous chapters, by suitable conventions we can let the state of networks of relay and switch contacts represent arbitrary information, such as logical propositions, yes and no votes of legislators, binary numbers, etc. In our computer, we are going to deal with information encoded as 13-bit (12 bits-plus-sign) binary numbers and processed by binary selection trees, memory cells, counters, adders, instruction decoders, etc.* Remember, however, that this decision was a somewhat arbitrary one, and may be easily generalized:

1) the binary numbers and the corresponding circuits could as easily have been specified as 8 bits long or perhaps 36 bits; computers come in all sizes, in all sizes, with "word lengths" ranging from 12-bits to 64-bits.

2) moreover, computers need not be restricted to binary numbers, but by suitable conventions can handle decimal digits or alphabetic characters as well.

The computer is kept small because we wish to minimize detail. In spite of its small size, however, this computer illustrates the fundamental concepts which give any digital computer its power and versatility. In principle, (with appropriate programming techniques) this little computer is capable of "doing anything" that any "general-purpose" computer can do, and is therefore general-purpose itself. Futhermore, each part of it could be built in a straightforward way using the same type of logic circuits that you have met already. How to arrange or organize these parts into an appropriate whole is the subject which concerns us here.

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*This type of computer is called a digital computer, in contrast to analog computers which represent information not by discrete numbers but by analogous continuous physical quantities such as length (slide rule) or electrical wave forms (electrical analog computer of Chapter B-4).

For the time being, then, we work with 12-bit-plus-sign numbers, which must be stored and manipulated in the computer. We must also design a means for "reading" numbers into the computer and "writing" numbers from the computer onto some external output mechanism.

2. COMPUTING BY MACHINE

Having information in the computer, we want to process or transform this this information in some way. For example, we may want to add a column of numbers to obtain their sum, or arrange (sort) numbers so that they are in descending order from largest to smallest. Computers perform such functions by taking input information and then processing it according to a prescription, or program. After the job is completed, the results are given as output. This overall process can be described by the diagram in Fig. 1. We say information is processed according to a plan to yield results.

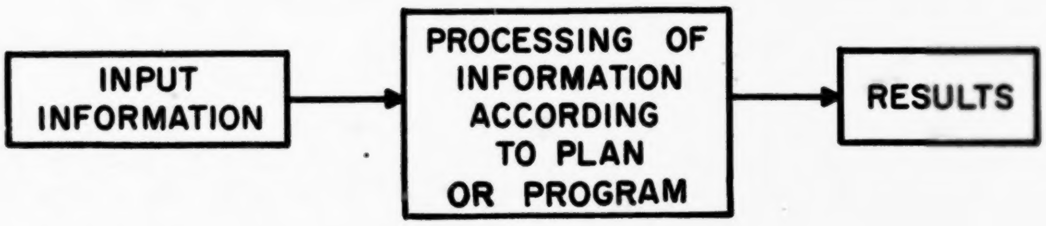


Fig. 1 What any computer does

Let us look at a few examples of information processing:

Input Information	Processing	Result
Binary number	Calculate decimal equivalent	Decimal number
Hours worked by employees during week	Calculate appropriate salaries	Payroll checks
Thrust sequence and mass of rocket	Calculate orbit	Prediction of satellite position
Geographical positions of cities to be visited	Calculate distances between cities	Route having minimum distance

Let us recognize that a computer in performing a complicated task makes use of the "building block" approach. We saw in the last three chapters how simple switches and relays could be combined to perform rather complicated tasks. This is an example of how many simple building blocks, properly assembled, can produce a whole that in some senses is more than a mere collection of parts. A brick building is more than a collection of bricks. It is shaped by the architect's plans into a useful and, one may hope, an esthetic form. Just as brick buildings do not usually resemble bricks, so machines made of switches and relays can perform feats far beyond those performed by the independent elements themselves. The computer, as an aggregate of switches and relays, is much more powerful than the sum of its parts.

In programming computers, we apply the same principle, specifying many simple steps to be executed in sequence in order to perform a complex task. The "program" is a plan for such a sequence of steps and is reminiscent of a cooking recipe. There, a sequence of not necessarily appetizing steps, taken in the proper order and properly executed, can lead to a culinary delight. The necessary repertoire of basic operations such as addition which computers perform is surprisingly small. To summarize, the basic principle is that the great power and utility of computers depends merely upon performing a sequence of relatively simple steps one after another. Later in this chapter we show how the computer "executes" the simple steps of a program, and in Chapter A-6 you learn how to solve problems by writing appropriate programs.

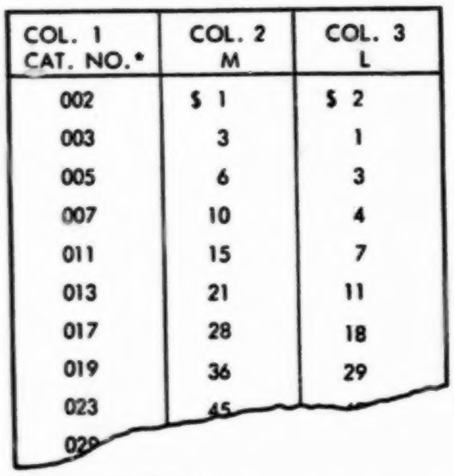
In order to establish what elements are necessary for a machine which repetitively executes simple steps, let us examine how a routine computing job might be done by a clerk.

Imagine you are the President of Gidget-Widget, Inc. You are publishing a new catalog and want to show the list price for each item. To get the list price you multiply the sum of the material cost, M, and the labor cost, L, by the factor 5.35. (This factor includes overhead, profit, markup for the wholesaler, distributor, and retailer and enough more so each customer can have a discount.) Before leaving for the French Riviera to do field observations on customer reactions to the latest G-W, you call in S. Fast Plodder.

"Stead, I'm going away on urgent business. While I'm gone, I'd like you to calculate new list prices for our entire line. These sheets contain the catalog numbers as well as the material and labor costs. This sheet has your instructions."

Plodder is one of your most trusted employees. He is a whiz with a desk calculator, works rapidly, and never makes a mistake. His only weak point is that he is not very good at figuring things out for himself; therefore you learned long ago that you must always give him an instruction sheet.

Let us take a closer look at the sheets. Fig. 2 shows one of the cost sheets and Fig. 3 shows the instruction sheet. We are going to watch Steadfast very closely as he does this calculation, and by paying close attention to his every move and sorting out the important details, we shall see, by analogy, what the functions of a computer should be, and how the components should interact.



COL. 1 CAT. NO.*	COL. 2 M	COL. 3 L
002	$ 1	$ 2
003	3	1
005	6	3
007	10	4
011	15	7
013	21	11
017	28	18
019	36	29
023	45	
029		

* "NO G-W EVER LEAVES OUR FACTORY IN OTHER THAN PRIME CONDITION"

Fig. 2 Cost sheet.

1. PREPARE A SHEET WITH TWO COLUMNS FOR THE CATALOG PRINTER. LABEL THE FIRST COLUMN "CATALOG NUMBER" AND THE SECOND "LIST PRICE".
2. COPY THE FIRST CATALOG NUMBER FROM THE COST SHEET ONTO THE PRINTER'S SHEET.
3. CLEAR YOUR DESK CALCULATOR AND PUT THE MATERIAL COST OF THIS ITEM INTO IT.
4. ADD THE LABOR COST OF THIS ITEM.
5. MULTIPLY BY 5.35.
6. WRITE THE PRODUCT ON THE PRINTER'S SHEET UNDER "LIST PRICE".
7. COPY THE NEXT CATALOG NUMBER ONTO THE PRINTER'S SHEET.
8. REPEAT STARTING FROM STEP 3 AS LONG AS THERE ARE UNUSED COST FIGURES. WHEN YOU HAVE FINISHED PUT THE PRINTER'S SHEET ON MY DESK.

Fig. 3 Instructions for S. F. P.

The block diagram of Fig. 4 summarizes the important actions which Steadfast takes; you should trace through it as you follow the text below.

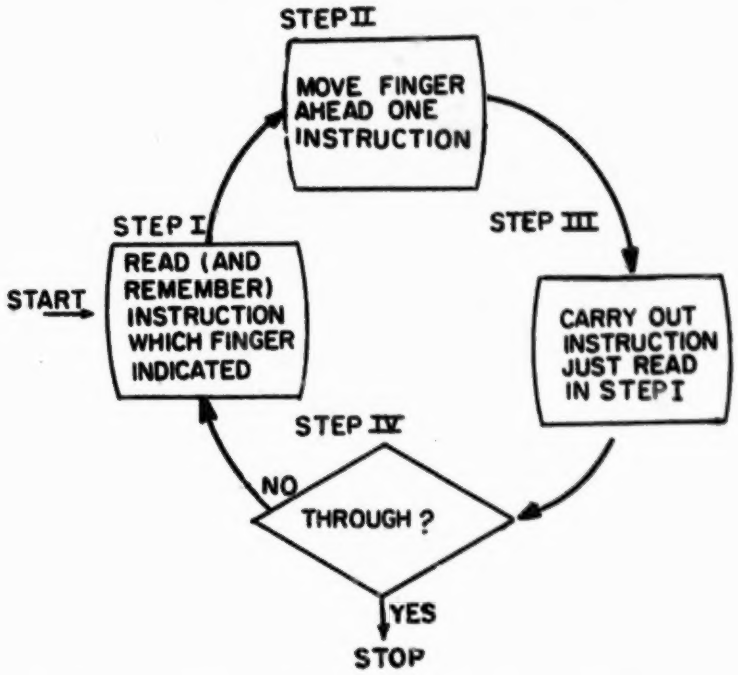

STEP II
MOVE FINGER AHEAD ONE INSTRUCTION
STEP III
STEP I
READ (AND REMEMBER) INSTRUCTION WHICH FINGER INDICATED
START
CARRY OUT INSTRUCTION JUST READ IN STEP I
STEP IV
NO
THROUGH ?
YES
STOP

Fig. 4 Plodder's operation cycle.

The first thing he does is sharpen his pencil. (This is unimportant), Next, he reads the first instruction (important, step I). Before executing this first instruction (#1), he wants to remember his place in the list of instructions, so he moves the index finger to his left hand ahead to the next instruction (#2) (without reading it), since #2 will be the one he will execute after the current one (#1). Having thus kept track of where he is (important, step II, he now executes the first instruction (important, step III) by taking a fresh sheet of paper and writing column headings "Catalog Number" and "List Price" on it. He now checks to see whether he is through (important, step IV). Meanwhile, his left index finger is still pointed at the next instruction (#2) which he now reads as the current instruction (important, step I). Again, before executing it, he first moves his finger ahead to what will soon be the next instruction (#3) (important, step II). Now he executes #2 by copying the first catalog number, 002, from the cost sheet to the printer's sheet which he prepared in instruction 1 (important, step III). He is not through (step IV), and since his finger is still pointed at the next instruction (#3), he now reads it as the current instruction (important, step I). He then moves his finger ahead to #4 (important, step II). Hearing thunder, he glances out the window and notices it is raining (unimportant). Still remembering #3, he executes it by clearing the calculator and entering the first material cost, $1.00, into it (all-important, step III). After checking to see he is not through (step IV), his finger tells him #4 is next, so he reads it (important, step I), moves his finger ahead to #5 (important, step II), and adds the first labor cost of $2.00 to give a total of $3.00 in the calculator (all-important, step III).

At this point let us stop and consider the evaluations that we have been making of the various activities as important or unimportant. For our purposes an activity is important if without it the computation cannot be done correctly. For instance, if Steadfast loses his place and repeats or misses an instruction, the list price he gets will be wrong; hence the finger on the instructions is important. From here on we omit Steadfast's actions which are "unimportant" by this definition.

Returning to our computation, we find that Steadfast reads instruction #5, moves his finger to instruction #6 and multiplies the $3.00 in the calculator by 5.35 to get $16.05. Following instruction #6, he puts the $16.05 on his printer's sheet. Using instruction #7, Steadfast writes 003, the next catalog number, on the printer's sheet.

Now observe that instruction #8 is different from the others. First Steadfast has to make a decision to repeat or not depending on whether he has used all the cost data. Second, if he has not yet finished, instruction #8 in effect tells him to move his finger back to instruction #3. That is, instruction #8 tells Steadfast to use instructions #3 to #8 again. Following instruction #8 Steadfast goes back to instruction #3 and starts to calculate the list price of the second item.

Let us now consider how we might design a computing machine to do this type of computation. Remember that although we illustrate the ideas with electromechanical relays and switches, modern digital computers are usually built with transistors and other solid state devices which are faster, smaller, cheaper, longer lived and use less power than relays. Their external behavior (as binary, or 2-state devices) is, however, the same.

In examining the essential features of Steadfast's performance, let us make a fundamental distinction between the information with which he worked and the apparatus and materials which he used in the process. The information and the operations on it are indicated in Fig. 5, which is very similar to Fig. 1.

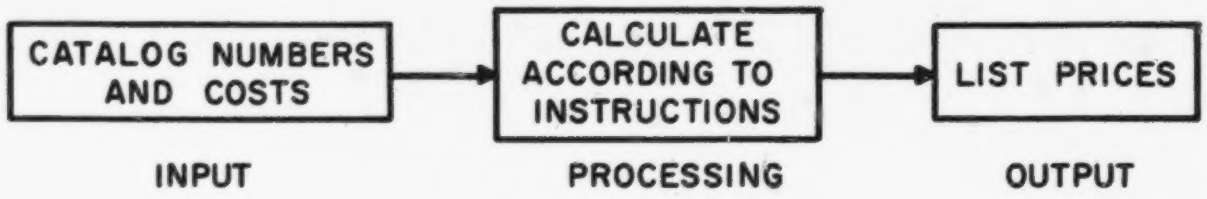

CATALOG NUMBERS AND COSTS
CALCULATE ACCORDING TO INSTRUCTIONS
LIST PRICES
INPUT
PROCESSING
OUTPUT

Fig. 5 Information Processing by S. Plodder

On the other hand, the apparatus and materials used by Steadfast were: pencils for writing, paper for recording (remembering) numbers, paper for the instruction list, and a desk calculator for doing arithmetic. A machine to duplicate Steadfast's performance must have corresponding equipment; namely, memory with a means for "writing" information into it electrically, memory for holding instructions, and a calculator to do arithmetic. The basic organization of such a machine is shown in Fig. 6. The Control box takes care of routing information appropriately.

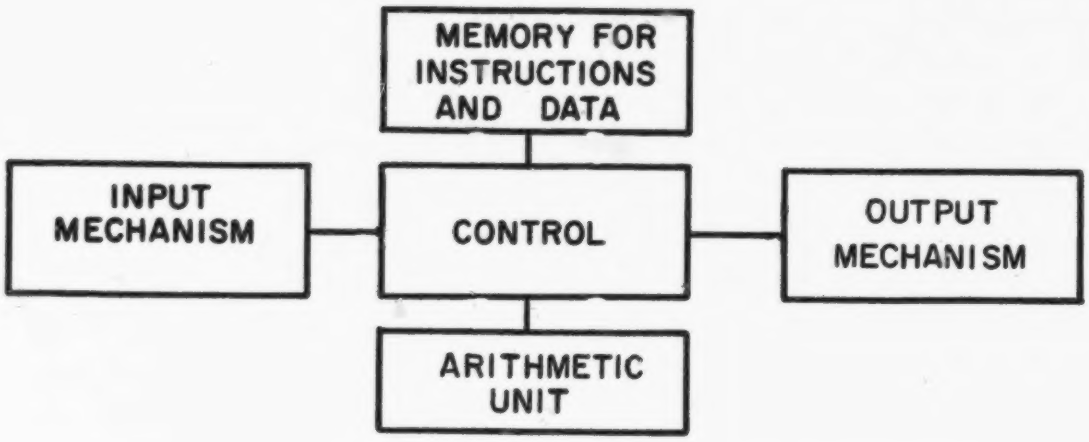

MEMORY FOR INSTRUCTIONS AND DATA
INPUT MECHANISM
CONTROL
OUTPUT MECHANISM
ARITHMETIC UNIT

Fig. 6 Block Diagram of Computer Functions

In thinking of the control function, it may be helpful to liken it to the operation of a telephone switchboard. The switchboard and operator in the telephone application connect people who want to talk to each other. The control function in the computer connects input, output, memory, and arithmetic unit when they want to "talk" to each other; that is, when information is to be transferred from one to the other. To duplicate Steadfast's performance, the machine must:

(1) Enter input information (data) into memory
(2) Calculate according to a plan of instructions in an orderly cycle such as that of Fig. 4.
(3) Deliver output information.

The five boxes shown in Fig. 6 are the basic components of any modern digital computer, as well as our own. Let us see how we use circuits previously designed and some new input/output "hardware" to implement these functions. You will find that these circuits are essentially unchanged; the only additions are some contacts to connect the circuits together and to control and time the flow of information between them.

3. INPUT AND OUTPUT OF INFORMATION

Information (data) can be stored in many forms other than on switch contacts. The most common form is the printed page, which in every sense is an efficient and convenient information storage medium. Information in this form can be easily read by most people (provided they know the language), but it can be read only with difficulty by a computer. Computers do not have eyes and brains with man's facility for reading the variety of type faces, sizes, and distortions commonly found in printing. A simple computer-readable information storage medium is the punched card.

A punched card is effectively divided into areas, each of which can either have a hole in it or not; that is, each area can be in one of two conditions, punched or unpunched. Our computer's card is divided into a single row of thirteen columns, thus providing thirteen punch areas to represent thirteen-bit binary numbers. A punched column represents a 1; an unpunched column represents a 0, as shown in Fig. 7(a).

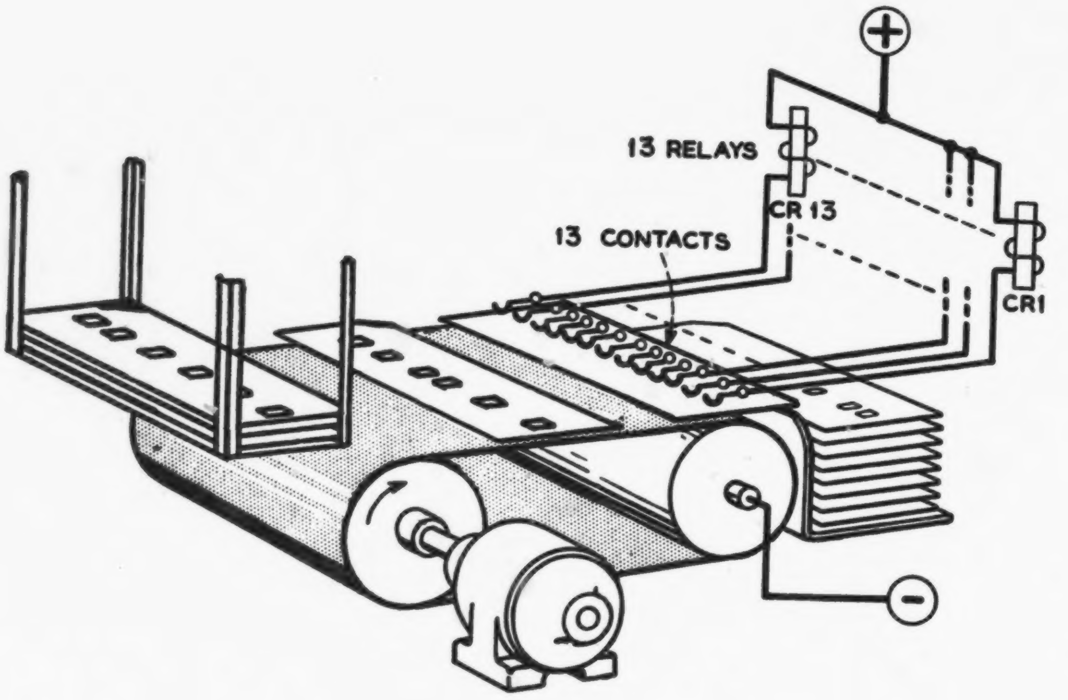


Fig. 7 Punched card and reader

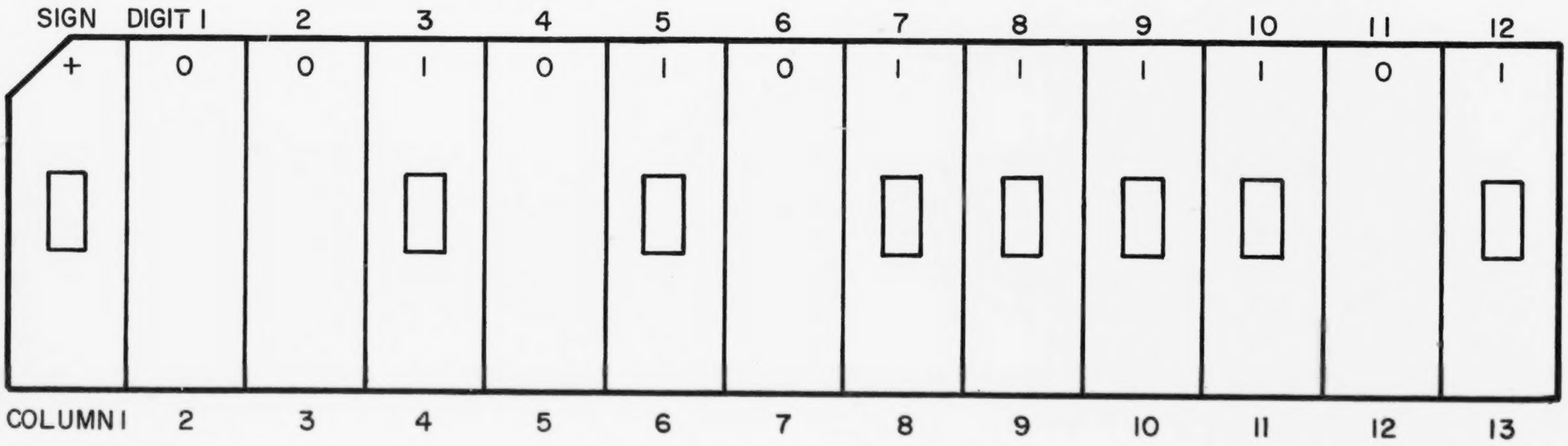

SIGN
DIGIT 1
2
3
4
5
6
7
8
9
10
11
12
+
0
0
1
0
1
0
1
1
1
1
0
1
COLUMN 1
2
3
4
5
6
7
8
9
10
11
12
13

Fig. 7(a) A 13 bit punched card for our machine (represents the number +001010111101).

Holes in cards are difficult for people to read. For that reason, most punched cards have the corresponding digits printed at the top of the card so that they are easily "man-readable". The computer reads the card by sensing the positions of the holes (see Fig. 7. A deck of cards is put into a stack, then the motor-driven belt moves them one at a time under a set of 13 contacts. At the instant that the contacts are aligned over the row containing the punches, a pulse of current is sent through the elctrical circuit which runs between the + and - terminals, and only those relays (CR_1 through CR_{13}) will be operated, which correspond to columns containing punches. Thereby, information can be transferred from a punched card into relays, or for that matter any other binary elements.

It is possible, too, to transfer information from relay contacts to punched cards by means of the card punch shown in Fig. 8.

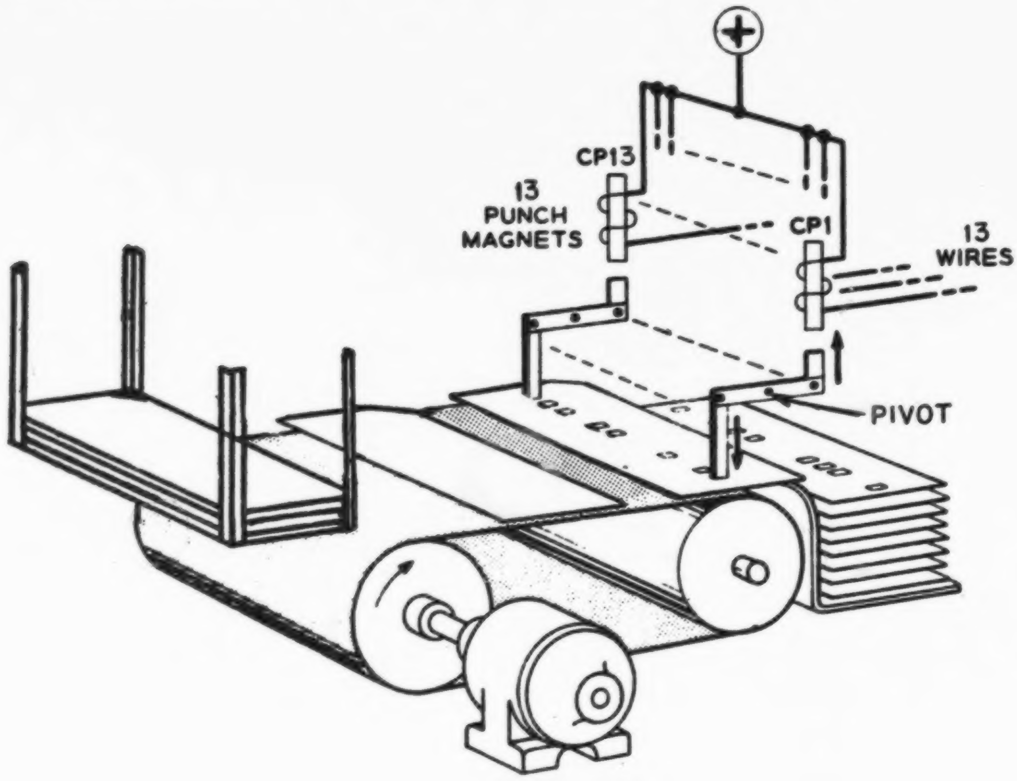


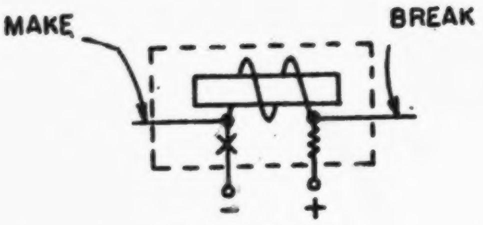
Fig. 8 Card punch.

The card punch has thirteen hole punches, each driven by an electromagnet similar to that in a relay, though stronger. During operation, a motor moves a blank card from the supply stack to the punches. When the punchers are aligned with the digit positions on the card, the thirteen electromagnets are selectively energized (in accordance with binary information) by an electrical pulse on their windings. This action causes the desired binary number to be punched in the card. Then the motor moves the card to the output stacker.

Information can also be stored in "machine-readable" form on punched paper tape or magnetic recording tape, or any one of many other media. All operate in essentially the same fashion as the punched card except for details of the recording mechanisms involved.

4. MEMORY

Our memory is an expanded version of the circuit of Fig. 6(b), discussed in Section 4 of Chapter 4, and reproduced below as Fig. 9(a). This block, i.e., a relay and its holding contact, can be considered as a single binary (two state) device whose left hand lead is the "1" ("make") input and whose right hand lead is the "0" ("break") input. Similarly, the roots of the make and break trees, identified as X_1 and Y_1, respectively, can be considered as input leads or terminals (for "1" and "0", respectively) to the entire 4 cell circuit. In our original discussion we stated that "c" and "d" contacts, which make up our selection trees, were operated by manual switches; but for our computer we control these contacts by relays. These relays are not illustrated in the diagram, but we must keep in mind the fact that somewhere in the computer relays C and D are present (with their own holding contacts to provide memory), and that by energizing these relays properly we may cause contacts "c" and "d" to select one of the memory cells (P, Q, R, S), where our information bits will be stored. Since the computer can't identify memory cells with symbolic letters such as P and Q, we refer to the binary settings of the tree relays as "addresses". In particular, C = 0, D = 0 corresponds to address 00. i. e., to relay P; similarly 01 corresponds to relay Q, 10 corresponds to relay R, and 11 corresponds to relay S. Thus every relay corresponds to a unique address, and vice-versa.



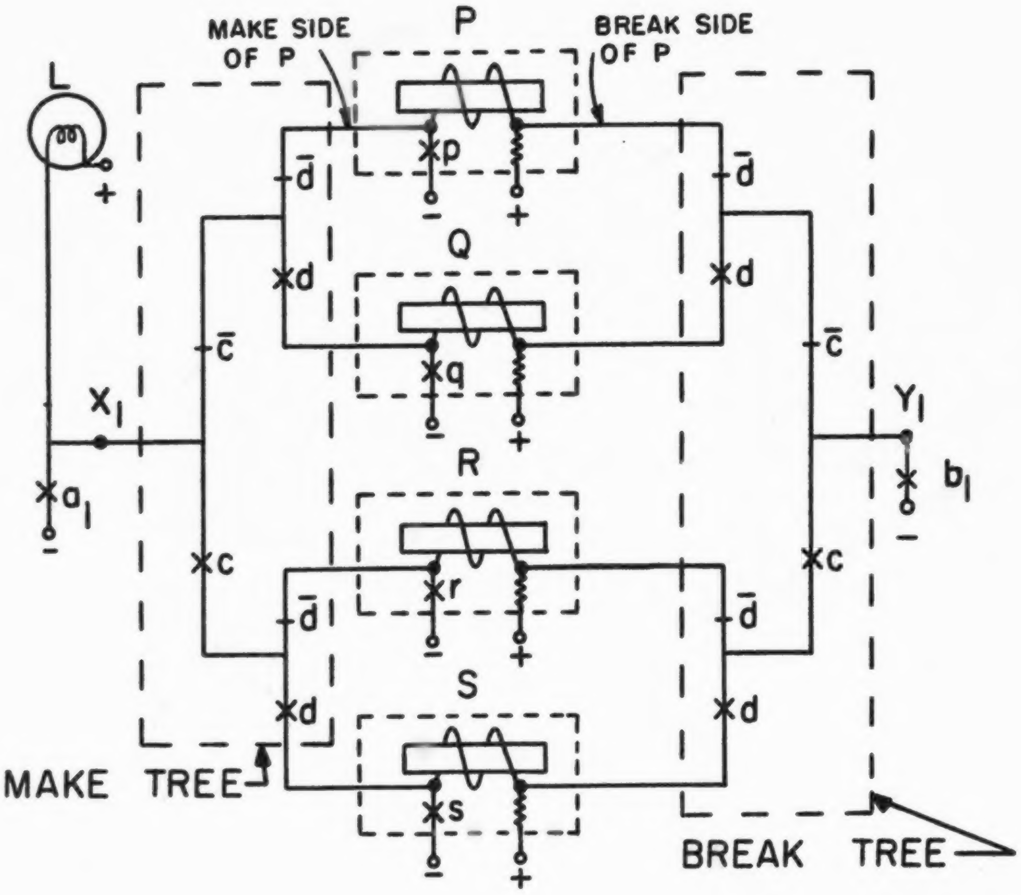
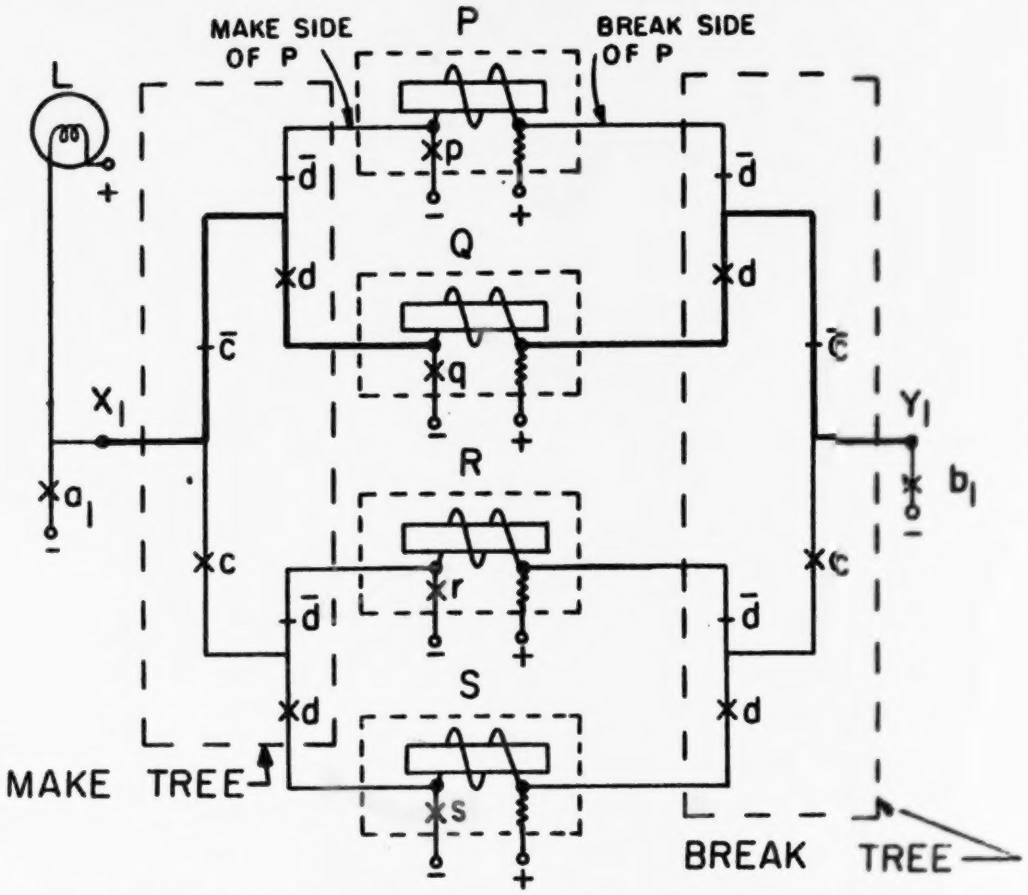


Fig. 9(a) A 4 cell, 1 bit memory with addressing
(The nodes X_1 and Y_1 are for future reference)

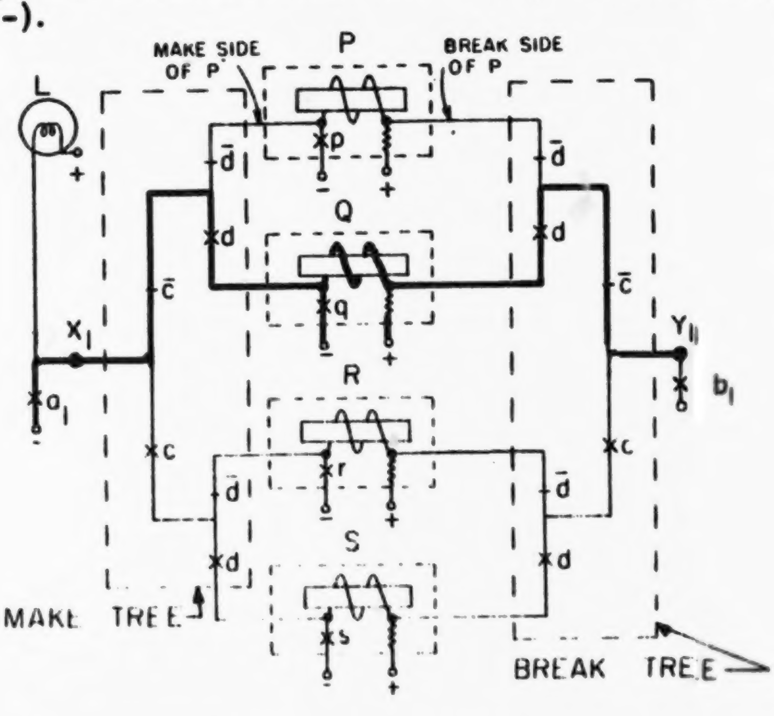
To store a "0" or "1" bit at a given address, you remember that we use a two step process: (1) select the address of the desired relay, and (2) set this relay to the desired value by connecting its make or break side to minus. Thus, to store a "1" in Q, i.e. at address 01, we first release C and operate D, to establish paths from the terminals X_1 and Y_1 to the relay Q. Fig. 9(b) shows the selected with heavy lines.



MAKE SIDE OF P
P
BREAK SIDE OF P
L
+
$\bar{d}$
d
p
Q
q
$\bar{c}$
c
X_1
Y_1
a_1
b_1
R
r
S
s
MAKE TREE
BREAK TREE

Fig. 9(b) Address 01 selected; contacts a_1 and b_1 remain open.

To set the state of Q to 1, we now operate the input A_1 to connect Q to minus. (Fig. 9(c)) shows a completed make path, and the operated relay Q. (The heavy line through the relay coil indicates that the relay is operated, note that only the left path is completed to -).



MAKE SIDE OF P
P
BREAK SIDE OF P
L
+
$\bar{d}$
d
p
Q
q
$\bar{c}$
c
X_1
Y_1
a_1
b_1
R
r
S
s
MAKE TREE
BREAK TREE

Fig. 9(c) Address 01 set to "1"; contact a_1 closed, b_1 remains released.

If we had chosen to store a "0" at Q instead, step 1 (selection) would have been the same, but step 2 (set) would have been to operate this input switch B, to release Q regardless of its previous setting. (Fig. 9(d) shows the completed break path.)

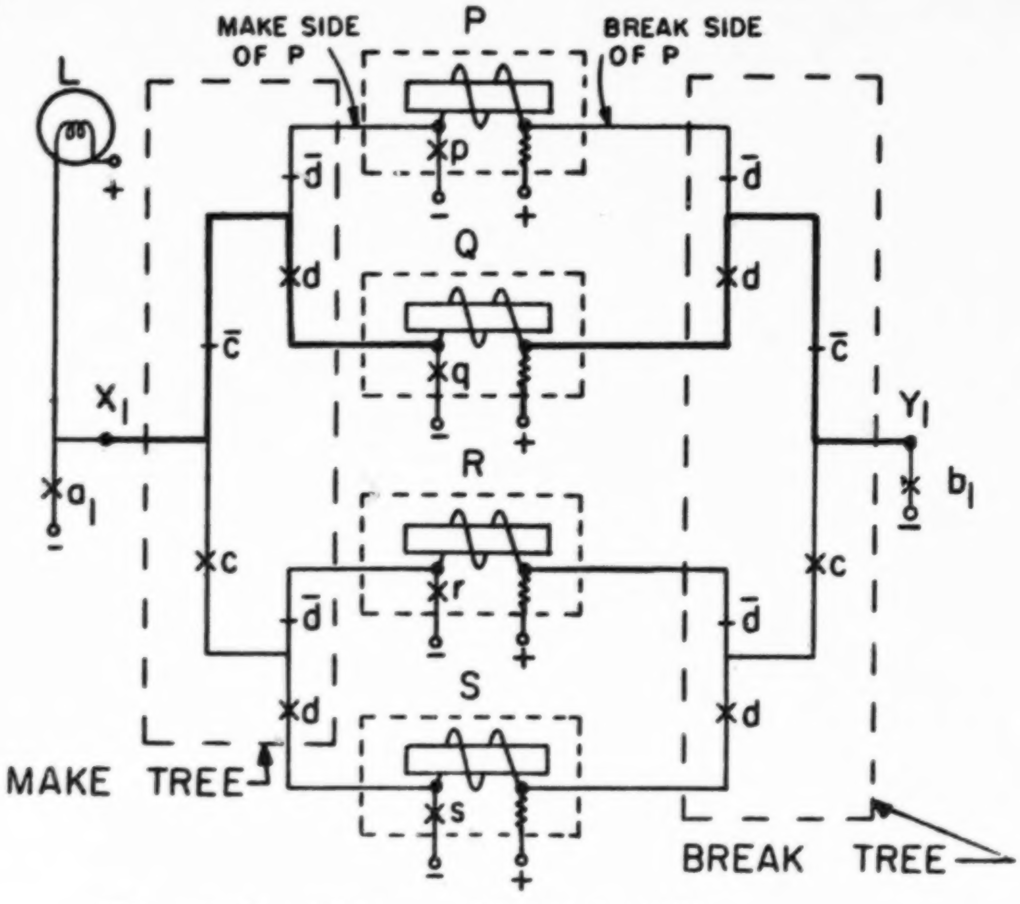


Fig. 9(d) Address 01 set to "0"

Notice that the input terminals X_1 and Y_1 are the only "handles" on this 4 cell circuit which can be used to change the states of any of the four cells.

To "read" the contents of a given address we simply select the cell as in Fig. 9(b): if X_1 is connected to an operated relay, X_1 will be at minus and the lamp will light. If X_1 is connected to a released relay, it is at plus, and the lamp will fail to light. See Fig. 9(e) Notice that the path from plus, through lamp L and contact X_1, is always present so that Figs. (b), (c), and (d) should have included that path - it was omitted for the sake of simplicity.

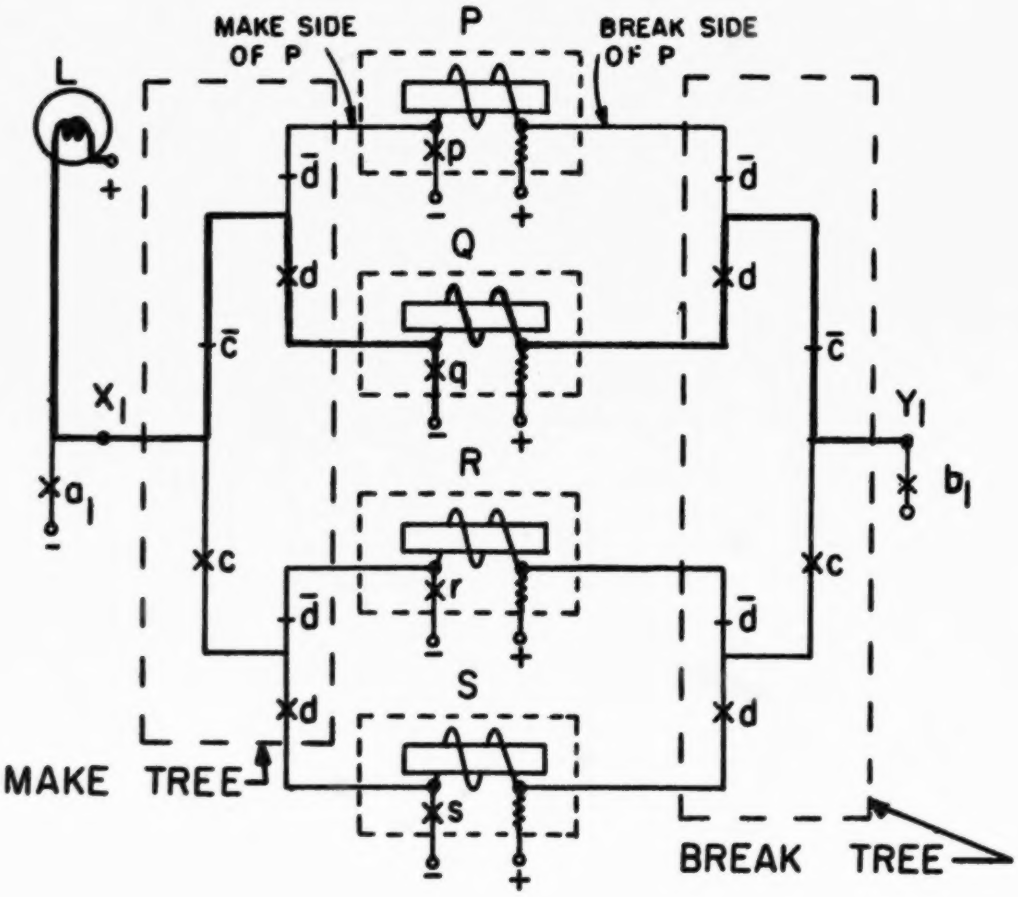

MAKE SIDE OF P
P
BREAK SIDE OF P
L
+
d̄
p
Q
d
q
c̄
X₁
Y₁
a₁
b₁
R
c
r
S
s
MAKE TREE
BREAK TREE

Fig. 9(e) Cell 01 selected and read (Q operated)

If we wish to store binary numbers more than 1 bit long in an addressable memory, we must use additional copies of this circuit. As an example, let us first store 3-bit numbers, again in one of 4 uniquely addressable locations. Fig. 5.10 shows 3 copies of the previous circuit, with symbolic boxes instead of the actual relays, and the setting contacts A_1 and B_1, etc., omitted.

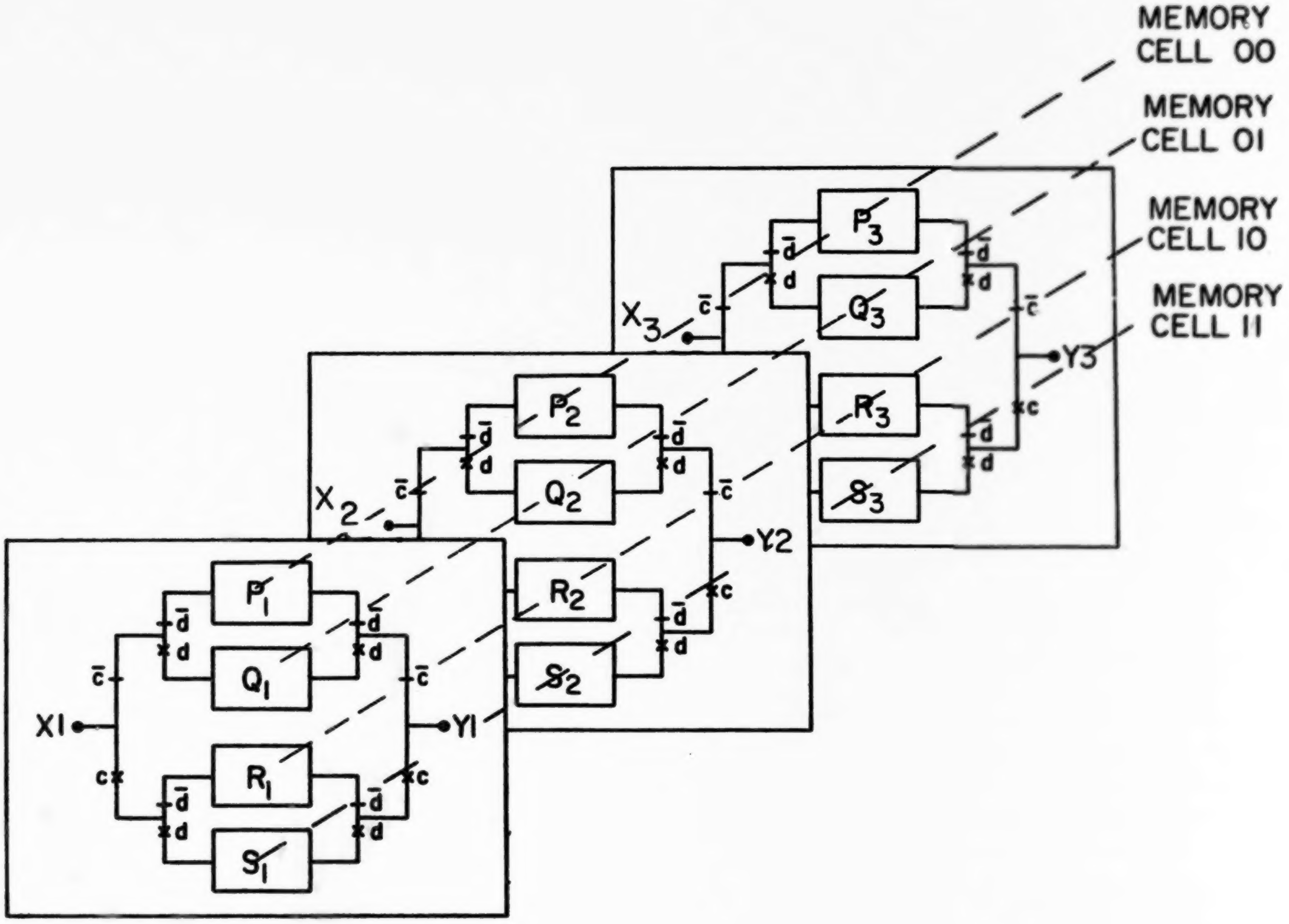


Fig. 10 A 3-bit, 4-cell memory (shown symbolically).

There is no direct electrical connection between these 3 circuits; there is, however, a logical one: a given setting of the C and D relays selects both sides of the corresponding cell in each of the 3 circuits (clearly, the C and D relays must have a large number of contacts). For instance, if we release C and operate D we select cell 01, the next to the topmost cell in each circuit. To symbolize the logical connection between those 3 cells, the three corresponding relays selected by a single address are designated by a letter (P, Q, R or S) with subscripts ($P_1P_2P_3$, Q_1Q_2...).

To store a 3-bit number, say 101, into cell Q ($Q_1Q_2Q_3$) we set 01 in the selection tree relays, and operate the recording relays A_1, B_2, and A_3, respectively. If we make the convention that B contacts are in effect break contacts on the A relays, rather than make contacts on separate relays, then we operate A_1, release A_2, and operate A_3 to store the pattern 101 on the Q relays. Thus we have copied the states of the A relays onto the corresponding Q relays. This copy action is the fundamental method by which information (numbers) is transferred in the computer.

Thus bits don't physically move from one place to another - addressed relays are set (regardless of their previous states) to agree with the originating relays, bit by bit, relay by relay, thereby copying the original.

For our 13-bit machine 13 replicas of the circuit are needed. Each group of 13 logically connected cells storing a single number ($Q_1 Q_2 \ldots Q_{13}$, for example) is referred to as a memory cell, memory word, or memory register. (The word "memory" if frequently omitted.) Again, a given word is located through the trees by a unique address (01, for example). Now, if we wished more than four words (i.e., more storage positions), we should expand the trees to include contacts on three relays: C, D and E, for instance. Three relays provide 2^3 possible paths, so that each tree would then select from 8 memory relays. We should then have 13 circuits, however, to provide for the storage of 13 bits: P_1 through P_{13}, Q_1 through Q_{13}, ..., V_1 through V_{13} and W_1 through W_{13}. In general, then, k tree relays and n copies of this type of circuit will provide 2^k words, each of which is n bits in length. Thus nine tree relays, and 13 circuits provide 512 13-bit cells.

5. INPUT AND OUTPUT CONNECTED TO MEMORY

For purposes of illustration, let us see how we might connect our card reader, card punch and a 4-cell 13-bit memory together. Fig. 11 shows a block diagram of the circuit. We have labeled the boxes (which stand for pieces of hardware) with general names but have indicated in parentheses the actual hardware used in this example. The solid lines with arrows running between boxes show the direction in which signals (binary numbers) are transmitted from box to box. For instance, a number can go from the card reader to a memory cell but not from the memory cell to the card reader. Relays I (Input, i.e. read a card) and O (Output, i.e. punch a card) are used to connect card reader or card punch at the appropriate time to the selected cell in memory.

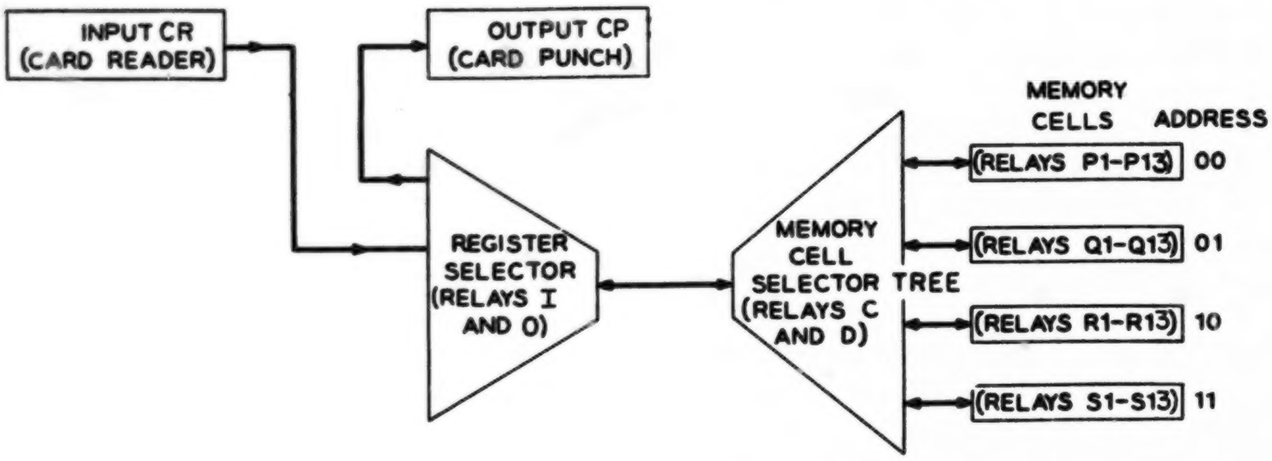


Fig. 11 Block diagram of a 4-cell memory connected to a card reader and a card punch. (The hardware associated with each block is shown in parentheses.)

An implementation of this block diagram is shown in Fig. 12. We have started with 13 copies of the circuit of Fig. 9. In this drawing we have removed the contacts on the relays A and B, which were connected to the information input nodes X_1 and Y_1, and replaced them by contacts which will carry information to and from the card reader or the card punch relays (CR 1 and CP 1, etc.). In addition, some control relays used for timing and selection purposes (I and O) have been introduced. The operation of this configuration, explained below, is naturally almost identical to that of Fig. 9.

To read a single 13-bit word from the card reader into a specific memory cell, we use the two step select-copy process. First we must select the address of the cell by selecting a path in the trees, and then read the first bit, 1 or 0 (hole punched or not punched, respectively), into the selected relay on the first circuit, the second bit into the corresponding relay on the second circuit, etc. We read the 13-bit word by copying the states of card reader relays onto cell relays. In detail, the action is as follows:

a) Clear: run the motor on the card reader to move a new card into the reading position

b) Select: choose a single memory cell by setting relays C and D to that cell's address

c) Copy: momentarily operate connection (timing) relay I (input: i.e., read a card), which will close contact i.

If there is a punched hole in the first column of the card, relay CR_1 on the card reader (Fig. 7(b) will be operated, and a path will be established from the "minus" terminal through make contact cr_1 through into node X_1, and then through the tree to the make side of the selected relay. Furthermore, break contact $\overline{cr}_1$ will open, and no path will exist from the minus terminal, through contact i, and node Y_1, to the break side of the selected relay. Thus, a 1 is stored in the selected relay. On the other hand, if there is no punched hold in the first column, relay CR_1 will not be operated, and the path from the minus terminal through the make contact cr_1, i, to node X_1, to the make side of the relay will be open. However, break contact $\overline{cr}_1$ will now be closed, so that there will exist a path from the minus terminal, through contact i, node Y_1, through the tree, to the break side of the selected relay. Regardless of its previous condition, the relay will therefore be released, and a O stored. Also note that there is no path from node X_1 through the make contact o, card punch relay CP_1 and the + terminal, since timing relay O is not operated while information is being read in memory.

The action described above is simultaneously duplicated for each of the other 12 card columns and 12 circuit copies. Since the copies are electrically independent, a pattern of 13 0's and 1's is thus stored in the selected set of relays (the selected memory cell) corresponding to the 13 card columns. Four numbers can thus be read into each of the four groups of cells by successive selection and copying cycles.

In order to connect our memory to the card punch, we have removed the lamps from Fig. 9. (These lamps, you remember, were used to observe the number stored in whichever cell was selected by relays C and D.) We have added another new relay whose make contacts connect the 13 punch magnets CP_1 through CP_{13} (see Fig. 8) to the trees where the lamps used to be. Then, to punch a card:

a) clear: run the motor to bring a new card under the punches

b) select: set an address by relays C and D

c) copy: momentarily operate timing relay O.

Whether relay CP_1 is now operated or not depends on the state of node X_1, which in turn depends on whether the selected relay is operated or not. If it is

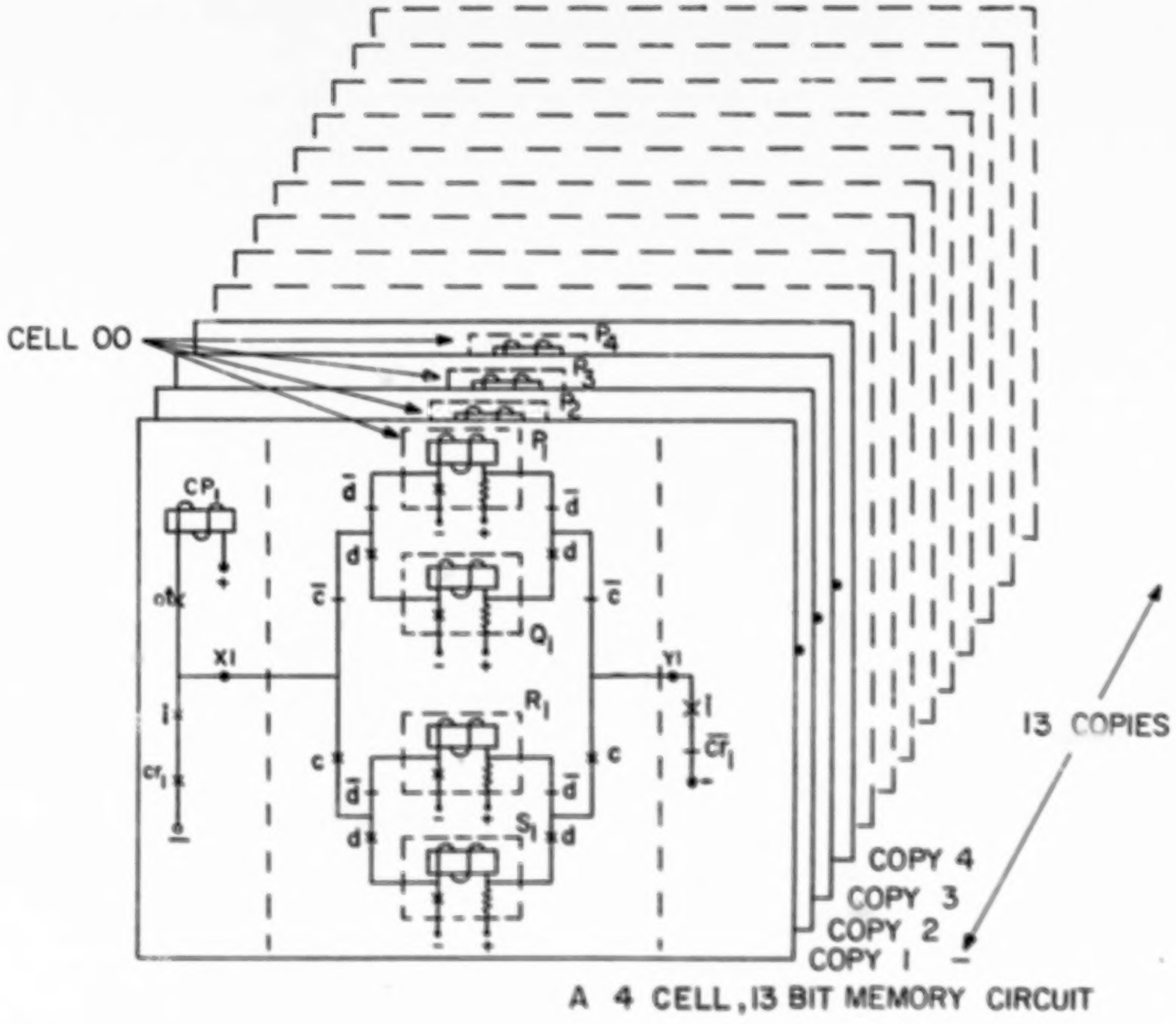

CELL 00
P4
P3
P2
P1
CP1
d̄
d
c̄
c
Q1
R1
S1
X1
Y1
i
cr1
x1
c̄r1
13 COPIES
COPY 4
COPY 3
COPY 2
COPY 1
A 4 CELL, 13 BIT MEMORY CIRCUIT

Fig. 12 Card reader and card punch connected to memory.

operated (if a 1 is stored), the node X_1 will be connected through the tree and the selected relay's make contact to the minus terminal. Since 0 is closed, CP_1 will then operate. If, however, the selected relay is released (a 0 is stored), the X_1 node is not connected to the minus terminal, and CP_1 cannot be operated. Similarly, CP_2 through CP_{13} are set to the correct values.

Before leaving this topic, let us for future reference list the control signals which we have used but for which no sources have been indicated. They are: pulses to start the reader and punch motors, pulses to set the C and D relays for the memory-cell selector tree, and pulses to operate the I and O timing relays in the register selector.

6. THE ACCUMULATOR

The part of a digital computer which is analogous to the desk calculator and which is used for performing arithmetic is called the arithmetic accumulator. The accumulator adds (or subtracts) one number to (from) another and "accumulates" the sum (difference). For our purpose the accumulator is a unit which consists of two registers (the A and B registers) for the temporary storage of the numbers to be added and a third register (the S register) which holds the sum of the numbers. The "sum" and "carry" circuits which perform the operation of addition and determine the sign are also included.

Fig. 13 is a reproduction of Fig. 9 of Chapter 3, a group of sum and carry circuits for a binary adder which adds three-digit binary numbers. A_1, A_2 and A_3 are the input switches and represent the three digits of one of the numbers, and B_1, B_2 and B_3 are the three switches which represent the second binary numbers.* With this method of representing the digits, operation of switch A_1 affects all a_1 contacts, A_2 controls all a_2 contacts, B_1 controls all b_1 contacts, and so on. Fig. 14 shows an addition example. The operation of the sum and carry circuits were previously treated in the discussion of Fig. 9, Chapter 3, but we are interested in the modification of this circuit to permit its use as an element suitable for a computer accumulator. For this function, we cannot depend upon the manual operation of switches. Relays are therefore required, and since we wish to add numbers each composed of twelve digits (and sign), it becomes necessary to use two sets of relays, each consisting of 12 individual relays, $A_1, A_2, \ldots A_{12}$ and $B_1, B_2, \ldots B_{12}$, and two special relays, A_s and B_s, which store the signs of the respective numbers but do not participate in the adder circuit itself (see Fig. 15(a)). For purposes of read-out of the sum of each pair of digits, we also replace the read-out lamps with 13 S register relays $S_0, S_1, S_2 \ldots S_{12}$ (S_0 stores possible "overflow") and a sign relay S_s.

*We have changed the "weighted" labelling scheme A_4, A_2, A_1 of Fig. 9, Ch. 3, to A_1, A_2, A_3, (A_3 least significant) to make the transition to the thirteen bit adder smoother. (In the weighted scheme, the most significant digit would have been A_{8192} or $A_{2^{13}}$.)

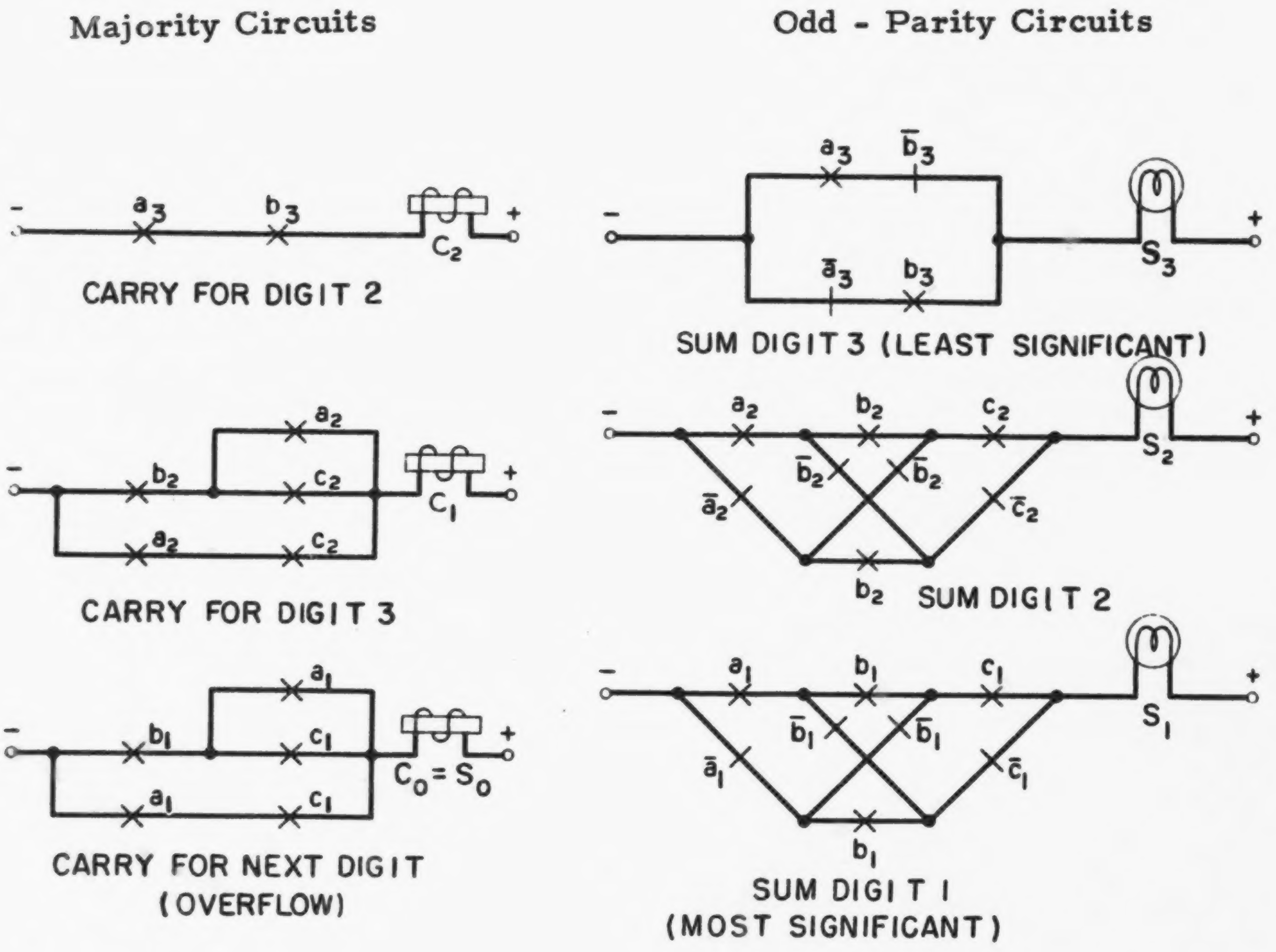

Majority Circuits
Odd - Parity Circuits
a3
b3
C2
CARRY FOR DIGIT 2
a2
b2
c2
C1
CARRY FOR DIGIT 3
a1
b1
c1
C0 = S0
CARRY FOR NEXT DIGIT
(OVERFLOW)
a3
b̄3
ā3
b3
S3
SUM DIGIT 3 (LEAST SIGNIFICANT)
a2
b2
c2
b̄2
ā2
c̄2
S2
b2
SUM DIGIT 2
a1
b1
c1
b̄1
ā1
c̄1
S1
SUM DIGIT 1
(MOST SIGNIFICANT)

Fig. 13 A three-digit binary adder.

CARRY DIGITS		C_0	C_1	C_2
REGISTER A		A_1	A_2	A_3
REGISTER B		B_1	B_2	B_3
REGISTER S	S_0	S_1	S_2	S_3

↑OVERFLOW DIGIT

	1	1	1	CARRY DIGITS
	1	0	1	FIRST NUMBER
	0	1	1	SECOND NUMBER
1	0	0	0	SUM

Fig. 14 Organization for a computer adder

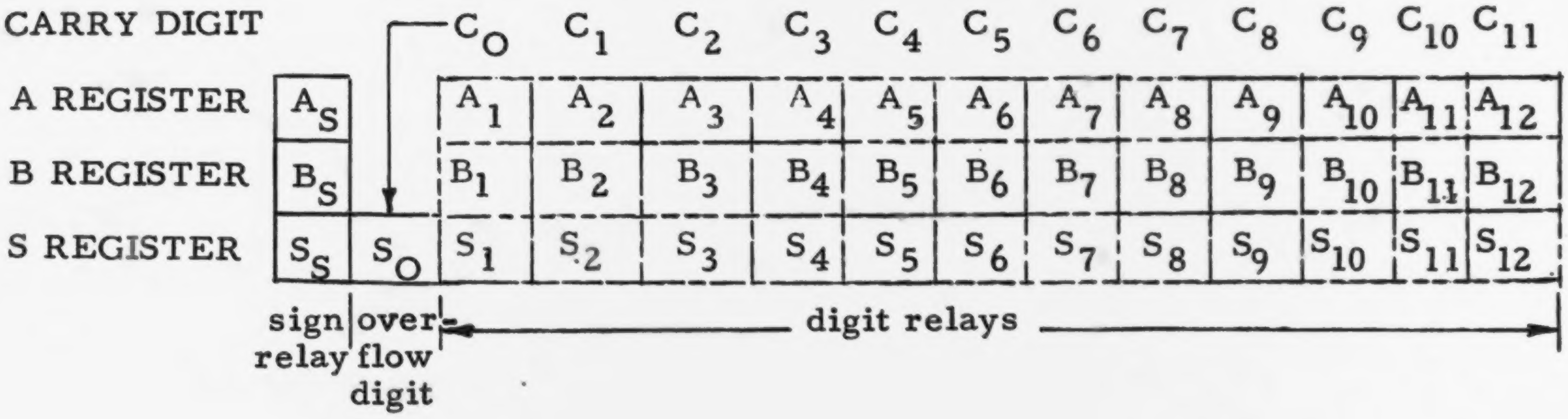

CARRY DIGIT
A REGISTER
B REGISTER
S REGISTER
C_0 C_1 C_2 C_3 C_4 C_5 C_6 C_7 C_8 C_9 C_{10} C_{11}
A_S A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10} A_{11} A_{12}
B_S B_1 B_2 B_3 B_4 B_5 B_6 B_7 B_8 B_9 B_{10} B_{11} B_{12}
S_S S_O S_1 S_2 S_3 S_4 S_5 S_6 S_7 S_8 S_9 S_{10} S_{11} S_{12}
sign relay
over flow digit
digit relays

Fig. 15(a) Registers for a 12-bits-plus-sign adder.

(There are also twelve carry relays $C_O, C_1, \ldots C_{11}$.) Fig. 15(b) shows the labelling correspondence between cards, cells and accumulator registers.

In addition to the register relays, we need some auxiliary control relays to move numbers into and out of the registers, as well as to clear them. As an example of the accumulator circuitry figure 16(a) shows the second stage of the accumulator, that is, the stage for calculating the second digit S_2 as the sum of A_2, B_2 and C_2. It is similar to all the other eleven stages except for stage twelve which is simpler, having no carry into it. Fig. 15(b) shows that it corresponds to the third copy of the memory circuit. In comparing this circuit to the 2nd stage of the accumulator in Fig. 13 we note that relays A_2, B_2 replace switches A_2, B_2 and relay S_2 replaces light L_2; also contacts of control relays for clearing (CLA, CLB, CLS), and connection relays (MA, MB, SM, SA) have been added.

Let us see how we use this accumulator to add numbers. Assume that the first number is stored in memory location 01 and the second in memory location 10.

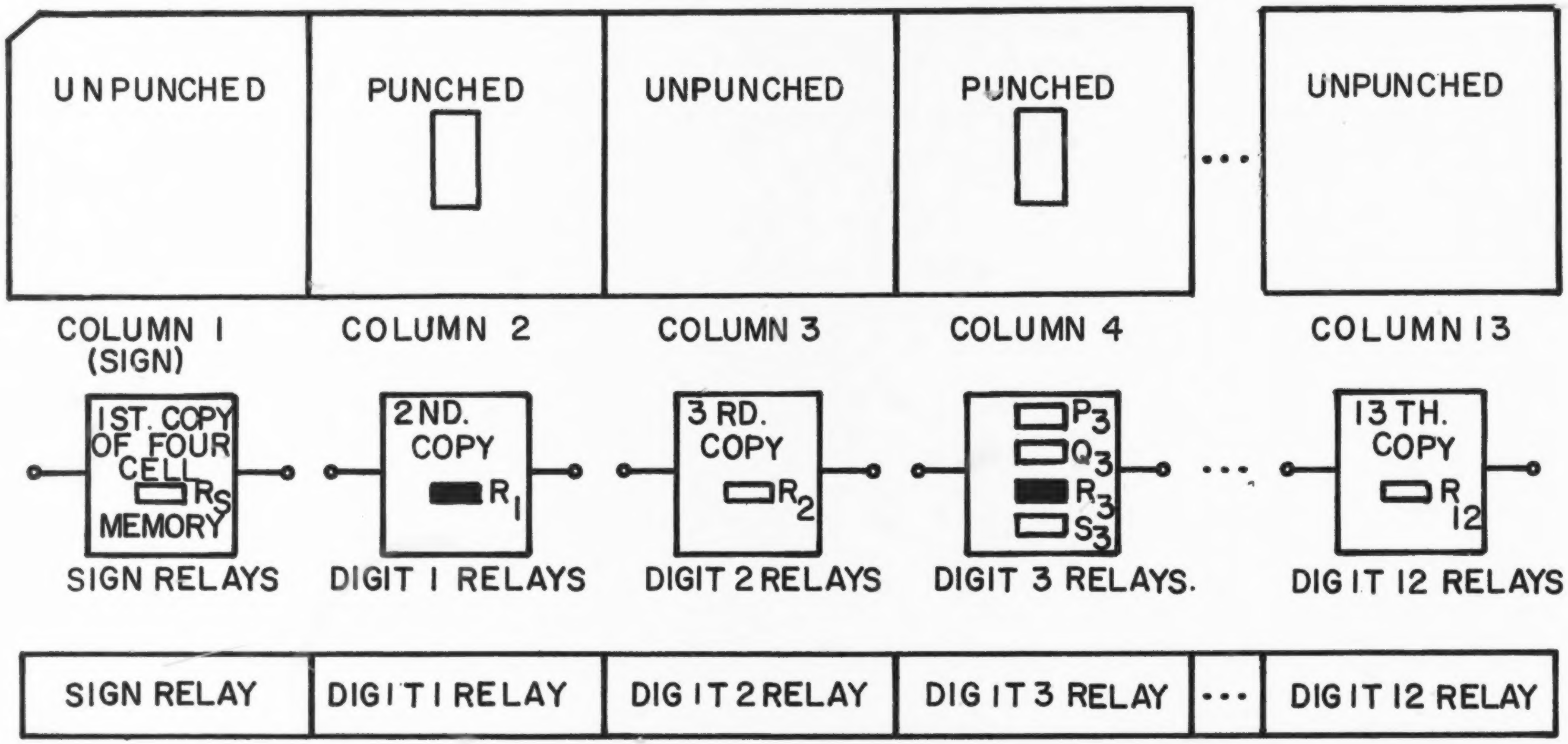
The sequence of action is remarkably similar to the two step select-copy process for reading or punching from a memory location. Since more than one number is involved, the switching required causes a small amount of additional complexity.

1) clear the A, B and S registers (0's are stored) prior to receiving the new information.

2) select the proper memory cell (first 01, then 10) in the memory selection tree (Fig. 12).

3) connect and set registers to the same value as the memory cells, digit by digit, thus copying the numbers from memory into the registers (01 in A, 10 in B).

4) add the numbers just loaded in the A and B registers.


UNPUNCHED
PUNCHED
UNPUNCHED
PUNCHED
...
UNPUNCHED
COLUMN 1 (SIGN)
COLUMN 2
COLUMN 3
COLUMN 4
COLUMN 13
1ST. COPY OF FOUR CELL MEMORY
R_S
2ND. COPY
R_1
3 RD. COPY
R_2
P_3
Q_3
R_3
S_3
...
13 TH. COPY
R_{12}
SIGN RELAYS
DIGIT 1 RELAYS
DIGIT 2 RELAYS
DIGIT 3 RELAYS.
DIGIT 12 RELAYS
SIGN RELAY
DIGIT 1 RELAY
DIGIT 2 RELAY
DIGIT 3 RELAY
...
DIGIT 12 RELAY

Example of a 13-bit Card
(number punched is -101. . . 0)

13-bit Memory Cells
(The R relays are
shown explicitly)

13-bit A or B Register
(The S register has an
additional S_0 overflow
relay)

Fig. 15(b). Labelling correspondance between punched card columns, (1-13), memory cell digits (S, 1-12), and register digits (S, 1-12).

NOTE: The memory cells have been re-labeled to reflect the fact that the first bit, formerly called P_1, Q_1, R_1, or S_1.

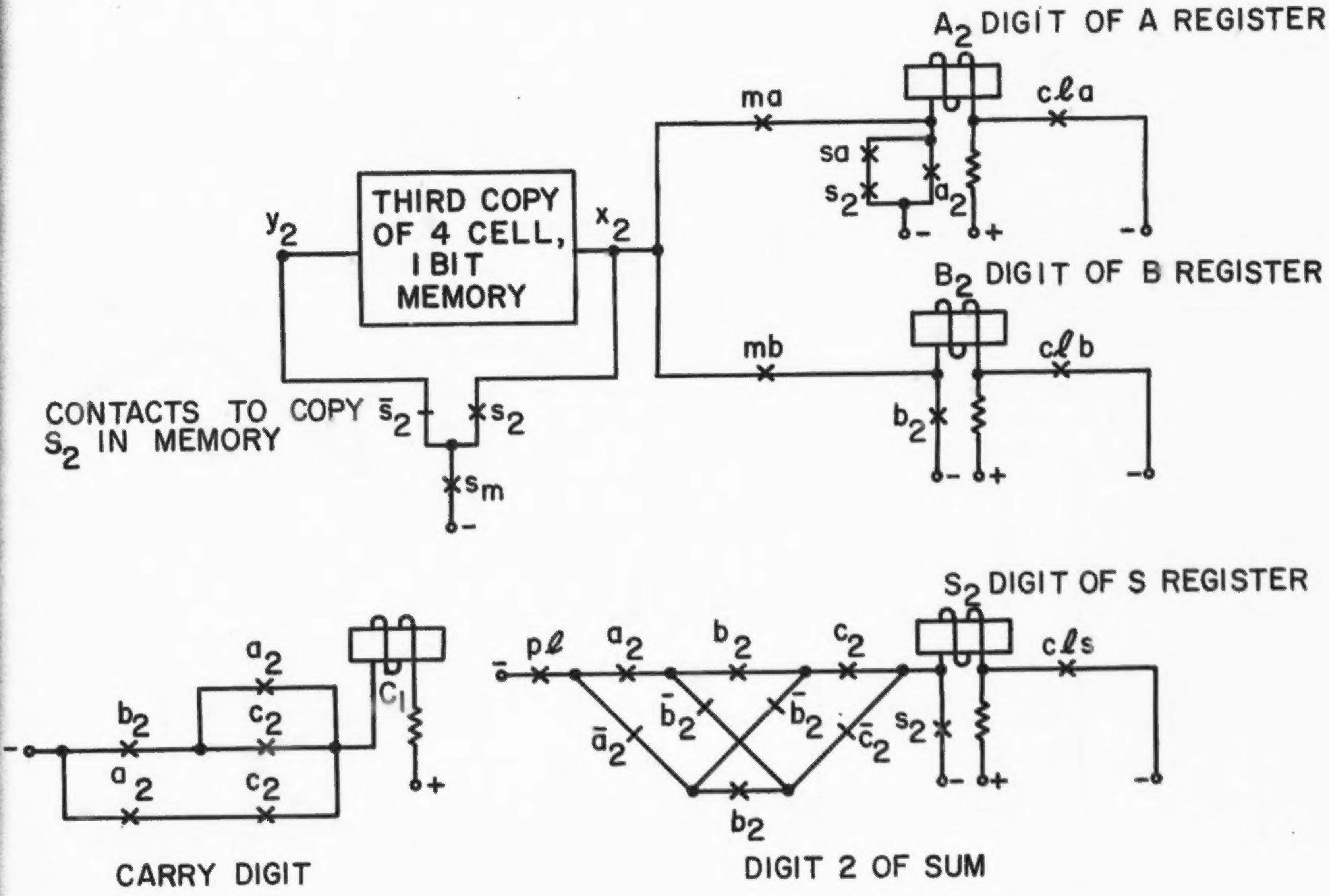
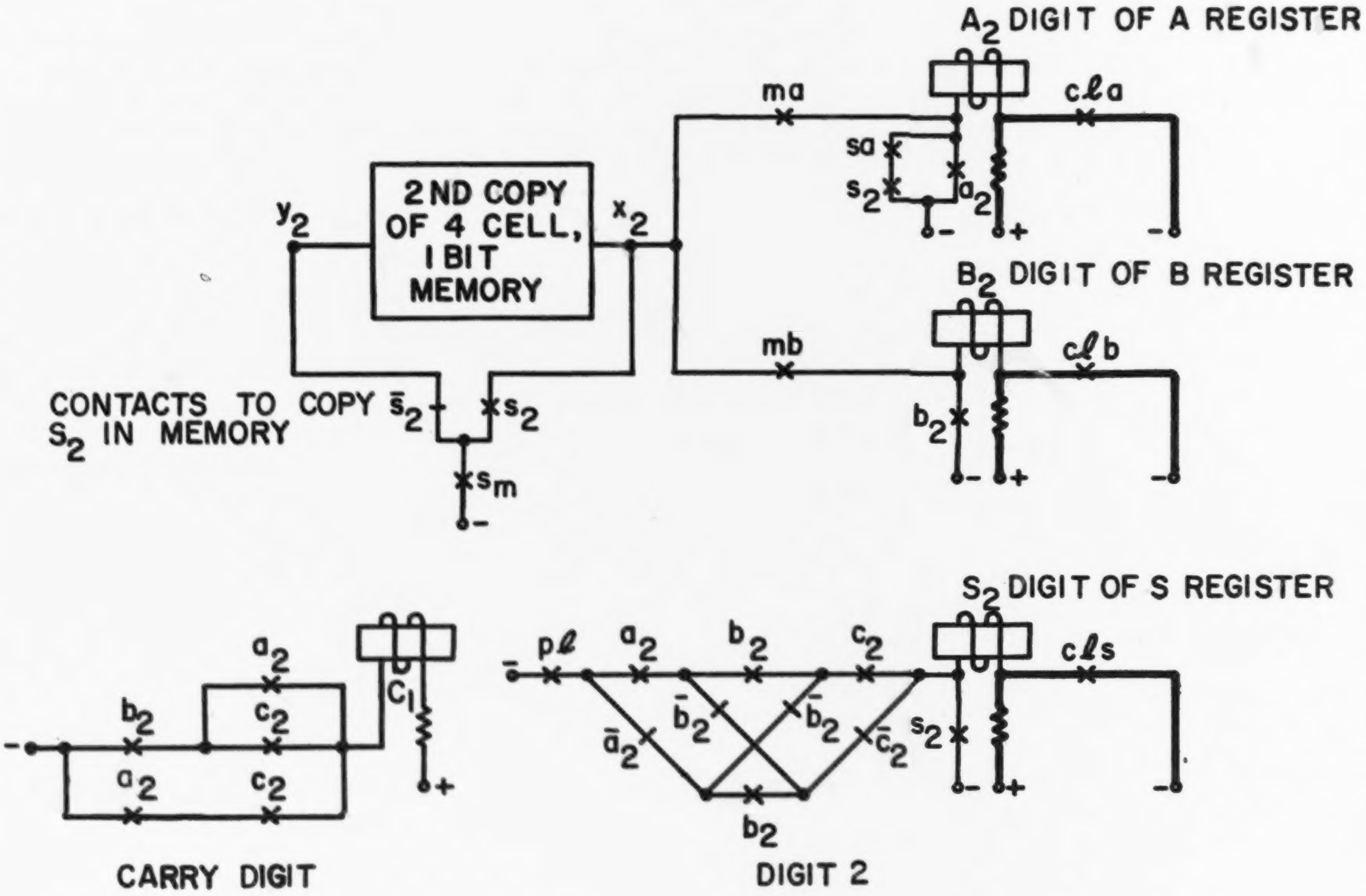

A$_2$ DIGIT OF A REGISTER
ma
cℓa
sa
s$_2$
a$_2$
THIRD COPY OF 4 CELL, 1 BIT MEMORY
y$_2$
x$_2$
B$_2$ DIGIT OF B REGISTER
mb
cℓb
b$_2$
CONTACTS TO COPY S$_2$ IN MEMORY
$\bar{s}_2$
s$_2$
s$_m$
S$_2$ DIGIT OF S REGISTER
a$_2$
b$_2$
c$_2$
C$_1$
a$_2$
c$_2$
pℓ
a$_2$
b$_2$
c$_2$
cℓs
$\bar{a}_2$
$\bar{b}_2$
$\bar{b}_2$
$\bar{c}_2$
s$_2$
b$_2$
CARRY DIGIT
DIGIT 2 OF SUM

Fig. 16(a) A, B, and S registers, and Communication with Memory.

CLA - Clear A
CLB - Clear B
CLS - Clear S
MA - Memory to A copy
MB - Memory to B copy
SM - S to Memory copy
SA - S to A copy
PL - Plus

In detail, the above sequence of action in the second stage takes place as follows:

1) clear: [Fig. 16(b)] momentarily operate relays CLA ("clear A"), CLB ("clear B") and CLS ("clear S"). These three shunt contacts cause the A, B and S relays to be connectdd to the minus terminal, thereby setting them to 0. (completed paths are shown heavy)


A2 DIGIT OF A REGISTER
ma
cℓa
sa
s2
a2
B2 DIGIT OF B REGISTER
mb
cℓb
b2
2ND COPY OF 4 CELL, 1 BIT MEMORY
y2
x2
CONTACTS TO COPY S2 IN MEMORY
s̄2
s2
sm
S2 DIGIT OF S REGISTER
pℓ
a2
b2
c2
cℓs
ā2
b̄2
b̄2
c̄2
s2
b2
DIGIT 2
a2
b2
c2
a2
c2
c1
CARRY DIGIT
+
-

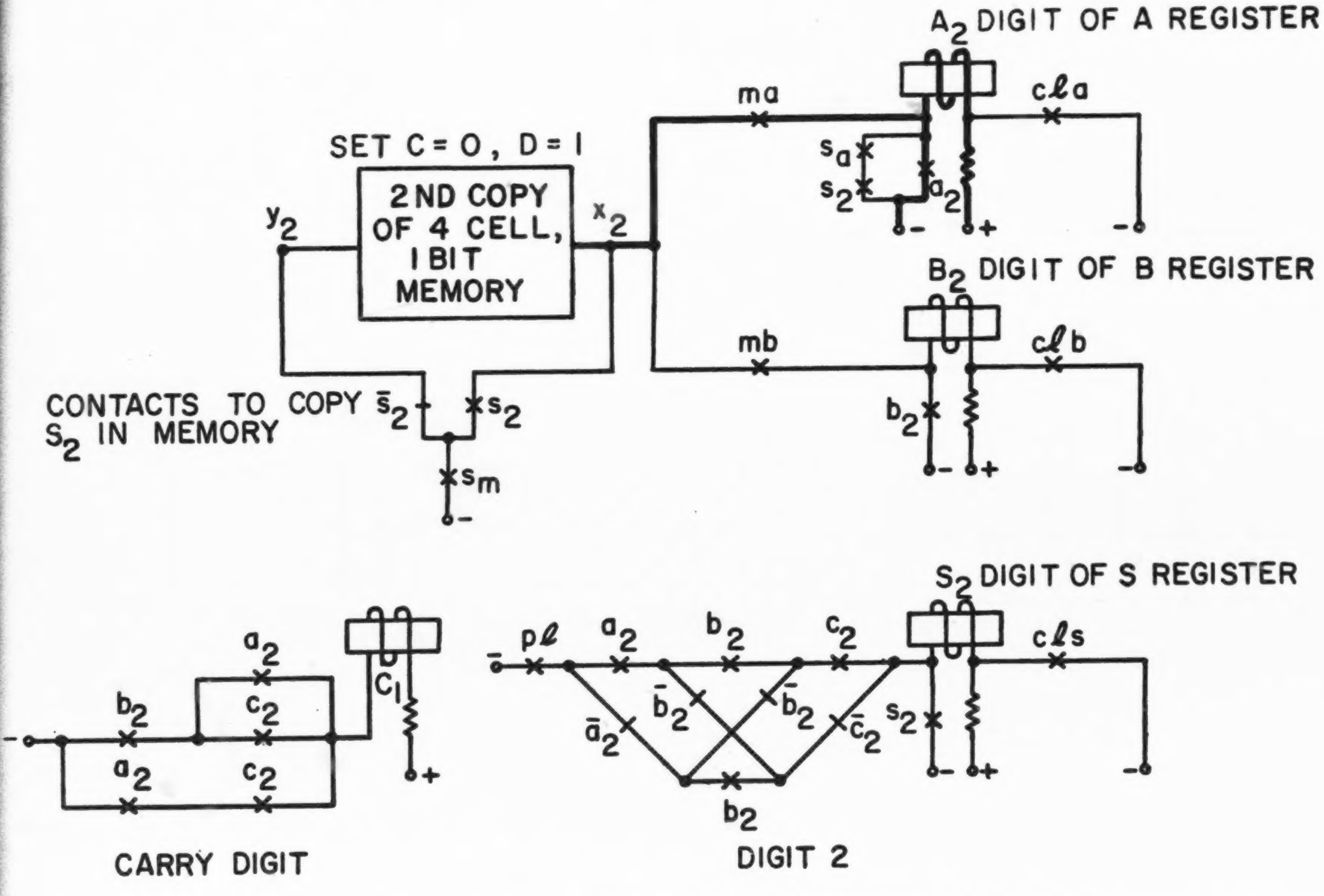
A, B, and S registers, and Communication with Memory

CLA - Clear A
CLB - Clear B
CLS - Clear S
MA - Memory to A copy
MB - Memory to B copy
SM - S to Memory copy
SA - S to A copy
PL - Plus

Fig. 16(b) Clear Registers.

2) select: [Fig. 16(c)] in the selection tree of Fig. 12, release relay C, and operate relay D, to connect the root of the tree (node X_2) to cell 01 (Q_S, Q_1, ... Q_{12}, where Q_S contains the sign bit, and Q_{12} the least significant digit).

3) copy: [Fig. 16(c)] momentarily operate relay MA ("memory to A") in Fig. 15. The make side of relay A_2 is now connected directly to X_2 and hence to the second digit relay of cell 01 (on circuit 2). If relay Q_2 were in an operated state, its make side would be at minus, as would be, therefore, node X_2 and the make side of relay A_2, thus causing A_2 to be operated; if, however, Q_2 were released, its make side would be at plus, as would be X_2 and A_2's make side, thus causing A_2 to remain released. Hence the two relays become identical in state, and the digit is copied. (a 1-bit in this example)


A_2 DIGIT OF A REGISTER
ma
cla
s_a
s_2
a_2
SET C = 0, D = 1
2ND COPY OF 4 CELL, 1 BIT MEMORY
y_2
x_2
B_2 DIGIT OF B REGISTER
mb
clb
b_2
CONTACTS TO COPY S_2 IN MEMORY
$\bar{s}_2$
s_2
s_m
S_2 DIGIT OF S REGISTER
cls
s_2
pl
a_2
b_2
c_2
$\bar{a}_2$
$\bar{b}_2$
$\bar{c}_2$
C_1
CARRY DIGIT
DIGIT 2

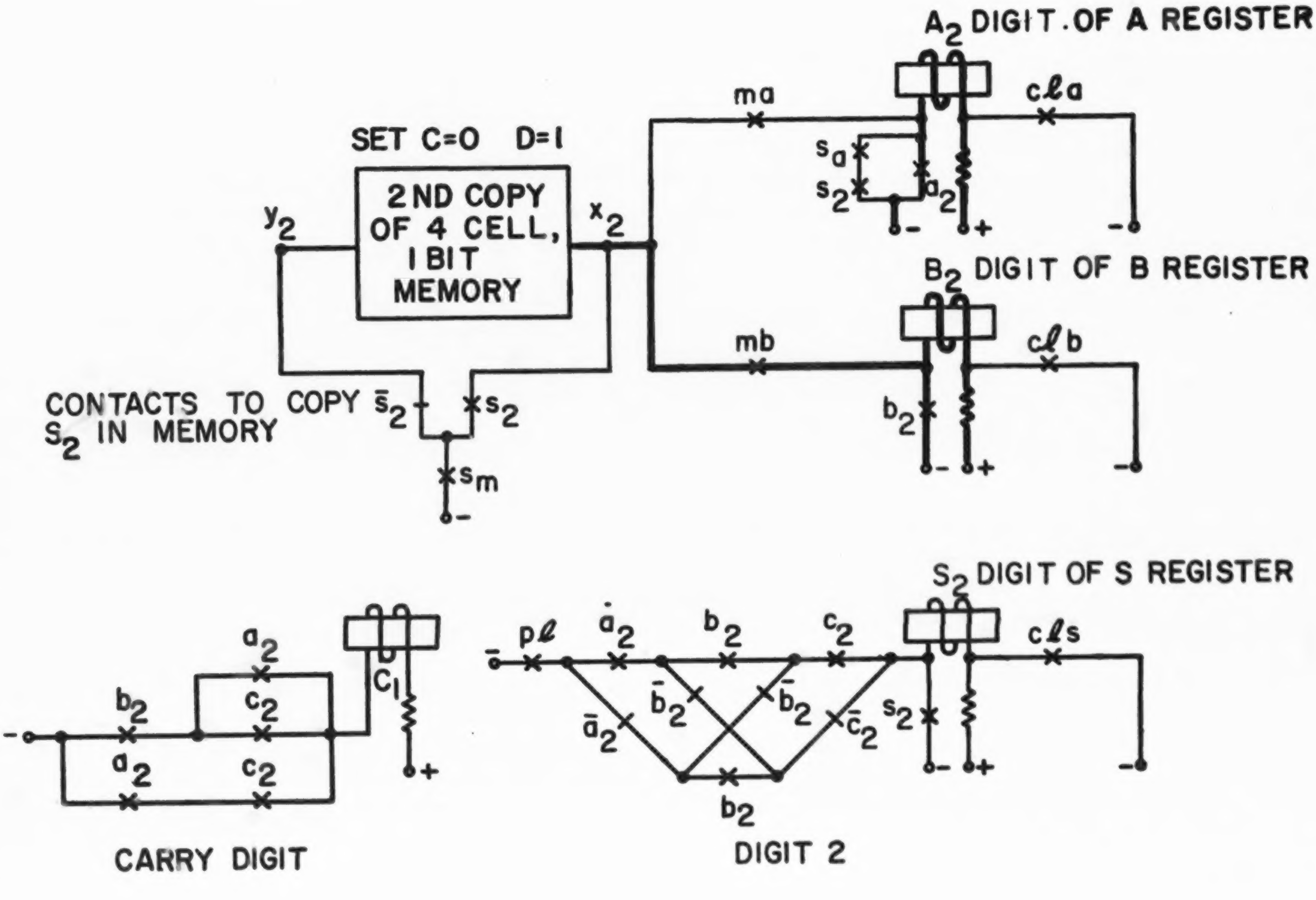
A, B, and S registers, and Communication with Memory

CLA - Clear A
CLB - Clear B
CLS - Clear S
MA - Memory to A copy
MB - Memory to B copy
SM - S to Memory copy
SA - S to A copy
PL - Plus

Fig. 16(c) Select and Copy cell 01 into A (Assume A_2 = 1).

4) select: [Fig. 16(d)] operate C and release D to connect the second digit relay in cell 10 to X_2.

5) copy: [Fig. 16(d)] operate MB ("memory to B") to set the R_2 and the B_2 relay to the same state, thus copying the digit. (Note that A_2 remains set)


A2 DIGIT.OF A REGISTER
ma
cℓa
sa
s2
a2
SET C=0 D=1
2ND COPY OF 4 CELL, 1 BIT MEMORY
y2
x2
B2 DIGIT OF B REGISTER
mb
cℓb
b2
CONTACTS TO COPY S2 IN MEMORY
s̄2
s2
sm
S2 DIGIT OF S REGISTER
pℓ
a2
b2
c2
cℓs
ā2
b̄2
b̄2
c̄2
s2
b2
b2
c2
a2
c2
C1
CARRY DIGIT
DIGIT 2

A, B, and S registers, and Communication with Memory

CLA - Clear A
CLB - Clear B
CLS - Clear S
MA - Memory to A copy
MB - Memory to B copy
SM - S to Memory copy
SA - S to A copy
PL - Plus into

Fig. 16(d) Select and Copy cell 10 into B (assume $B_2 = 1$).

6) add: [Fig. 16(e)] now the A_2 and B_2 relays have been set, and similarly the entire A and B registers have been loaded; we can now add them using the sum and carry circuits we discussed before. Momentarily operate relay PL ("Plus") which initiates the action of the adder, and causes the results to be stored on the S relays. (Note that the addition at stage 2 involves the carry C_2 previously calculated in the S_3 stage not shown explicitly in the diagram. Similarly, the carry C_1, is calculated for the next stage.)

In order to add a third number to the previous two, the contents of the S register must be transferred to A or B prior to the addition. Let's load it into A. The other steps are identical to the ones above.

1) clear: momentarily operate relays CLA and CLB. This makes certain that registers A and B both have zero in them. (Register S, however, still has the sum in it.)

2) select: in the memory access tree, Fig. 12, operate relay C and relay D. This selects memory cell 11 containing the third number.

3) copy: momentarily operate relays MB and SA ("S to A"). This makes the states of the B relays correspond to the bits stored in the memory cell 11, thus copying the number. It also makes the states of the A relays the same as those of the S relays.

4) clear: momentarily operate relay CLS. This clears register S of the number which has been transferred to A in step (c).

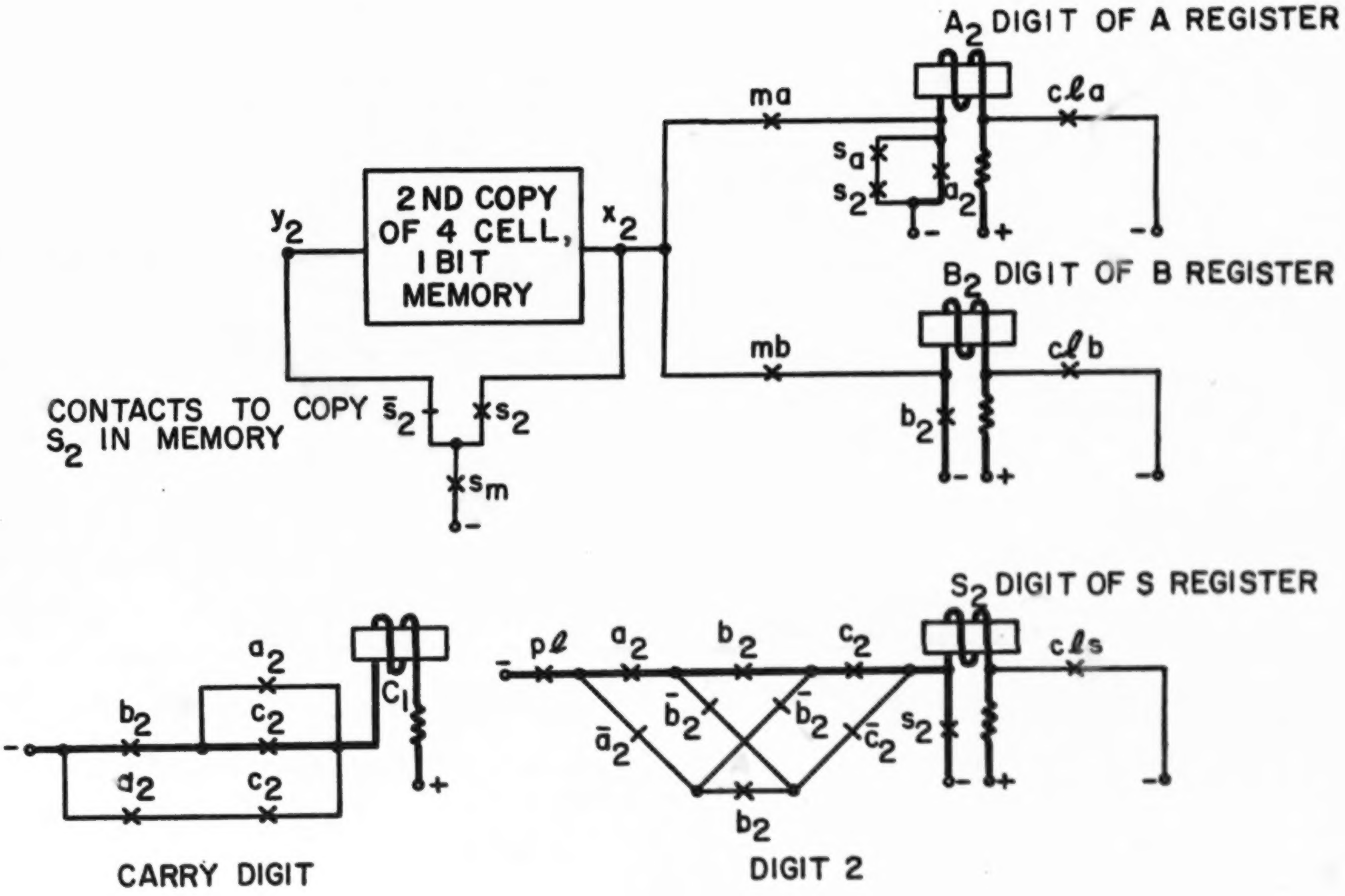
5) add: momentarily operate relay PL. This puts the sum in register S.

Suppose that we wanted to copy this sum in memory cell 00. Two steps are required:

1) select: in the memory access tree release both relays C and D to select memory cell 00.

2) copy: [Fig. 16(f)] momentarily operate relay SM ("S to memory"). This makes the state of memory cell 00 the same as the state of register S.

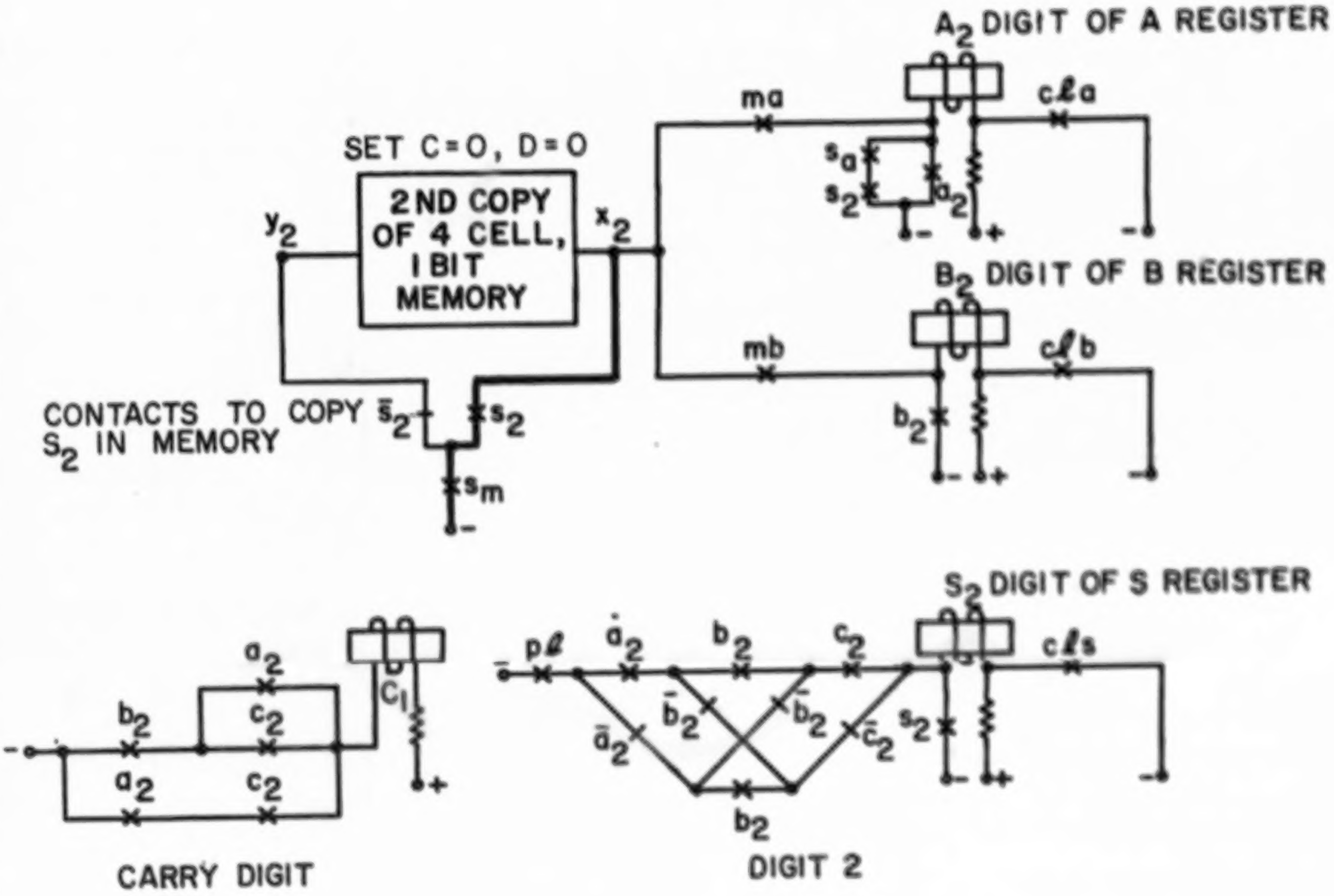
It is possible to build a multiplier circuit into the accumulator; in fact, most large digital computers have such a circuit. Multiplication can be accomplished by a combination of shifting and adding, as we saw in Section 5 of Chapter 4. While we shall incorporate shift register hardware in our computer, we shall do multiplication by "programming" repeated addition, rather than by including extra multiplier logic in the computer. Without going into detail, let us then assume that the accumulator has shift register (and subtractor) circuits in addition to the above adder circuits.


A$_2$ DIGIT OF A REGISTER
ma
cℓa
s$_a$
s$_2$
a$_2$
y$_2$
2ND COPY OF 4 CELL, 1 BIT MEMORY
x$_2$
B$_2$ DIGIT OF B REGISTER
mb
cℓb
b$_2$
CONTACTS TO COPY S$_2$ IN MEMORY
$\bar{s}_2$
s$_2$
s$_m$
S$_2$ DIGIT OF S REGISTER
pℓ
a$_2$
b$_2$
c$_2$
$\bar{a}_2$
$\bar{b}_2$
$\bar{b}_2$
$\bar{c}_2$
s$_2$
cℓs
b$_2$
C$_1$
a$_2$
b$_2$
c$_2$
a$_2$
c$_2$
CARRY DIGIT
DIGIT 2

A, B, and S registers, and Communication with Memory

CLA - Clear A
CLB - Clear B
CLS - Clear S
MA - Memory to A copy
MB - Memory to B copy
SM - S to Memory copy
SA - S to A copy
PL - Plus

Fig. 5.16(e) Add A ar 3 in S (Assume $A_2 = 1$, $B_2 = 1$, $C_2 = 1$)


A_2 DIGIT OF A REGISTER
ma
cℓa
SET C=0, D=0
2ND COPY OF 4 CELL, 1 BIT MEMORY
y_2
x_2
s_a
s_2
a_2
B_2 DIGIT OF B REGISTER
mb
cℓb
b_2
CONTACTS TO COPY S_2 IN MEMORY
$\bar{s}_2$
s_2
s_m
S_2 DIGIT OF S REGISTER
pℓ
$\bar{a}_2$
b_2
c_2
cℓs
$\bar{b}_2$
$\bar{c}_2$
s_2
a_2
c_2
C_1
CARRY DIGIT
DIGIT 2

A, B, and S registers, and Communications with Memory

CLA - Clear A
CLB - Clear B
CLS - Clear S
MA - Memory to A copy
MB - Memory to B copy
SM - S to Memory copy
SA - S to A copy
PL - Plus

Fig. 5.16(f) Select and Copy S into cell 00.

Let us now add the accumulator to our block diagram. The result is shown in Fig. 17. All the relay contacts represented in Fig. 16, except those of MB, MA and SM are included in the block labeled "Accumulator". Contacts of MB, MA and SM are part of the "Register Selector" block, since they are used in moving numbers between the accumulator and the memory.*

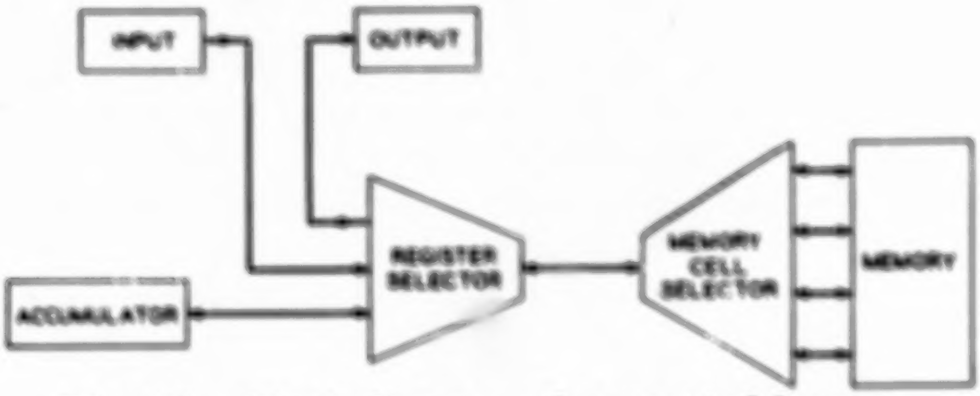


Fig. 17 - Block diagram of connected input, output, accumulator, and memory.

7. INSTRUCTIONS

At this point let us pause for breath, stand back and see how far we have progressed with out plans for a computer. We have gained a good idea of how numbers can be put into the machine, how we do arithmetic with them and how we read them out.

We must now decide how instructions are to be presented to the computer. One method would require taking the English text used as instructions for Steadfast and converting this to Morse Code. This code could certainly be put into a machine by having dot stored, for instance, as 0, and dash as 1. This, however, would be poor practice. Not only is it needlessly bulky (it takes 150 dots and dashes for instruction 3 in Steadfast's list), but there are thousands** (if not millions) of ways of giving instructions in English. Steadfast can understand and obey any one of these variations, but to have the computer do so is not practical. We require a unique way of expressing a given instruction.

To accomplish this, we prepare a list of useful and required instructions which we use consistently to command the machine, and we assign a unique binary number to each of these instructions. This method provides a useful abbreviation of some fairly complex commands.

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*Note once again the role of contacts MA, MB, SA, SM, and PL in the accumulator circuit; they are selection contacts which do not store information but allow connections between circuits to be made at designated times. They are similar to the i and o contacts on 4-cell memory circuit and effectively address the appropriate relay at the appropriate time. (Note that S_A and S_2 play a very different role with respect to relay A_2: S_A is an addressing contact whereas S_2 is the information storing contact).

**Here are 1024 ways. By making the 10 choices in all possible ways, you get 2^{10} different sentences. $\begin{pmatrix}\text{Clear}\\ \text{Reset}\end{pmatrix}$ $\begin{pmatrix}\text{your}\\ \text{the}\end{pmatrix}$ $\begin{pmatrix}\text{desk}\\ \text{manual}\end{pmatrix}$ $\begin{pmatrix}\text{calculator}\\ \text{computer}\end{pmatrix}$ and $\begin{pmatrix}\text{put}\\ \text{enter}\end{pmatrix}$ the $\begin{pmatrix}\text{material cost}\\ \text{cost of material}\end{pmatrix}$ $\begin{pmatrix}\text{of}\\ \text{for}\end{pmatrix}$ $\begin{pmatrix}\text{this}\\ \text{the present}\end{pmatrix}$ $\begin{pmatrix}\text{item}\\ \text{part}\end{pmatrix}$ $\begin{pmatrix}\text{into}\\ \text{in}\end{pmatrix}$ it.

Referring to the block diagram of Fig. 17 and recalling our discussion of the circuits, the following list of operations has been encountered so far:

a) Read: copy the contents of a punched card into a memory cell.

b) Punch: copy a card from a memory cell.

c) Clear and Add: Clear the accumulator (by reading 0-bits into the A, B, and S registers). Then copy a number from its address in the memory into the A register in the accumulator. Finally transfer the contents of the A register via the "add" circuits into register S so as to copy the number from memory into the accumulator (add A and 0 into S).

d) Add: Without clearing the Accumulator, add a number to it from a memory cell. *

e) Subtract: Subtract from the Accumulator a number taken from a memory cell.

f) Store: Put a copy of the Accumulator into a memory cell.

g) Shift: Shift a number in the Accumulator left Y places (or right Z places), putting 0's in the vacated digit positions.

Note that all but the last instruction refer to specific memory cells. We shall therefore assign binary numbers to machine instructions which consist of two parts. One part of the binary instruction, called the operation code ("read", "punch", or "clear and add", etc.), indicates the operation to be performed. The second part indicates the address of the cell in the memory to which we refer. The operation of shifting requires no reference to a memory location, so that we can use the address portion of the instruction to indicate, instead, the number of places the digits should be shifted and the direction in which the shift should take place. If we have 16 or fewer operations, four binary digits (bits) are enough to specify uniquely a particular operation: 0000 for the first operation (0 on our list), 0001 for the second operation, etc.

If we represent our instructions as 13-bit numbers, 9 bits will be available for for addressing, so we could have 512 memory cells, numbered 000000000 to 111111111 consecutively. These sixteen operations and 512 memory cells are more than adequate for our purpose. Thus 0010000000101,

$$\underbrace{\overbrace{0010}^{=2}}_{\text{operation code}}\,\underbrace{\overbrace{000000101}^{=5}}_{\text{address}}$$

a 13-bit number, might symbolize the machine instruction "without clearing the accumulator, add the contents of memory cell #5 to it".

**Note that we have not included the instruction which adds two numbers from memory: our computer instructions will be standardized to involve at most one number, as you will see shortly. Thus to add two numbers, you "clear and add" the first, then "add" the second to it.

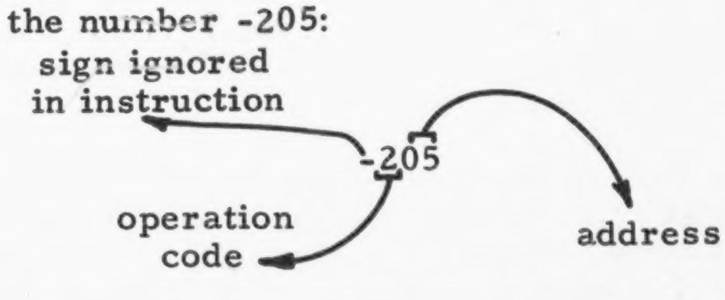
The representation of instructions with unique binary numbers offers an extremely important additional advantage. Represented as binary numbers, instructions can be stored in the machine's memory and, from this memory, these instructions may be withdrawn in a sequential fashion to guide the operation of the entire computer. All that is needed is to store the instructions in a block of consecutively numbered memory cells and to design the machine to read its instructions one after another, and to execute the commands in this sequence. In other words, the computer is designed to store not only the data which are involved in a computation, but also the sequence of instructions necessary to carry out the computation. Note that there is no *a priori* way of distinguishing ordinary 13-bit numbers used as data, from 13-bit numbers to be used by the machine as its instructions. In section 9 we see how this difficulty is resolved.

8. DECIMAL VERSUS BINARY CIRCUITS

Before we continue to discuss the problem of executing instructions, we must make a simplifying assumption about the fundamental nature of our computer. Until now our logic and our circuits have been binary-valued. It is possible, however, to build a circuit components which can be in one of ten states, say, rather than in one of two. As a matter of fact, the world's first electronic computer, the ENIAC built in 1944, was a decimal machine in that its data, instructions and circuits were all founded on base 10 rather than on base 2.

While we won't discuss such circuits here*, we note that their operation is entirely similar to that of binary circuits - compare, for instance, the action of a binary adder and a decimal adder. The individual circuit which represents a single decimal digit, since it must be 10-valued, is clearly more complex than the binary relay (it may be a group of relays or a special device such as a 10-position rotary switch. Now notice that the decimal representation of a given number is much shorter than the binary one: it takes only three decimal digits to express most 10-bit binary numbers (1111100111 being the largest such binary number). The number of digit positions required to represent a given numeral in decimal memory cells or registers is thus considerably smaller than in the binary machine, at the cost of increased complexity for each digit position. Nonetheless, it is easier to discuss numbers in a familiar number system using only 3 digits than in an unfamiliar one using 10 or 13 digits in base 2, and we therefore avail ourselves of this notational convenience hereafter. If at any time you wonder what is really happening in the decimal computer, just remember the binary machine we set out to construct, and reason by analogy - the operating principles are identical.

Thus we turn to our decimal computer, whose cells for convenience have been picked to be three-digits-plus-sign long. An instruction, then, consists of a sign plus a 1-digit operation-code part, and a 2-digit address part:



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*See the discussion of Section 10.

might symbolize the "add from cell #5" instruction of the previous section. Since there are two address digits in an instruction, we can specify one of 100 addresses labelled 00 through 99, each holding, of course, 3 decimal digits plus sign, data or instruction.

9. INSTRUCTION CYCLE AND CONTROL UNIT

Now that we have a way of encoding and storing (decimal) instructions in memory, let us devise a systematic method for having our computer carry them out one after another (execute them). Remembering the discussion of Section 2, we can see that the very orderly cycle of the execution of Steadfast's instructions, diagrammed in Figs. 4 and 18(a), can be redrawn in computer terminology as in Fig. 18(b). For instance, in Step II, if we assume for the moment that the instructions are numbered, then clearly a counter can replace Steadfast's finger as a way of keeping our place in the list of instructions. In Step IV Steadfast checks to see that in executing the current instruction he was (or was not yet) finished (i.e., was it instruction 8, and, if so, did he jump back to 3, or did he decide to halt?) If he decides to halt in Step IV, he will finish the job by putting pencil and paper away, clearing the calculator for the next person's use, and similar housekeeping chores. Our computer must have a definite piece of hardware for just this function, called the "run-stop switch". It is set to "stop" by a special "test and halt" machine instruction similar to Steadfast's #8.

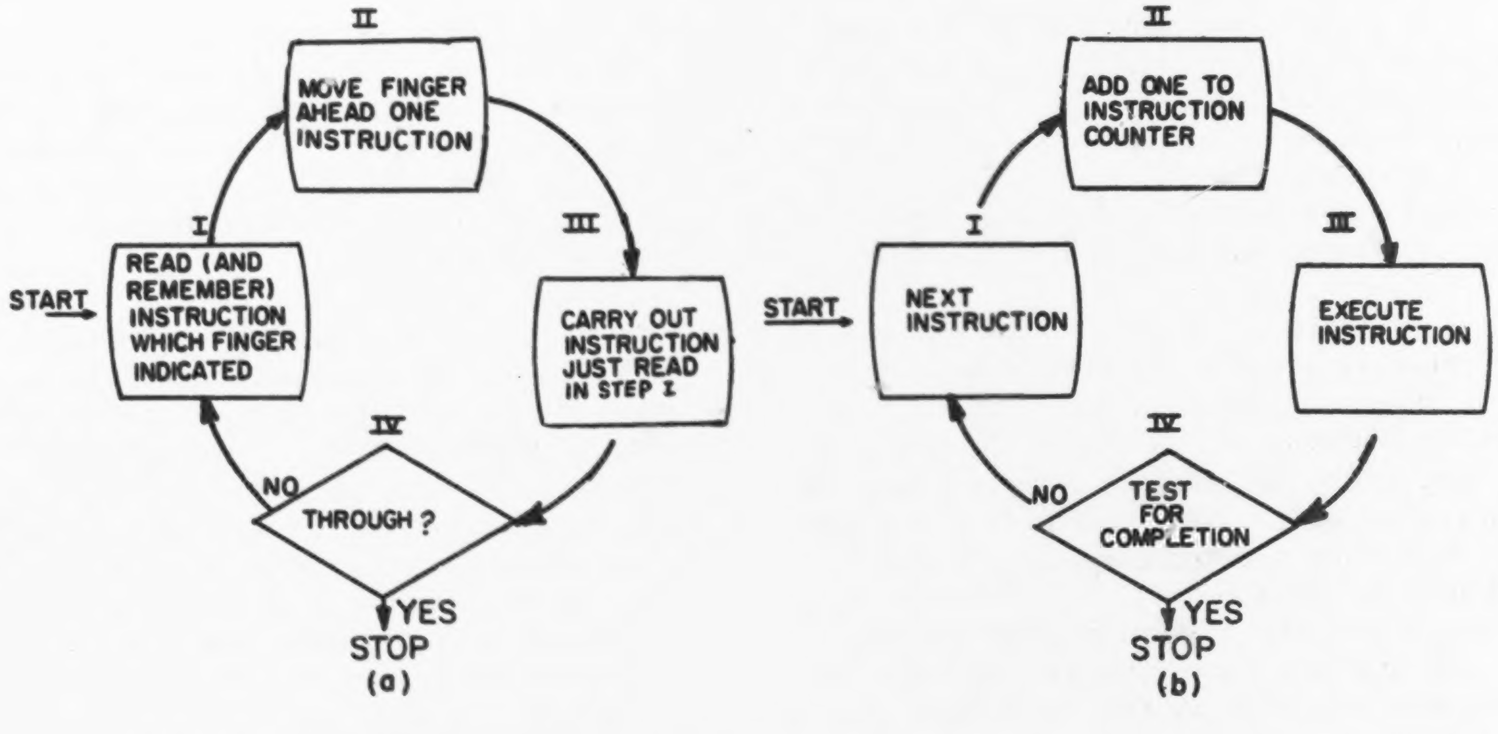
II
MOVE FINGER AHEAD ONE INSTRUCTION
I
READ (AND REMEMBER) INSTRUCTION WHICH FINGER INDICATED
START
III
CARRY OUT INSTRUCTION JUST READ IN STEP I
IV
THROUGH ?
NO
YES
STOP
(a)
II
ADD ONE TO INSTRUCTION COUNTER
I
NEXT INSTRUCTION
START
III
EXECUTE INSTRUCTION
IV
TEST FOR COMPLETION
NO
YES
STOP
(b)

Fig. 18 Flow chart for instruction cycle of
(a) Steadfast and (b) computer

Let us now look at the "instruction cycle" of 18(b) in some detail. First of all, if we are to make the cycling automatic, we have to have an "instruction cycle" logic circuit which will go through steps I, II, III and IV of the cycle in repetitious clocklike fashion. In addition to this circuit, we need the "instruction decoding" logic circuit mentioned in Section 7, Chapter 4 (the Morse Coder) whose function is the decoding and actual execution of the instruction itself. Be sure to distinguish between stepping through the entire instruction cycle (steps I through IV), and "executing" the current instruction only, as symbolized in Fig. 19. In other words, as we saw in tracing Steadfast's actions, there is a lot more to "executing" a list of instructions than just doing them (Step III) - we also need to do the "bookkeeping" (Steps I, II and IV) before and after each instruction, and to set up for the next instruction (if any). Thus we have two new circuits - the instruction cycler, and the instruction decoder, the latter operating under control of the former at the appropriate time.

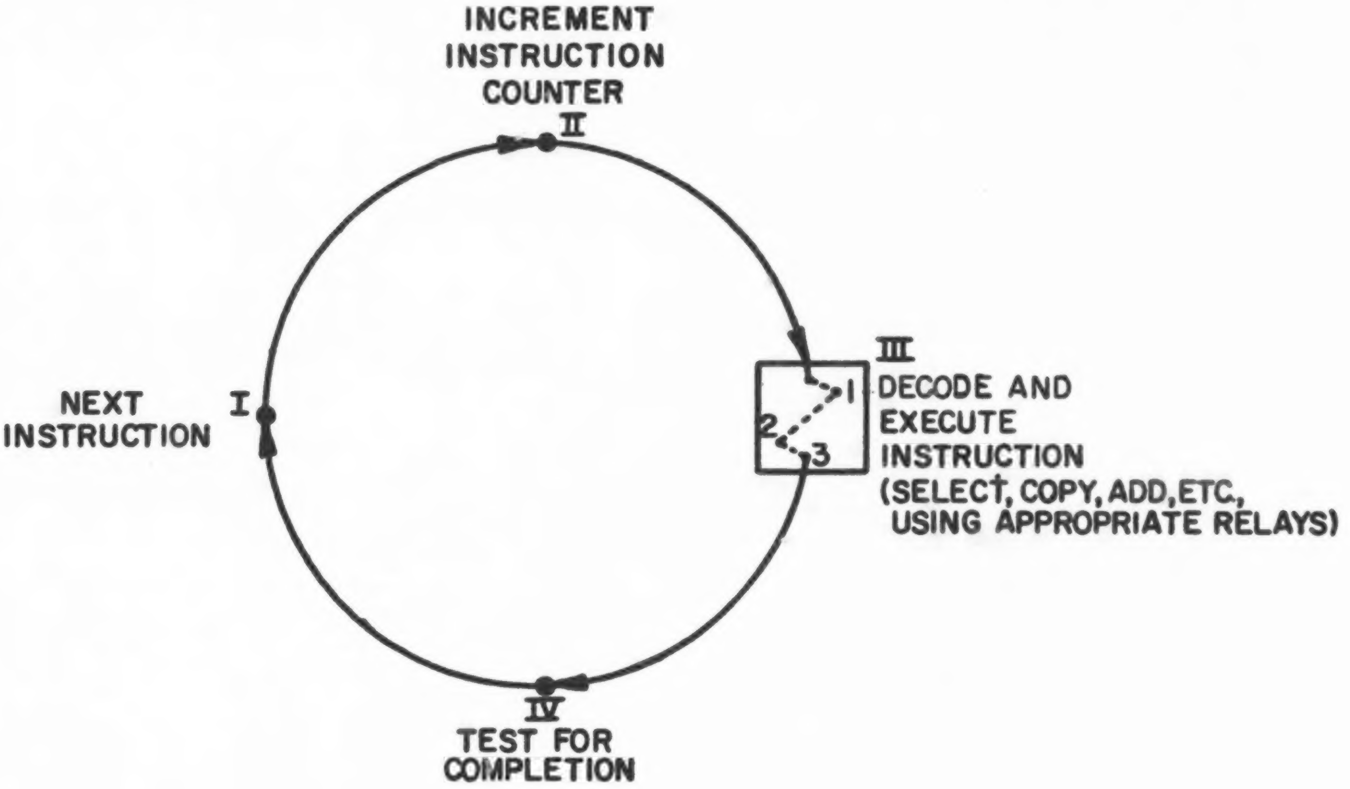
INCREMENT INSTRUCTION COUNTER
II
III
1
2
3
DECODE AND EXECUTE INSTRUCTION (SELECT, COPY, ADD, ETC., USING APPROPRIATE RELAYS)
IV
TEST FOR COMPLETION
NEXT INSTRUCTION
I

Fig. 19 Instruction Decoding Sequence within the Instruction Cycle Sequence: "the play within a play"

Since both of these circuits "control" (one in the larger sense of controlling the entire cycle, and one in the smaller sense of controlling the execution of the instruction itself), it is customary to build these two circuits into the same physical unit, called the CONTROL UNIT.

In a sense you might say that the Control Unit is the heart of our computer (or of any other) since it sets the stage for the execution of the instructions and supervises this execution at the appointed time. Figure 20 shows the completed computer. The two selectors and devices connected to them are as in Fig. 17. We have added the Control Unit in a central position to indicate its importance. Also, there are two new registers, the Instruction Counter (IC) and Instruction Register (IR) to help with controlling the instruction cycle. The Instruction Counter "remembers which instruction comes next" (corresponding to Steadfast's index finger). It is essentially a copy of the circuit shown in Fig. 12 in Chapter 4. This circuit can initially be set to a given instruction number and then used as a counter to keep track of the consecutive instructions executed. The Instruction Register is simply a set of relays whose job is to "read and remember this instruction" up to the time (and while) it is being executed, in steps I, II and III.

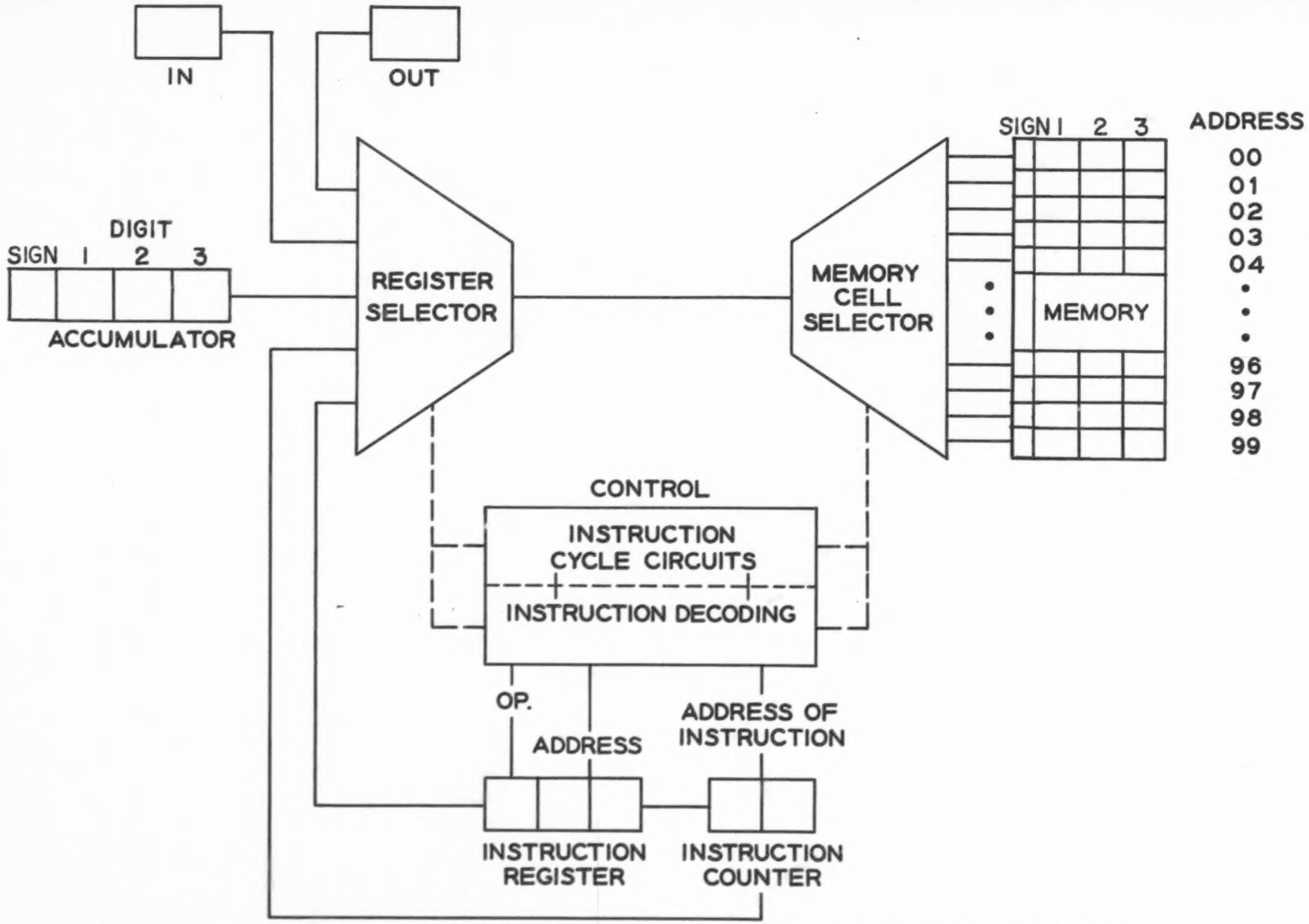
IN
OUT
DIGIT
SIGN
1
2
3
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
SIGN 1 2 3
ADDRESS
00
01
02
03
04
MEMORY
96
97
98
99
CONTROL
INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 20. Block diagram of a 3-decimal-digit computer, with a 1-digit operation code (OP.), 2-digit address, one hundred 3-digit-plus-sign memory cells (addresses 00 through 99). (Broken lines mean control of information flow; solid lines mean flow of information.)

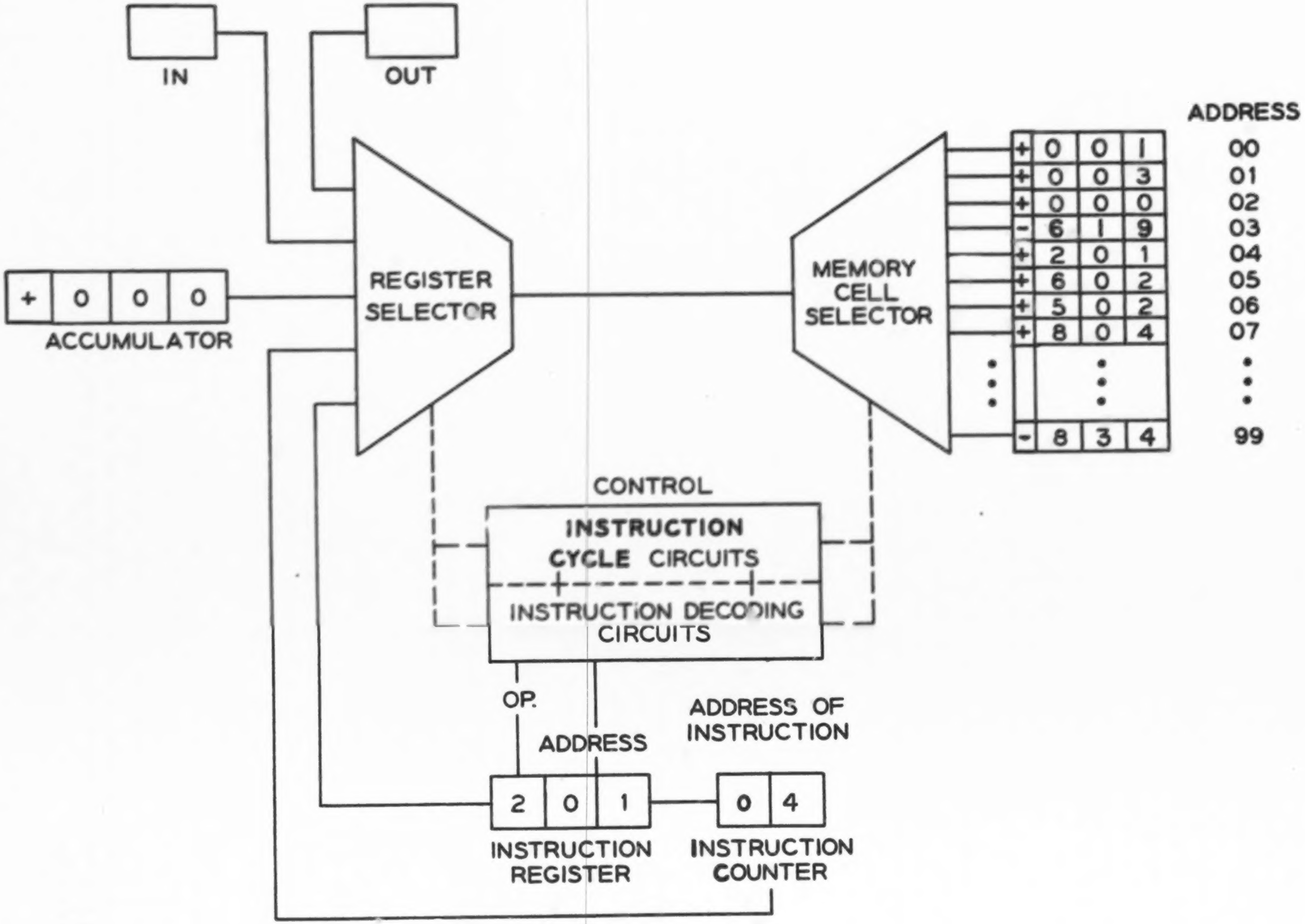
IN
OUT
+ 0 0 0
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
ADDRESS
+ 0 0 1 00
+ 0 0 3 01
+ 0 0 0 02
- 6 1 9 03
+ 2 0 1 04
+ 6 0 2 05
+ 5 0 2 06
+ 8 0 4 07
- 8 3 4 99
CONTROL
INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING CIRCUITS
OP.
ADDRESS
ADDRESS OF INSTRUCTION
2 0 1
INSTRUCTION REGISTER
0 4
INSTRUCTION COUNTER

Fig. 21 Instruction Counter and Instruction Register

Just as we saw in the previous section that an instruction has two parts, the operation code and the address portion, so we now think of this register as having two parts, a one-digit operation code part and a two-digit address part.

The Instruction Counter holds the two-digit address of the instruction. All this is symbolized in Fig. 20, which shows the instruction from address 04 in the Instruction Register. Notice that 201 is regarded by Control as being composed of an operation code, 2, and an address (of data), 01 in this case. Be sure to distinguish between the address of an instruction (i. e., its location within the the block of instructions in memory, cell 04 in this case) "pointed out" by the Instruction Counter, and the address portion of the instruction (i. e., where in memory the instruction's data are to be found or to be stored, (cell 01 in this case) pointed out by the last two digits of the Instruction Register.

In order to understand how the two Control circuits (Instruction Cycle and Instruction Decoding) function together to execute instructions, it is useful to see in some detail what has to happen. We cannot show you how to "build" these two circuits as we did for our memory cell selector and memory, or our accumulator, since they have far too many components. Their operating principles, however, are remarkably similar to that of the Morse Code Transmitter of section 7, Chapter 4. You may remember that the simultaneous setting of three switches (see Fig. 14, Chapter 4), produced a timed sequence of actions (lighting of dot or dash lamps in this case), one particular sequence for each individual setting of the three switches. Both Control circuits behave in a way similar to the Morse Code Transmitter in that they too cause a timed sequence of actions (timed, as before, by a sequence of "clock pulses" from a counter). In the case of the instruction cycle circuit, this is the timed sequence of steps, I through IV. In contrast, the instruction decoding circuit actually takes charge of step III and causes, in general, a rather long sequence of small operations (clearing registers, copying numbers, adding A and B registers in the accumulator, etc.) necessary to execute the individual instruction in step III. (See again Fig. 19.) Thus you see that the "timed sequence of actions" may include something much more complicated than the momentary lighting of a bulb.

The best way in which to make these essentially simple (though seemingly complicated) instruction cycle - and decoding-sequences clear, is to take "snapshots" of our machine during its operation. We did this in words and pictures when we discussed the detailed operation of the accumulator - and we do it again for the machine as a whole.

Along with each picture there is a description of the action, as well as an explanation of the circuit details in an idented paragraph. You may skip such details or come back to them after things have become clearer. In fact, you will probably want to reread all of this current section in the light of the picture sequence.

Let us start with Fig. 22(a), an annotated version of Fig. 20. Here we show the same number, +201, in location 04, but have stored several other numbers in other locations.* Since the Instruction Counter is a circuit similar to that of

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*The problem of loading these numbers (some of which will be instructions, and some of which will be data) in the machine is postponed until Section 10, Chapter 6.

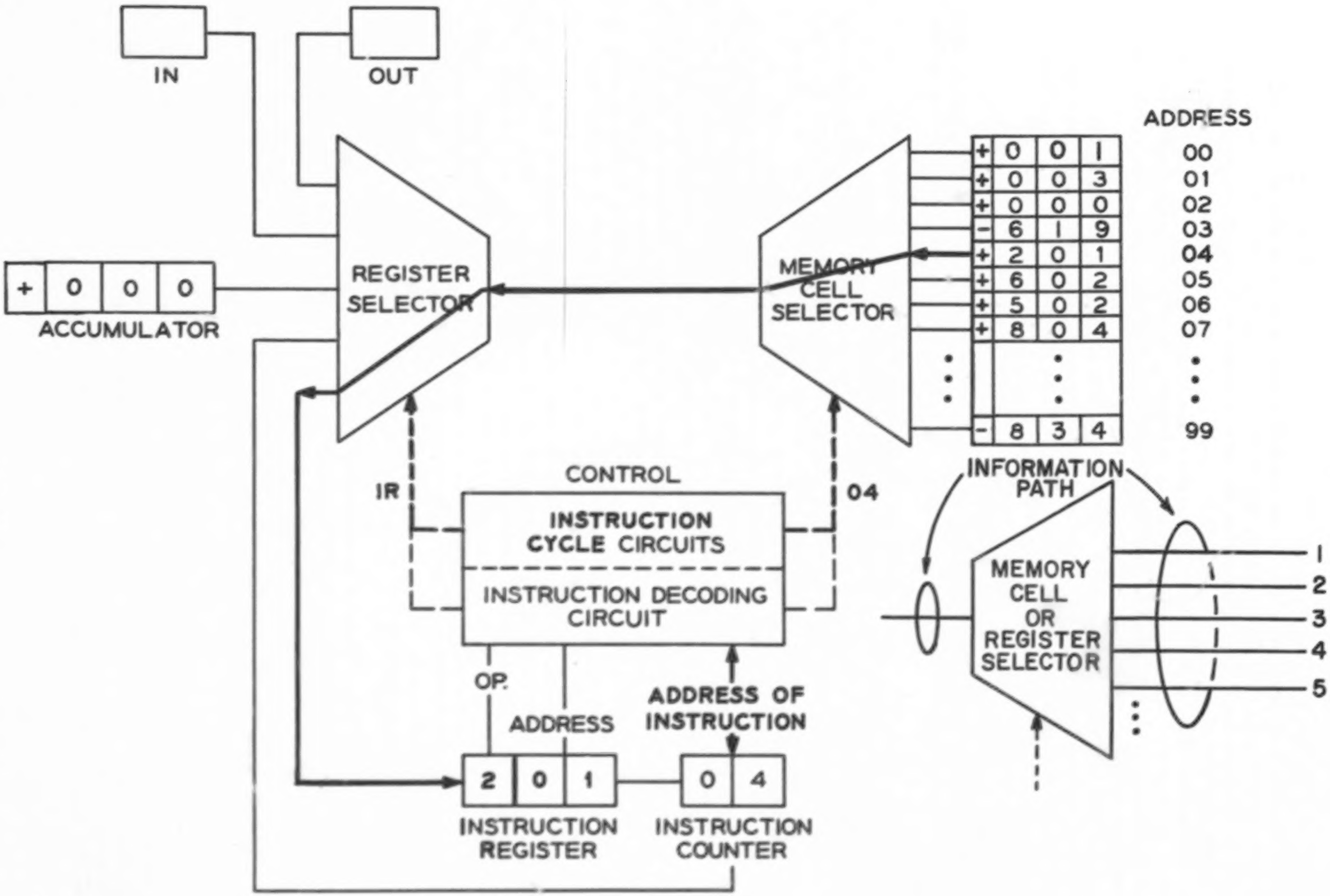
IN
OUT
+ 0 0 0
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
ADDRESS
+ 0 0 1 00
+ 0 0 3 01
+ 0 0 0 02
- 6 1 9 03
+ 2 0 1 04
+ 6 0 2 05
+ 5 0 2 06
+ 8 0 4 07
- 8 3 4 99
CONTROL
IR
04
INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING CIRCUIT
OP.
ADDRESS
ADDRESS OF INSTRUCTION
2 0 1
0 4
INSTRUCTION REGISTER
INSTRUCTION COUNTER
INFORMATION PATH
MEMORY CELL OR REGISTER SELECTOR
1
2
3
4
5

Fig. 22(a) Step I of Instruction Cycle
"Fetch Instruction from Memory"

Fig. 12, Chapter 4, we can set it to an initial value, 04 for instance, thereby indicating to Control that 04 contains the first instruction to be "fetched", i.e., brought to (copied into) the Instruction Register prior to being decoded in Step III. After the machine has finished "fetching" in Step I, we take our first snapshot. (Step I is indicated in heavy ink above the words "Instruction Cycle", also heavy.) In this and the following figures, the computer is diagrammed at the end of the indicated step. Please follow Fig. 18(b) simultaneously. Also, paths for information (solid lines) and control (dashed lines) used in each step are shown heavy, as are the affected circuits and registers and their contents. The settings of the selectors as they are set by the Control Unit are shown alongside the corresponding dashed line. (The selectors are only networks of contacts of Control Unit relays.)

Thus in Step I the memory cell selector was set to the address of the instruction to be fetched (04), and the register selector to the Instruction Register (IR) which received the contents 201. The direction of the flow is indicated by the arrows.

In detail: Assume that CONTROL has a built-in counter-driven "clock pulser" as did the Morse Coder (Lab. Experiment #6) to regulate and time all events in lockstep with these pulses. For the duration of clock-pulses 1 and 2, the computer is in step I. a) select: during both these counts the tree contacts in the memory cell selector are set to address 04, as dictated by the setting of "04" on the Instruction Counter relays. b) clear: also at clock-pulse 1, a pulse operates a "clear" relay, whose make contact shunts the Instruction Register. c) copy: at count 2 a pulse to a special MIR (Memory to Instruction Register) relay in the Control closes the MIR contact in the register selector circuit, thus causing the instruction register to be connected, relay for relay, to the selected memory cell. (What is shown as a single connecting line is in actuality a 3-line bundle for the decimal case, or a 12-line bundle for the binary case.) Thus, in a manner familiar to us from our study of the accumulator, the result of pulses 1 and 2 and the action of the Instruction Cycle Circuit is that a copy of the contents of 04 is placed in the Instruction Register. Please note that the direction of the information flow indicated by arrows is an artifice - as you know, digits don't actually travel from one place to another. Again, the states of the relays on the "receiving end" of the arrow are set to the same states as the corresponding relays on the "sending end". The matching of states takes this "direction" since the "receiving" relays are always released (cleared) first.

In step II of Fig. 18(b) it is specified that 1 be added to the Instruction Counter (IC) - i.e., that it be "incremented" to the "next" instruction's address. Fig. 22(b) shows that the contents of IC have been incremented from 04 to 05.

In detail: Since the IC is indeed a counter, it is simple for Control to increment it during clock pulse 3, in the manner described in Section 6, Chapter 4.

We have now arrived at the crucial step of the instruction cycle - the execution of the current instruction as it is contained in the Instruction Register. In a sense, the instruction cycle circuit now passes responsibility to the instruction decoding circuit, so that it may execute its timed sequence of actions. Again, just as the Morse Coder's action sequence depended upon the setting of 3 switches,

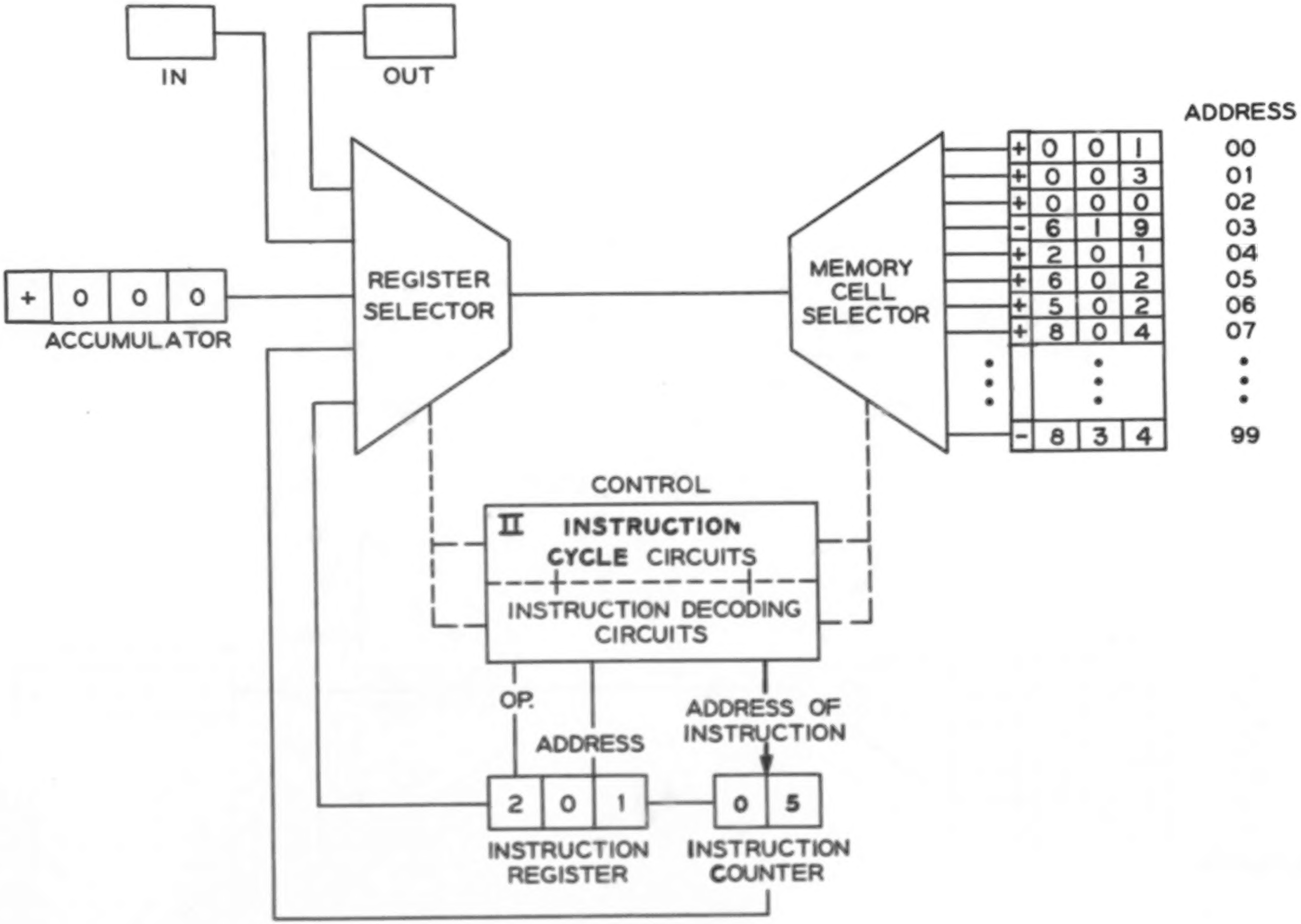
IN
OUT
+ 0 0 0
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
ADDRESS
00 + 0 0 1
01 + 0 0 3
02 + 0 0 0
03 - 6 1 9
04 + 2 0 1
05 + 6 0 2
06 + 5 0 2
07 + 8 0 4
99 - 8 3 4
CONTROL
II INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING CIRCUITS
OP.
ADDRESS
ADDRESS OF INSTRUCTION
2 0 1
INSTRUCTION REGISTER
0 5
INSTRUCTION COUNTER

Fig. 22(b) Step II of Instruction Cycle
"Increment Instruction Counter"

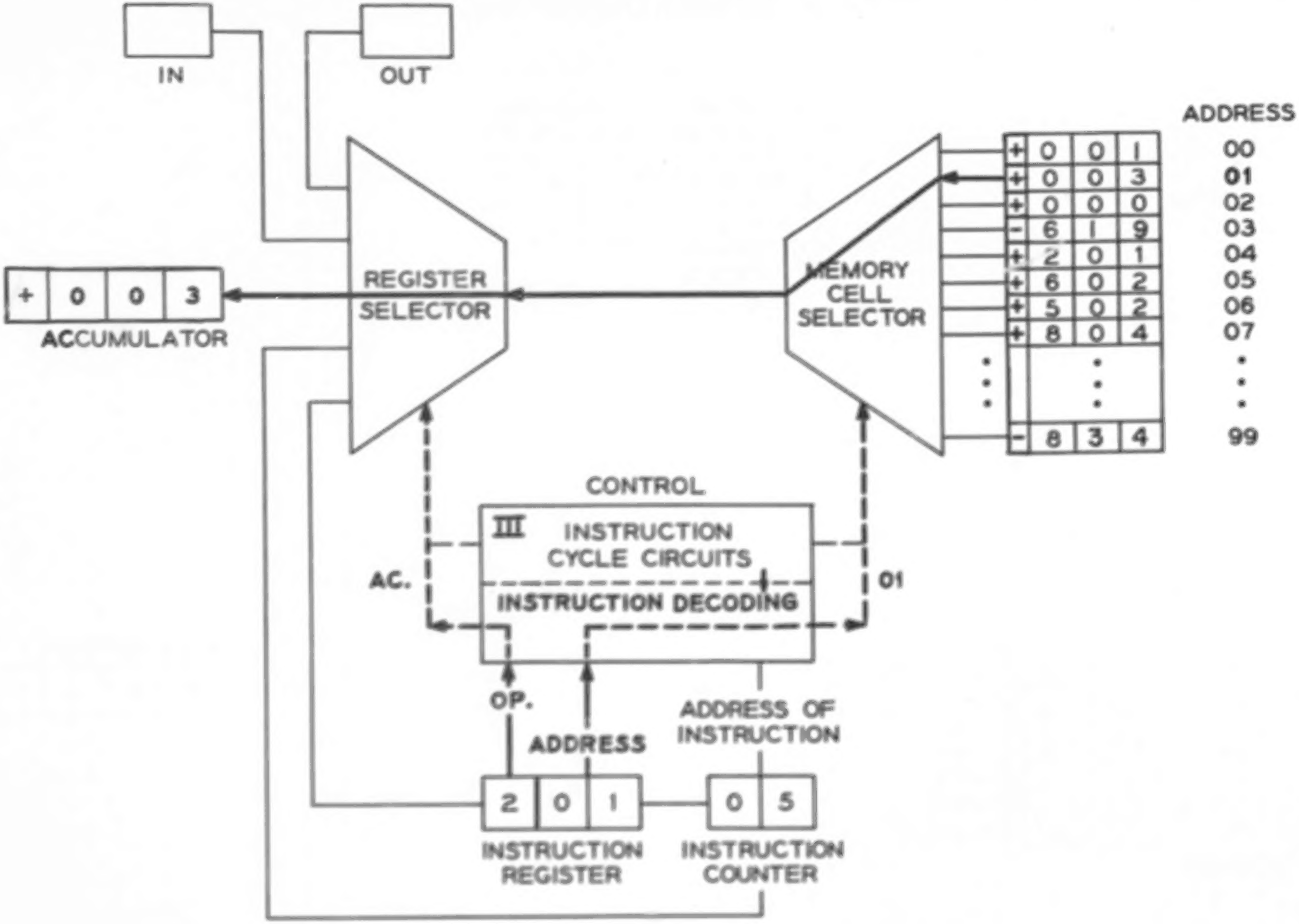
IN
OUT
ADDRESS
00
01
02
03
04
05
06
07
99
+ 0 0 1
+ 0 0 3
+ 0 0 0
- 6 1 9
+ 2 0 1
+ 6 0 2
+ 5 0 2
+ 8 0 4
- 8 3 4
+ 0 0 3
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
CONTROL
III
INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
AC.
01
OP.
ADDRESS
ADDRESS OF INSTRUCTION
2 0 1
0 5
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(c) Step III of Instruction Cycle
"Decode and Execute Instruction"

the instruction-decoding-execution depends on the setting of the IR relays. In particular, let us assume that a first digit of "2" is interpreted (by us, *and* by the computer) as the operation code (op code) for "add to the accumulator the contents of the cell specified in the last 2 digits of the instruction". The setting of 2 on the op code relays causes the register selector to be set to Accumulator (AC), while the setting of 01 on the address relays of the Instruction Register causes the memory cell selector to be set to 01. Thus a path is established from 01 to the accumulator, with the result that 000 (old AC contents) and 003 (old 01's contents) are added to accumulate the sum, 003, in the AC. Since numbers are only copied, it is clear that 01's contents are not changed by this operation.

> *In detail:* During step III, the next group of clockpulses, 4 through 7, causes appropriate auxiliary relays to operate in the card reader, punch, accumulator, or register selector, according to the setting of the op code relays. In this case, a path is selected from memory to the accumulator and addition is performed. a) *clear:* according to the discussion of Section 6 in the paragraph starting with "In order to add a third number to the sum accumulated in S, . . ." (page A-5.21), we see that on pulse 4, the CLA and CLB relays are operated, by the Instruction Decoding circuit. b) *select:* Also on pulse 4, the *memory cell selector* is set according to the address relays of the IR, 01 in this case*. c) *copy:* on pulse 5 relays MB (Memory to Register B of the accumulator) and SA are operated. Note that the setting of the register selector is not actually the first action in Step III, though we described it as such above. d) *clear:* on pulse 6 CLS is operated. e) *add:* pulse 7 concludes the instruction execution by causing the operation of relay PL to produce the sum in the S register or accumulator.
>
> Note that, strictly speaking, the clock pulses only serve to time events, not really to cause them. The settings of the op code relays determine which of the many auxiliary relays are to be operated and when.
>
> Let us now look at the operation decoding problem in more detail. The specifications for the Morse Coder can be interpreted as a list of pseudo "op codes" (letters A through H, or alternatively the equivalent XYZ switch settings 000 through 111). Depending on these settings, the circuit can cause operation (or release) of two lamps (we could equally well use relays) at clockpulse-specified times. You can see, then, that we could construct similar specifications for a list of computer op codes, 0 through 9, but with a much larger number of alternative actions (i.e., operation of more than 2 relays). The *row* for op code 2 of such a table of specifications might look like this:

---

*You may wonder how the instruction counter relays (in step II) *and* the address relays of the instruction register (in step III) can both determine the settings of the contacts of the memory cell selector tree. In reality, there is an intermediate "current address selector" set of relays (not shown) whose contacts actually form the memory cell selector tree. The setting of these current address relays is controlled in the familiar way by either the IC or IR address relays, which are therefore in direct control of the memory cell selector tree.

I. Op code	II. Binary Input Code				III. Relay Action (Output)			
	Relay 0	Relay 1	Relay 2	Relay 3	clock	pulse	number	
2	0	0	1	0	4 CLA, CLB Address 01	5 MB, SA	6 CLS	7 PL

Clock pulses (start with #4) 0 1 0 1 0 1 0 1
off on . . . .

Note: 1) instead of counting time intervals as we did in the Morse Coder, we equivalently count clock pulses - there is an "off" between every two "on"'s and vice versa; 2) "address 01" in the clock pulse 4 column stands for "operate relay in the current address selector (not shown) to set up address 01 in the contacts of the memory cell selector tree.")

The other 9 rows in the table of specifications would look similar. How would the specification table for the instruction cycle circuit look? Very similar - it would have 4 rows, one for each step, and an analogous selection of relays to be operated at specified times. In row III, it would show no action at clock pulses 4 through 7 since the instruction decoding circuit would be active then.

In step IV since the op code of step III was not "halt" (defined as 9), the Instruction cycle circuit and the clock pulser were reset to start a new cycle with step I. If the op code had been 9, then the "run-stop" switch would have been set to "stop". In step IV, the machine would have done such housekeeping as turning off the clock pulser and clearing the accumulator and instruction register; the computer would be in a "wait" state, waiting for someone to start it up again by pushing the "run-stop" switch to "run". Since we have not included a halt in this sample program, we hereafter omit the rather obvious step IV snapshot and go directly to Step I of the next cycle.

In Step I, the contents of cell 05, namely +602, were fetched as the current instruction (the sign, as usual, is ignored).

In Step II, the counter was incremented to 06, the address of the next instruction.

Once again it is time to execute the current instruction. By definition, it is whatever the contents of the Instruction Register specify. As shown in Fig. 22(g), 602 was decoded as "Store the contents of the accumulator in the memory cell specified in the address portion of the Instruction". Thus we see that the sum, +003, previously accumulated during the execution of instruction 201, is stored in cell 02.

In detail: a) select: according to the description of storing from the accumulator in Section 6, we see that on clock pulse 4, the memory selector tree is set according to the address relays to 02; b) copy: on clock pulse 5, contact SM is closed in the register selector, making the state of cell 02 the same as that of the S register in the accumulator - i.e., the number is copied. (We omit step IV.)

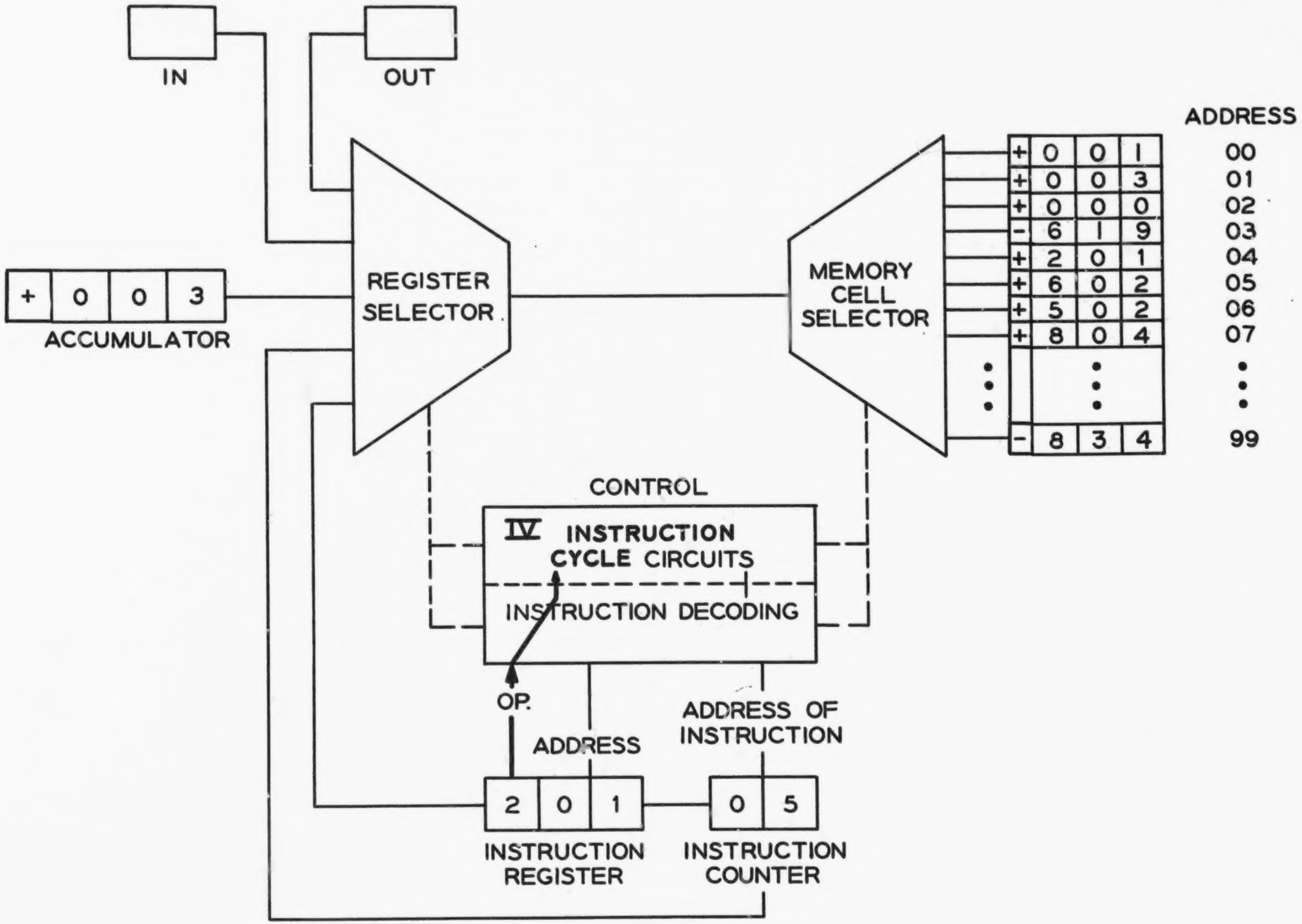
IN
OUT
+ 0 0 3
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
ADDRESS
00 + 0 0 1
01 + 0 0 3
02 + 0 0 0
03 - 6 1 9
04 + 2 0 1
05 + 6 0 2
06 + 5 0 2
07 + 8 0 4
99 - 8 3 4
CONTROL
IV INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
2 0 1
INSTRUCTION REGISTER
0 5
INSTRUCTION COUNTER

Fig. 22(d) Step IV of Instruction Cycle
"Test for Completion"

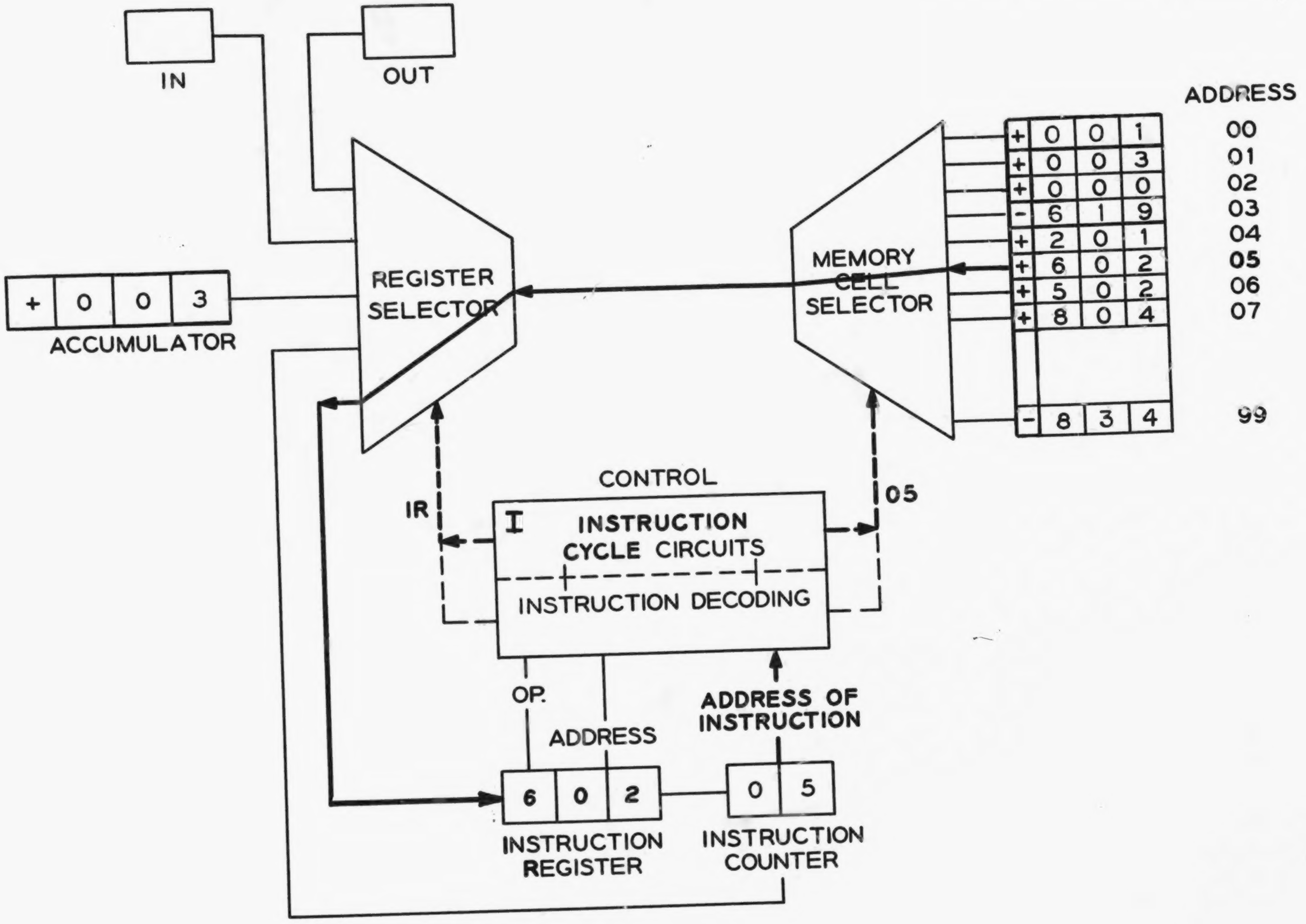
IN
OUT
ADDRESS
+ 0 0 1 — 00
+ 0 0 3 — 01
+ 0 0 0 — 02
- 6 1 9 — 03
+ 2 0 1 — 04
+ 6 0 2 — 05
+ 5 0 2 — 06
+ 8 0 4 — 07
- 8 3 4 — 99
REGISTER SELECTOR
MEMORY CELL SELECTOR
+ 0 0 3
ACCUMULATOR
CONTROL
IR
05
I
INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
6 0 2
0 5
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(e) Step I of Instruction Cycle
"Fetch Instruction from Memory"

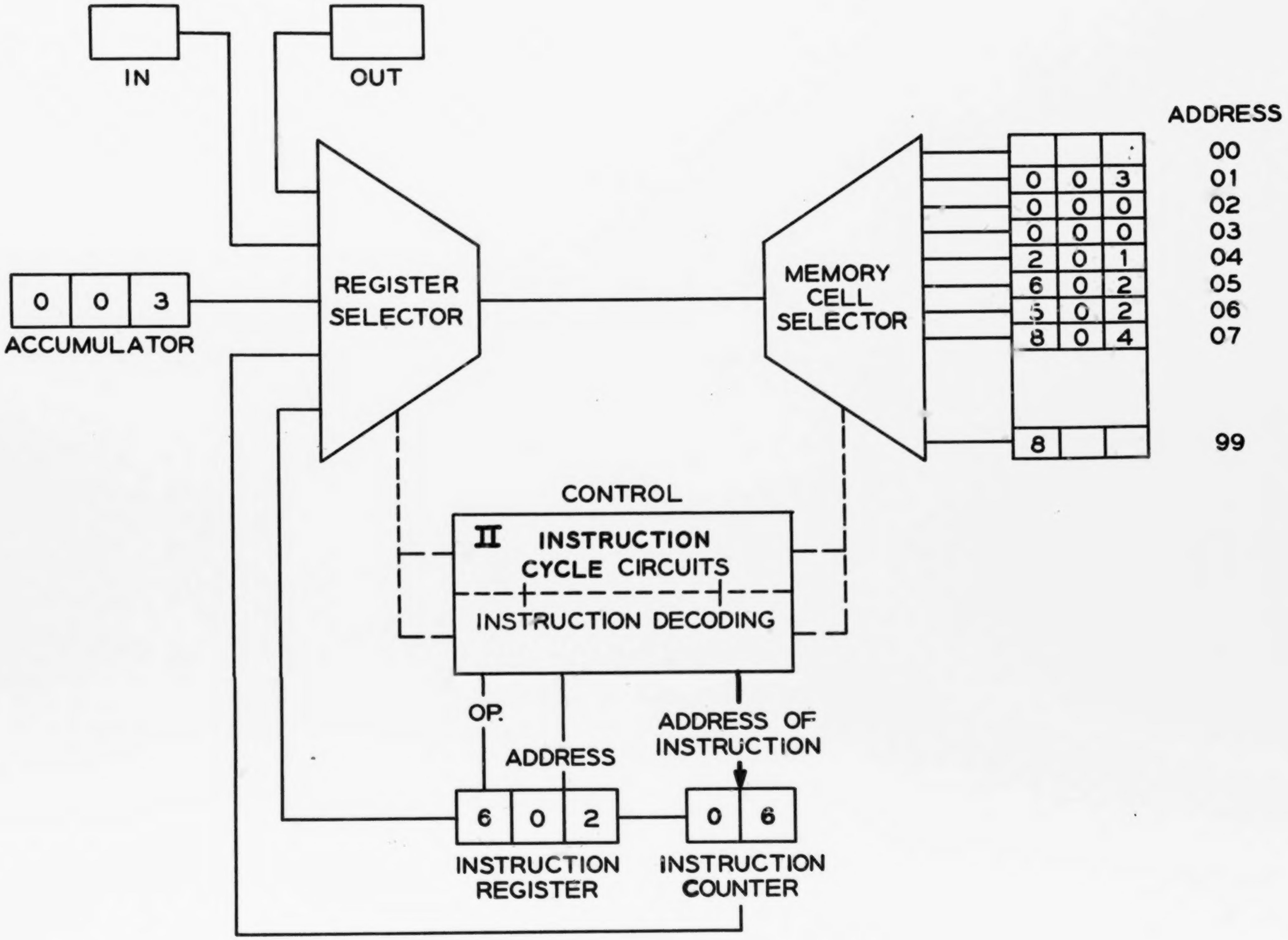
IN
OUT
ADDRESS
00
01
02
03
04
05
06
07
99
0 0 3
0 0 3
0 0 0
0 0 0
2 0 1
6 0 2
5 0 2
8 0 4
8
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
CONTROL
II INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
6 0 2
0 6
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(f) Step II of Instruction Cycle
"Increment Instruction Counter"

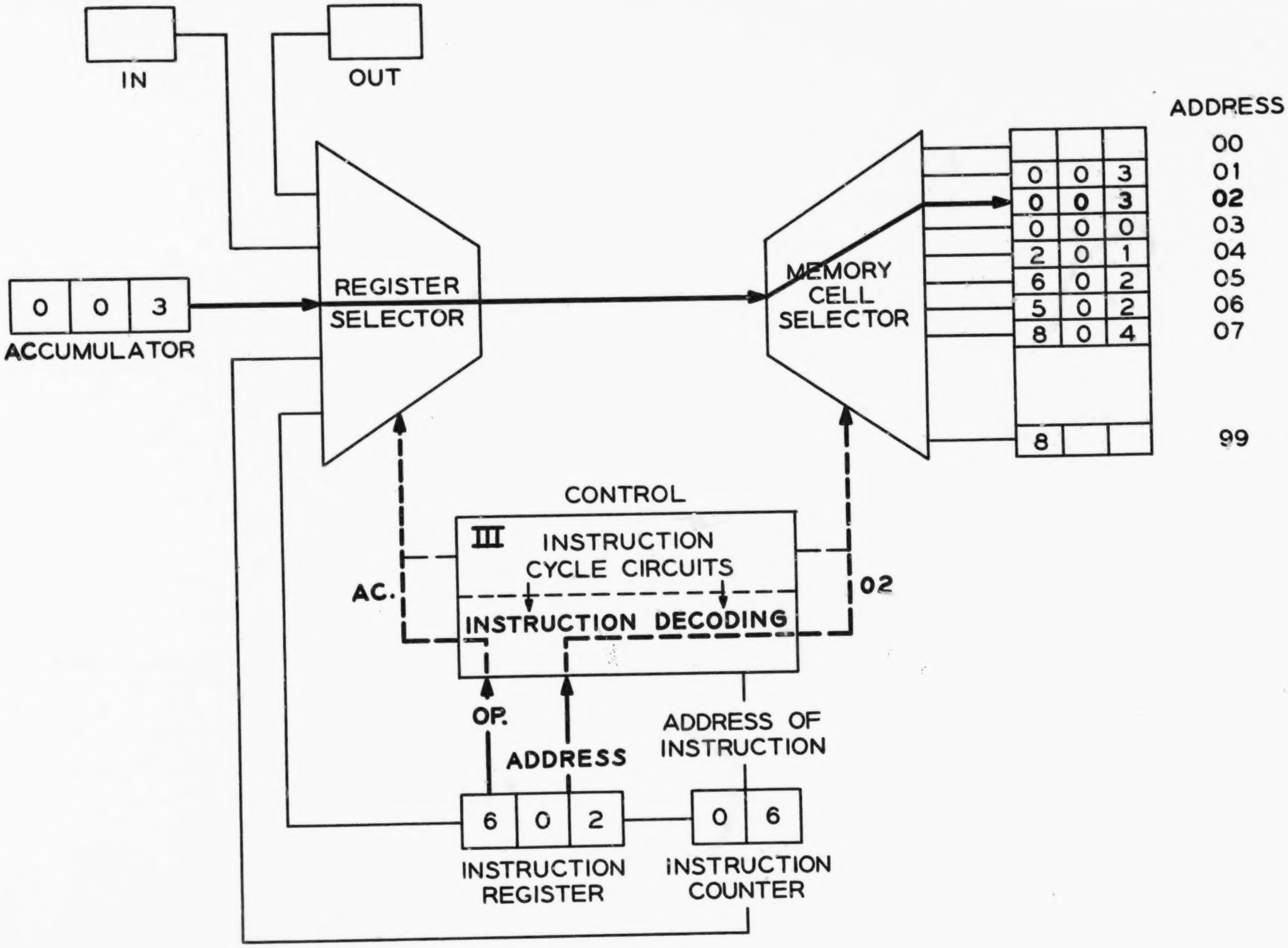
IN
OUT
ADDRESS
00
01
02
03
04
05
06
07
99
0 0 3
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
0 0 3
0 0 3
0 0 0
2 0 1
6 0 2
5 0 2
8 0 4
8
CONTROL
III INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
AC.
02
OP.
ADDRESS
ADDRESS OF INSTRUCTION
6 0 2
0 6
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(g) Step III of Instruction Cycle
"Decode and Execute Instruction"

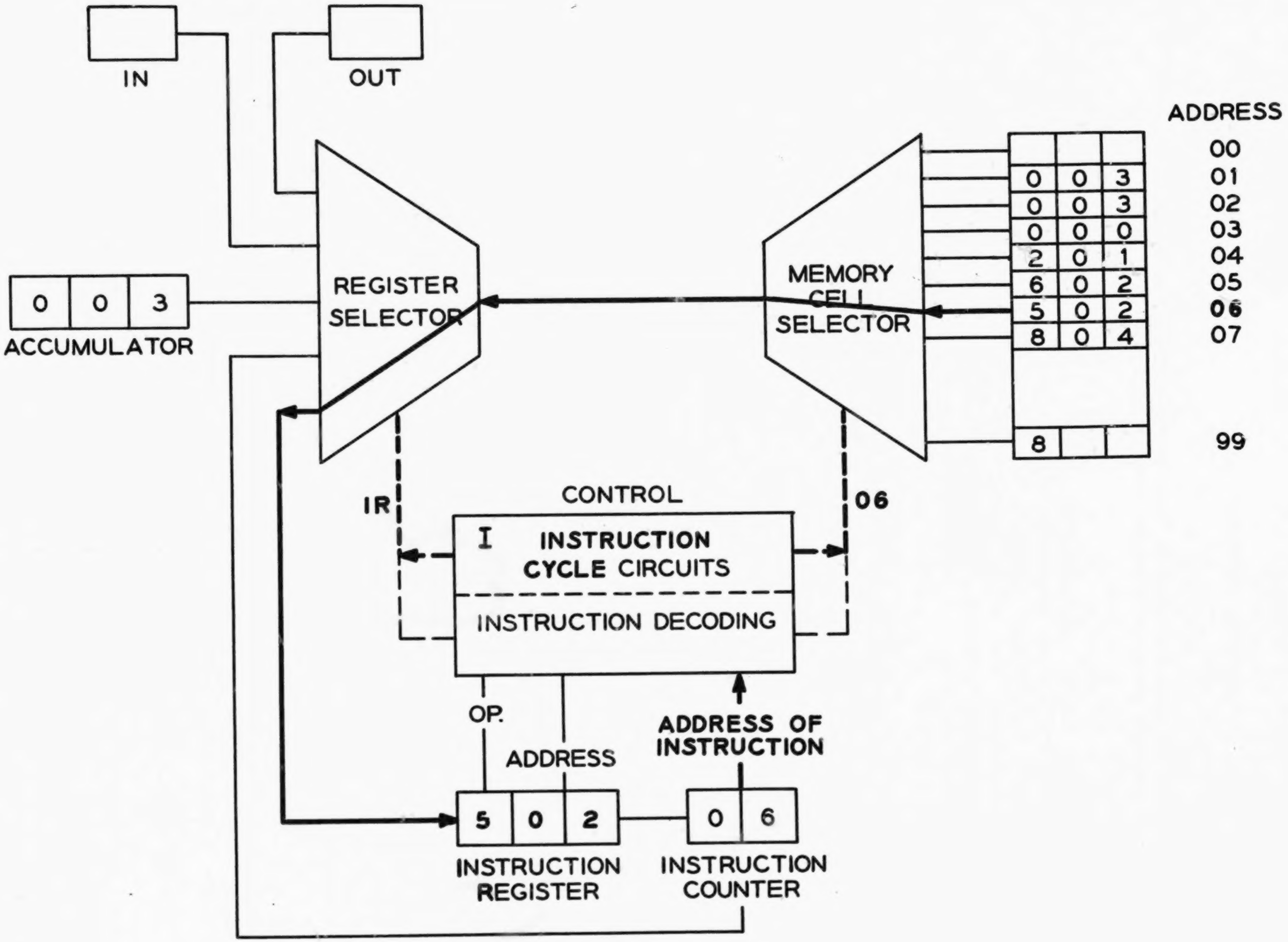
IN
OUT
ADDRESS
00
01
02
03
04
05
06
07
99
0 0 3
0 0 3
0 0 0
2 0 1
6 0 2
5 0 2
8 0 4
8
0 0 3
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
IR
CONTROL
06
I INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
5 0 2
0 6
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(h) Step I of Instruction Cycle
"Fetch Instruction from Memory"

In step I the current instruction 502 was fetched from cell 06.

In step II the counter was incremented to 07, the address of the next instruction.

In step III, 502 was decoded as "punch the contents of the memory cell specified in the address portion of the instruction". Thus +003, the contents of 02, was punched out.

> In detail: a) clear: according to the discussion of "Input and Output connected to Memory", Section 5, we see that on clock pulse 4, relays which run the motor are operated. b) select: at the same time, the memory selector tree relays are set to 02. c) copy: on pulse 5, operation of relay (output) closes contact 0 in the register selector to connect memory to the punches (CP1 through CP13 for our true binary machine). Step IV is omitted.

In step I, 804 was fetched from 07 as the current instruction.

In step II, the counter was incremented to the address of the next instruction, 08.

What is the purpose of instruction 804? Until now we have added a number (+003) to the accumulator (instruction 201), stored the new sum (instruction 602), and punched it out (502). Now suppose that we wished to continue this pattern of instructions indefinitely, generating thereby a sequence of cards each with a number 003 greater than the previous one. What we need is to be able to "jump back" to the instruction in memory location 04. But this is equivalent to saying "we need to be in step I of the instruction cycle, with 04 in the Instruction Counter, rather than 08." (Then, in step III, by definition, 201 will be executed as it was in the first snapshot, since it was fetched in step I.) Instruction 804 does precisely this - you'll notice that it caused 04 to appear in the Instruction Counter. 804 was therefore decoded as "take the address specified by the address portion of the Instruction Register and put that address (04 in this case) in the Instruction Counter".

> In detail: a) clear: at clock pulse 4, a special relay CLIC (Clear Instruction Counter, now shown) shorts the Instruction Counter relays to minus, thereby releasing them. b) copy: on pulse 5 the address relays are connected through contacts on another special relay REC (Register to Counter, also not shown) to the counter relays, thereby transferring the address to the counter.*

At the end of step IV of the current cycle, we go to a new cycle starting with instruction 201. Would a step I snapshot look any different from Fig. 22(a)? Only in this detail: +003 in the accumulator and in memory cell 02. In the next four cycles, the accumulator's contents would be increased to +006, etc.

> Note: In the final analysis, only usage determines the distinction between data (+003 in call 01), instructions (+201 through +804 in cells 04 through 07 respectively), temporary results (000, 003, etc., in cell 02) and unused numbers (+001 in 00, -619 in 03, etc.). If initially, after pushing the run-

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*The contents of cell 99 are also modified in accordance with the description of the 8 instructions of Section 3, Chapter 6.

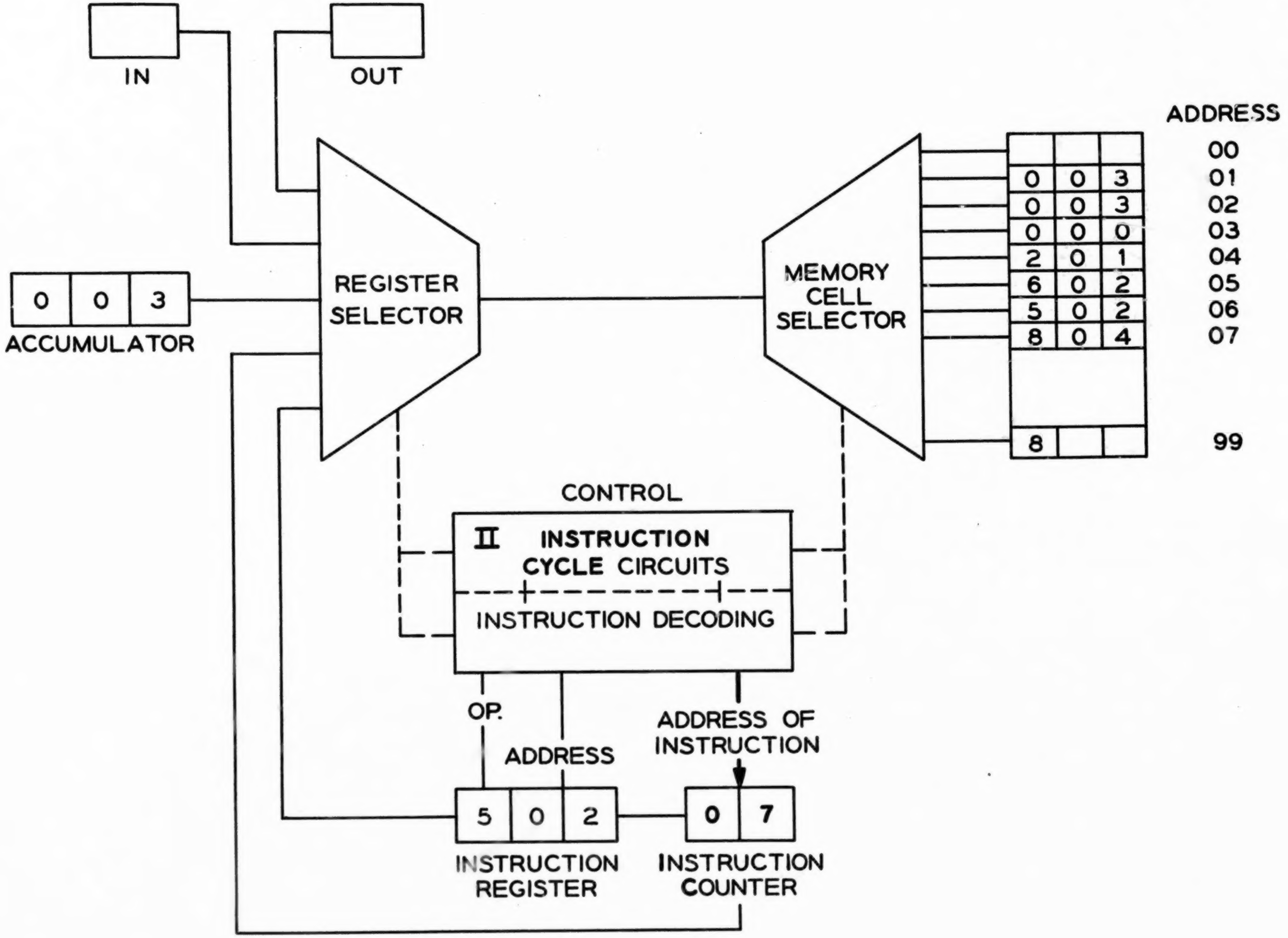
IN
OUT
ADDRESS
00
01
02
03
04
05
06
07
99
0 0 3
0 0 3
0 0 0
2 0 1
6 0 2
5 0 2
8 0 4
8
0 0 3
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
CONTROL
II INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
5 0 2
0 7
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(i) Step II of Instruction Cycle
"Increment Instruction Counter"

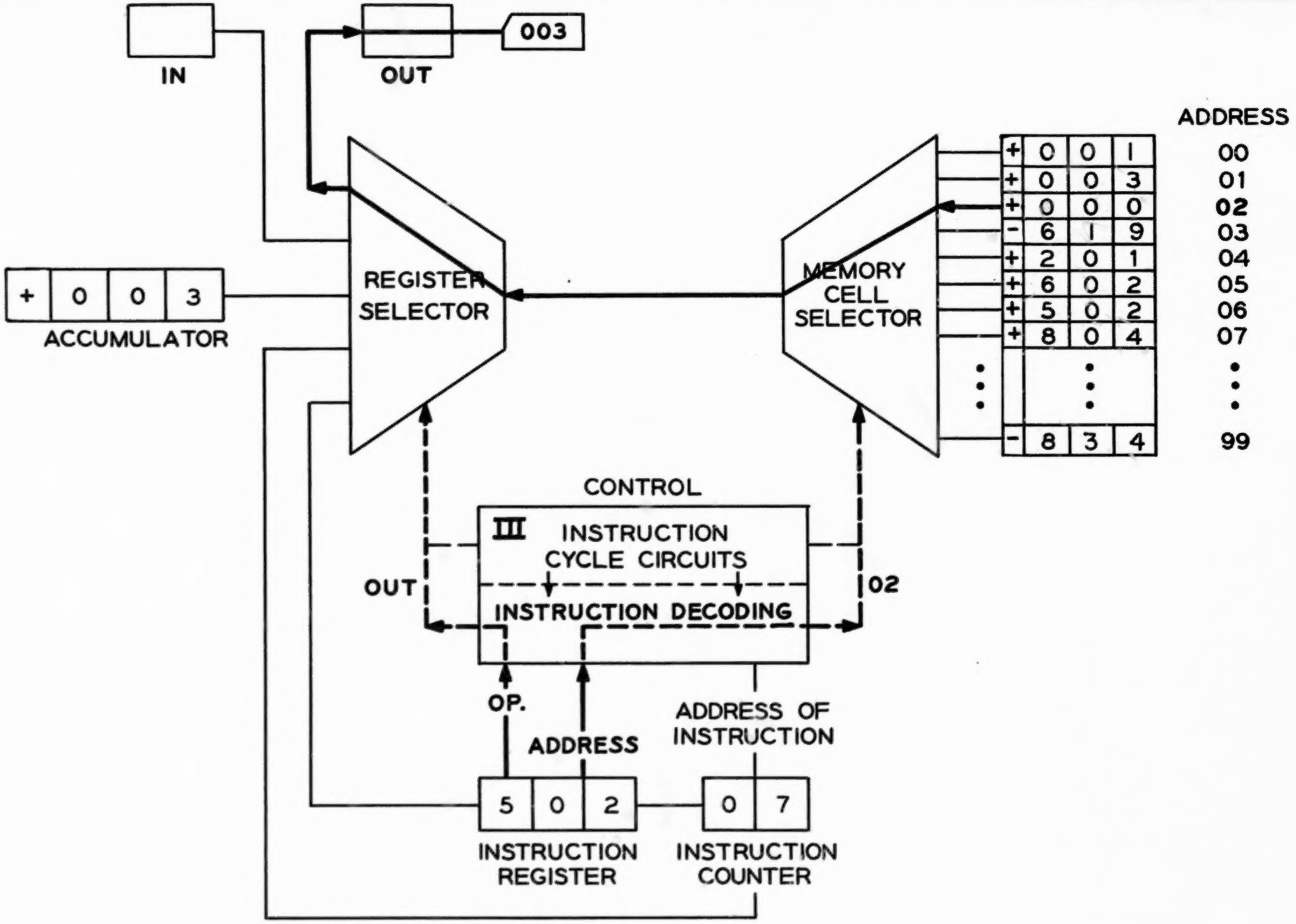
IN
OUT
003
ACCUMULATOR
+ 0 0 3
REGISTER SELECTOR
MEMORY CELL SELECTOR
ADDRESS
00 + 0 0 1
01 + 0 0 3
02 + 0 0 0
03 - 6 1 9
04 + 2 0 1
05 + 6 0 2
06 + 5 0 2
07 + 8 0 4
99 - 8 3 4
CONTROL
III INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OUT
02
OP.
ADDRESS
ADDRESS OF INSTRUCTION
5 0 2
0 7
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(j) Step III of Instruction Cycle
"Decode and Execute Instruction"

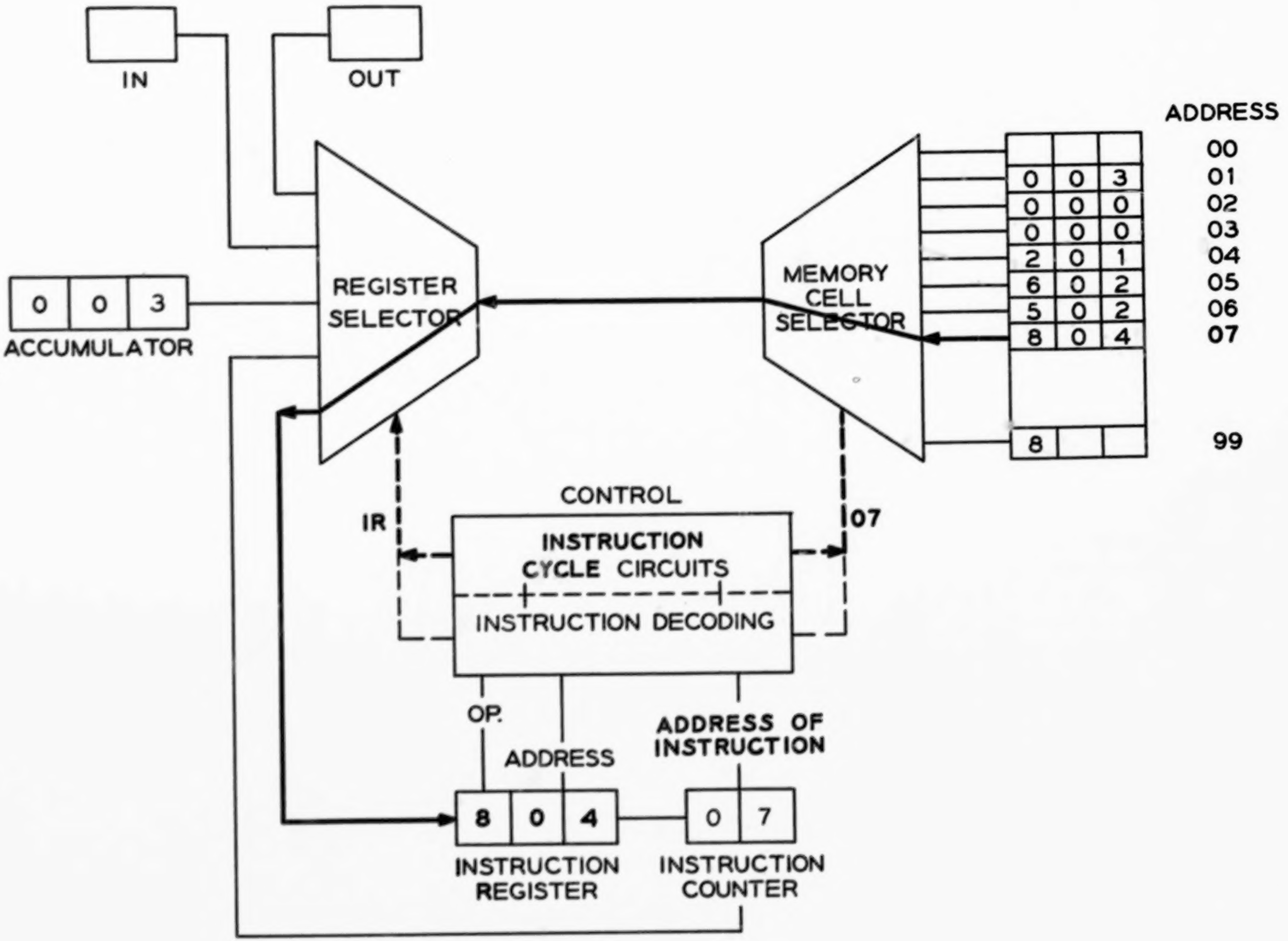
IN
OUT
ACCUMULATOR
0 0 3
REGISTER SELECTOR
MEMORY CELL SELECTOR
ADDRESS
00
01
02
03
04
05
06
07
99
0 0 3
0 0 0
0 0 0
2 0 1
6 0 2
5 0 2
8 0 4
8
CONTROL
IR
07
INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
8 0 4
0 7
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(k) Step I of Instruction Cycle

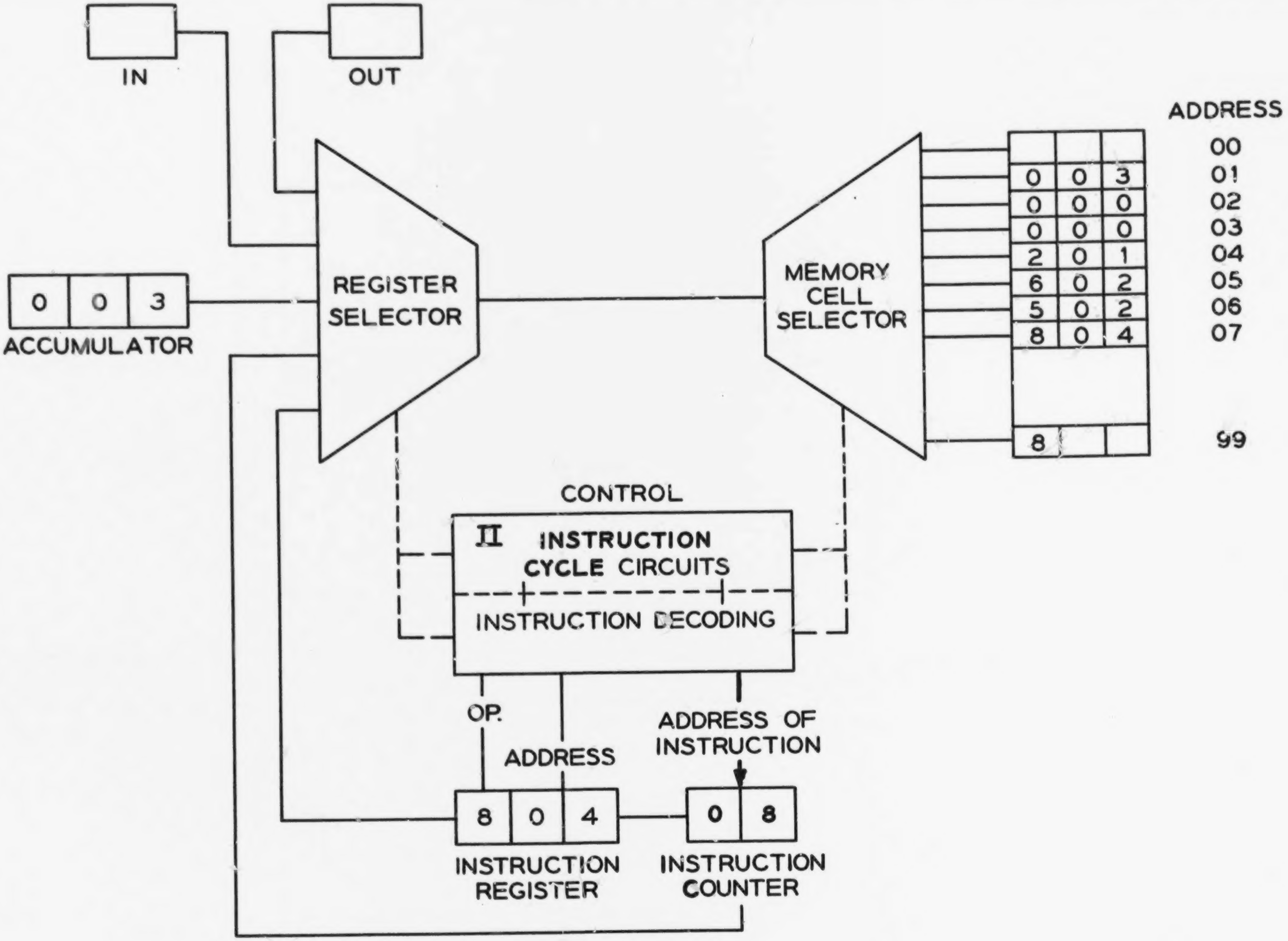
IN
OUT
ADDRESS
00
01
02
03
04
05
06
07
99
0 0 3
0 0 3
0 0 0
0 0 0
2 0 1
6 0 2
5 0 2
8 0 4
8
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
CONTROL
II INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
8 0 4
0 8
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(1) Step II of Instruction Cycle
"Increment Instruction Counter"

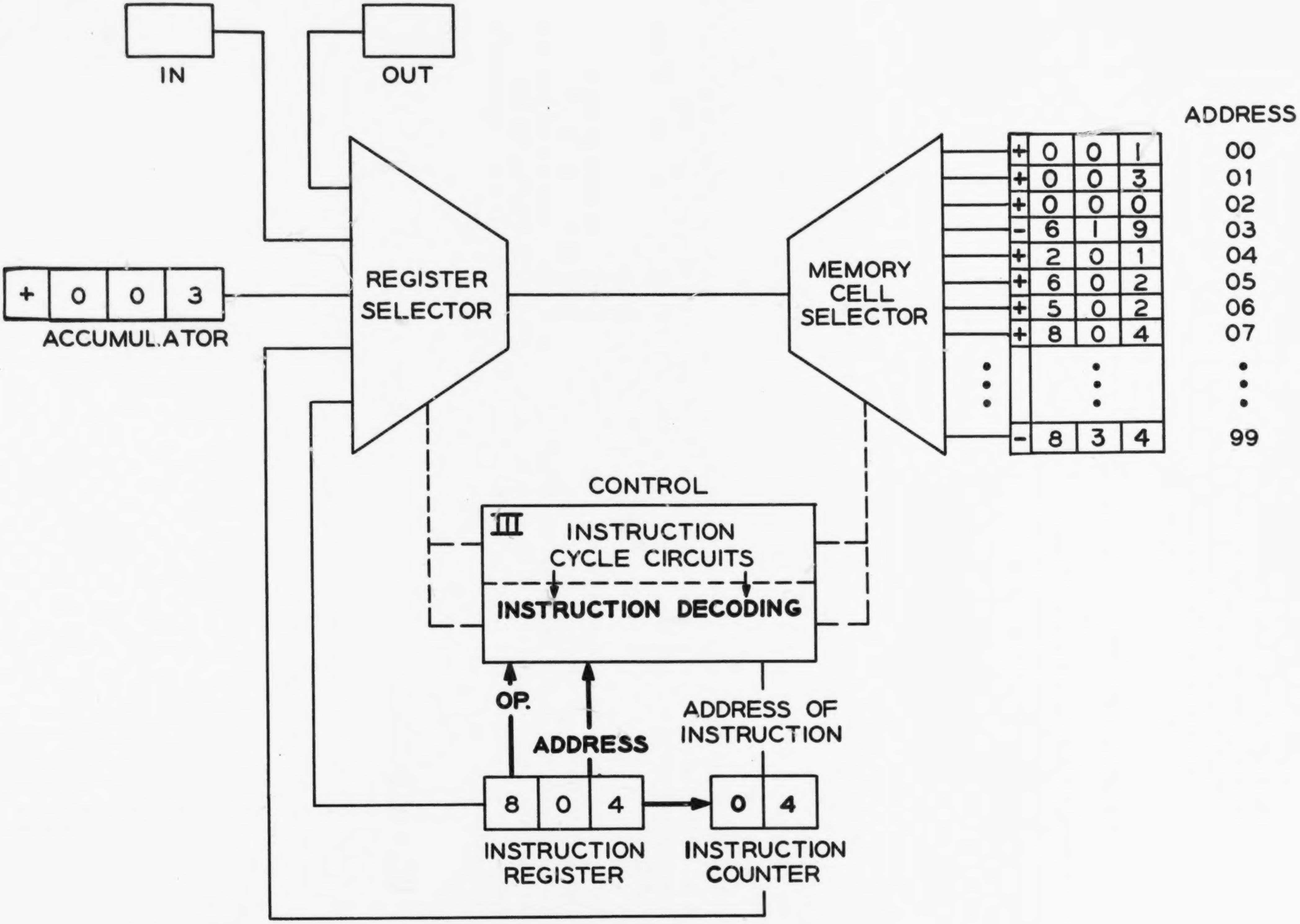
IN
OUT
ADDRESS
+ 0 0 3
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
+ 0 0 1 00
+ 0 0 3 01
+ 0 0 0 02
- 6 1 9 03
+ 2 0 1 04
+ 6 0 2 05
+ 5 0 2 06
+ 8 0 4 07
- 8 3 4 99
CONTROL
III
INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
8 0 4
0 4
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 22(m) Step III of Instruction Cycle
"Decode and Execute Instruction"

stop switch to "run", we had started the instruction counter in step I at 00 instead of at 04, the machine would have executed 001, "read a card into cell 01", followed by 003, "read a card into cell 03", etc. Only the programmer knows for sure what the meaning of each individual cell in memory is, and it is his responsibility to put the right things in the right places, as well as to start the machine at the right place.

10. GENERALIZATIONS AND EXTENSIONS

Since we have now given a basic idea of how a computer is put together and what types of instructions it handles, it is fair to ask what relationship our machine bears to the real world. As we said in Section 1, our original machine handles binary numbers only, but could be made to handle realistic alphabets of decimals or alphabetic characters. Let us see for a moment how we could use a binary machine to represent (i.e., to encode) such characters.

Binary encoding permits computers, in effect, to process and store letters, decimal numbers, logical truth or falsity, the outcome of legislative votes, etc., by making use of the contacts in binary logic circuits. Of course, one must be careful not to apply these circuits indiscriminately, i.e., we should not want to add one "letter" to another, or to a number for that matter. Nevertheless, this idea of representation or encoding is basic to the use of computers for a variety of tasks.

The fundamental point is that modern computers consist of two-state elements; computers do not contain numbers or letters or any other quantity, but the binary elements can represent, or symbolize, any one of these or many other things at the choice of the user depending on his objectives.

Thus the process of encoding or representation involves establishing a correspondence between two sets of items. As an example, Table I gives a correspondence between the decimal numbers and their binary equivalents for the first sixteen decimal numerals. If we take each "0" and "1" to represent the condition, or state, of a switch contact, then Table I is a coding for the numerals (up to 16) suited for use in a computer with binary circuits.

Table I

Decimal Numeral	Binary Numeral
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

While we represent a closed contact by "1" and an open contact by "0" so that we can talk conveniently about the various things that contacts can represent, you should note that not only this correspondence but Table I as well is entirely arbitrary. Table I is a correspondence that conforms to the binary number system. However, we could choose to represent the decimal numerals by any other set of symbols. For example, another coding that is sometimes useful in computers involves replacing each individual digit of a decimal numeral by its binary equivalent. Table II shows several examples.

Table II

Decimal Numeral	Binary-Coded Decimal (BCD) Numeral	Binary Numeral
128	0001 0010 1000	10000000
20	0010 0000	10100
6	0110	0110

If we wanted to use BCD coding for our 3-decimal-digit-plus-sign computer, we would need 1 bit for the sign and three times 4 bits per digit (13 bits all together). Would these 13 bits function in the same way as the 13 bits of our original binary computer? Indeed not, as the example of decimal 128 in Table II shows. Also, the 13-bit binary adder of Section 6 works only for straight 13-bit binary and not for 13-bit BCD. To construct logic circuits for a BCD machine would be a complex but entirely possible task: each group of 4 bits, representing an individual digit, would have to be treated as a unit, and the BCD adder would not have the easy similarity of stages of the binary adder. An alternative, which is sometimes used in modern-day computers, is to store decimal digits in BCD, and to use logic circuits to convert the BCD representation of the decimal number to a straight binary representation, prior to arithmetic operations, reconverting the result back to BCD to store it. In fact, BCD-type coding is usually extended to include both letters and decimal digits, as shown in Table III. Each letter, numeral or symbol is represented by six bits, and the encoded set includes not only the upper-case letters and numerals but some useful special symbols as well. This correspondence was established by merely listing the numerals, letters and symbols in a convenient order and then numbering them consecutively with the six-digit binary numbers. Thus, a multiple-digit decimal numeral, such as 796, is written in 6-bit binary-coded decimal form as 000111 001001 000110.

Notice that n bits can represent 2^n distinct objects labelled $\underbrace{000\ldots0}_{n}$ through $\underbrace{111\ldots1}_{n}$. To show this, consider an individual n-bit number to have been composed by picking 0 or 1 for n consecutive digit positions. Thus there were 2 choices for the first, 2 choices for the second, . . . , and 2 choices for the nth digit position.* Hence there are $\underbrace{2.2.2.\ldots.2}_{n} = 2^n$ distinct sequences of choices. Try it out for n = 3, for instance. Table III holds 64 (or 26) characters for 6-bit BCD numbers.

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*As we did in the footnote on page A.5.22, we can symbolize these successive choices by $\underbrace{\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\ldots\begin{pmatrix}0\\1\end{pmatrix}}_{n}$.

TABLE III
SIX-BIT BCD CODE

0	000000
1	000001
2	000010
3	000011
4	000100
5	000101
6	000110
7	000111
8	001000
9	001001
[	001010
#	001011
@	001100
:	001101
>	001110
?	001111
ƀ	010000
A	010001
B	010010
C	010011
D	010100
E	010101
F	010110
G	010111
H	011000
I	011001
&	011010
.	011011
]	011100
(	011101
<	011110
\	011111
↑	100000
J	100001
K	100010
L	100011
M	100100
N	100101
O	100110
P	100111
Q	101000
R	101001
-	101010
$	101011
*	101100
)	101101
;	101110
'	101111
+	110000
/	110001
S	110010
T	110011
U	110100
V	110101
W	110110
X	110111
Y	111000
Z	111001
←	111010
,	111011
%	111100
=	111101
"	111110
!	111111

TABLE IV
SEVEN-BIT BCD CODE

0	0000000
1	1000001
2	1000010
3	0000011
4	1000100
5	0000101
6	0000110
7	1000111
8	1001000
9	0001001
[	0001010
#	1001011
@	0001100
:	1001101
>	1001110
?	0001111
ƀ	1010000
A	0010001
B	0010010
C	1010011
D	0010100
E	1010101
F	1010110
G	0010111
H	0011000
I	1011001
&	1011010
.	0011011
]	1011100
(	0011101
<	0011110
\	1011111
'	1100000
J	0100001
K	0100010
L	1100011
M	0100100
N	1100101
O	1100110
P	0100111
Q	0101000
R	1101001
-	1101010
$	0101011
*	1101100
)	0101101
;	0101110
'	1101111
+	0110000
/	1110001
S	1110010
T	0110011
U	1110100
V	0110101
W	0110110
X	1110111
Y	1111000
Z	0111001
←	0111010
,	1111011
%	0111100
=	1111101
"	1111110
!	0111111

One difficulty with codings such as that in Table III lies in the possibility of making errors. With a long string of 0's and 1's, it is all too easy for a person to substitute a "1" for a "0" when copying. Relay and switch contacts, too, can malfunction and be open when they are supposed to be closed or vice-versa. One way of protecting against errors is to add to the binary string an extra digit whose value, 0 or 1, is arranged so that the total number of "1's" in each 7-bit string is even; that is, 0, 2, 4, or 6. Table IV is the same as Table III with the additional bit appearing at the left end of the string. Note that all the seven-bit strings have an even number of ones. Should a string appear with an odd number of 1's during a computation, you can be sure that an error has occurred in transcription of the initial coding.

This 7-bit representation is an example of an error-detecting code, and such codes are often used where great amounts of data and symbols are to be handled. Computers are one example, and error detection is vital for them since their elements can fail, even though such failures are infrequent.

This particular error-detecting code always works when a single digit is in error. It may not work if two or more digits are wrong. For example, a "1" can be interchanged with a "0" in any of the 7-bit strings without violating the "even number-of-ones conditions." However, single errors are the predominant kind in many situations and so our single "parity-check" error-detection code, as it is called, can be quite useful. Codes which can detect multiple errors and can also be used to correct them have been devised for applications where higher reliability is necessary. All such codes involve adding additional bits to the binary words so that certain prescribed relations exist among the bits making up the words.

Having seen how to store characters as bits, you may now understand how the conventional punched card (usually referred to as an "IBM card") shown in Fig. 23 is encoded in a binary machine. Each of the columns represents one digit or character. A commonly-used code for representing characters and numbers is indicated on the card. In this representation each column can have at most three punches; and the correspondence is indicated by the pattern of punches beneath the printed character at the top of the card. In effect, each character is represented by a 12-digit binary number (if we equate a hole in the card to "1" and no hole to "0") in which only 3 bits at most can be "one". This particular code takes 12 bits rather than the 6 bits of Table I, because of this restriction. That is, the 12-bit punched card code is not a minimum representation. A single character therefore takes up an entire 13-bit word in this representation, and the machine could have logic circuits to convert two consecutive words of 12-bit characters into a single word with two 6-bit BCD codes. Note that the character appears on the top row; it is produced by a special device, such as a "keypunch" or an "interpreter", whose logic circuits can cause typewriter-like keys to print the translated 12-bit code.

To sum up: we have seen how a binary computer, using only binary logic elements, can represent and manipulate arbitrarily complex information if suitable encoding, conversion and operation circuits exist. In what sense is our simple binary machine (or its decimal equivalent) a general-purpose computer, as mentioned in the introduction to this chapter? The answer is not immediately

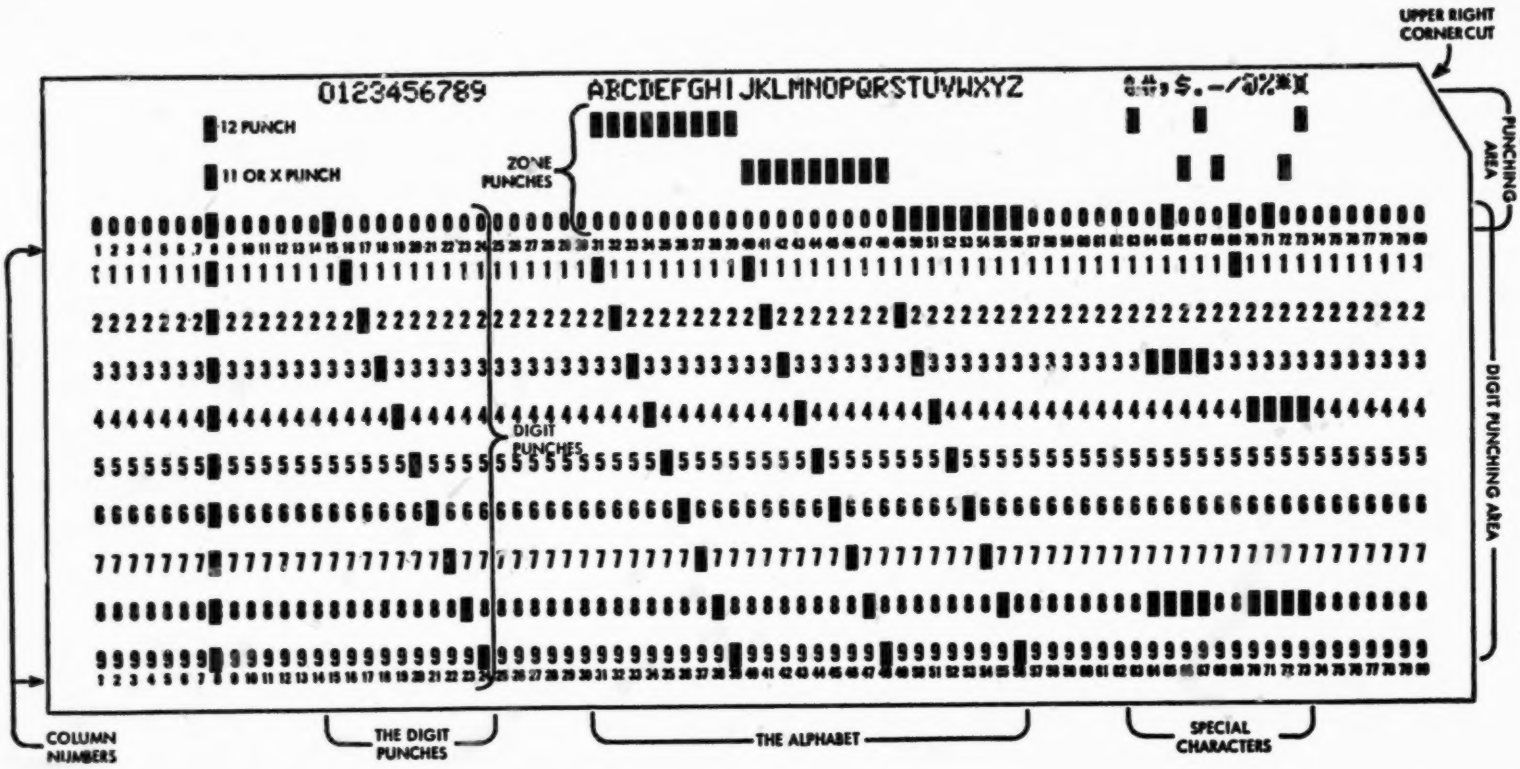
UPPER RIGHT CORNERCUT
0123456789
ABCDEFGHIJKLMNOPQRSTUVWXYZ
12 PUNCH
11 OR X PUNCH
ZONE PUNCHES
ZONE PUNCHING AREA
DIGIT PUNCHES
DIGIT PUNCHING AREA
COLUMN NUMBERS
THE DIGIT PUNCHES
THE ALPHABET
SPECIAL CHARACTERS

Fig. A-5. 23 Punched card

obvious, and its proof will have to be demonstrated by example in the following chapters; however, it is, at least, simple. A binary (decimal) machine is general-purpose if it can 1) input and output, 2) do simple arithmetic (addition and subtraction are enough, since multiplication and division can be done by repeated addition and subtraction), 3) make simple two-way comparisons between (binary) numbers and react by choosing one of two alternative "next instructions". This last capability is added to our machine in the next chapter. It turns out that any arbitrarily complex procedure can be assembled from such primitive operations executed by binary (or decimal) logic circuits. The detailed specification of such procedures is the task of the programmer; we see the fundamentals of his art in the next chapter.

PROBLEMS

5.1 Write the decimal number 396 in binary-coded decimal form.

5.2 Write the binary number 1011010 in a binary-coded decimal form.

5.3 Decode the following message according to the correspondence of Table III

100001 110100 100011 111000 000100 111011 000001 000111
000110 000110

5.4 Which of the following coded symbols are in error according to the error detection scheme of Table IV?

(a) 0000001
(b) 1000100
(c) 1101100
(d) 1001100
(e) 1111111
(f) 0111111

5.5 Construct (draw) a logic circuit which when properly connected to a card reader of the kind shown in Fig. 7 will turn on a light when there is a parity error on the card being read by it. The card is punched with the code of Table IV.

5.6 Construct (draw) a logic circuit which when properly connected to a card punch (Fig. 8) will punch the proper parity check bit onto each card as it is punched. The information on the card punch relays conforms to Table III.

5.7 Write out a sequence of instructions for your assistant, Steadfast Plodder, to calculate the cost of a competitor's products (labor plus materials) from his price list, assuming that his markup is the same as yours. Make out an instruction sheet similar to Fig. 3.

5.8 Assume that you have a memory of 1024 cells storing 32 bits each. What is the total number of bits stored by such a memory? How many relays are required to access each cell independently? (Assume relays with any number of contacts are available.) How many bits are required in the address for each cell?

5.9 During each instruction cycle of a computer:

(a) how many times is the instruction counter incremented
(b) how many times is the instruction register changed?

Chapter A-6

PROGRAMMING

1. Introduction

Programming is problem analysis followed step-by-step implementation of the mathematical or logical solution to the problem in a computer-executable language.

"Analysis" in this context does not necessarily refer to the initial parts of the problem-solving process in which the laws of natural (or social) science and the techniques of mathematics are brought together. The programmer need not always be a scientist or mathematician himself; whether he is or not, he usually takes a vaguely expressed mathematical or logical solution - his or some other person's - and translates it into an algorithm*: a tightly organized, unambiguous sequence of instructions whose mechanical execution will yield proper answers in a finite amount of time.

This finite instruction sequence is often detailed by means of a flowchart, and the last step of translating the contents of flowchart boxes to machine language instructions for a specific computer is referred to as coding. In this chapter we will concentrate on learning how to code for our specific machine, but it must be emphasized that the programmer's primary task is not the essentially mechanical one of coding from flowcharts, but rather the composition of the flowcharts themselves, i.e., the construction of algorithms. The examples chosen are such that you will be able to perform all steps in the analysis and implementation.

2. Review

Before we discuss coding technique, let us briefly review the properties of the little computer we designed in the last chapter. Referring to Fig. 1 of Chapter 6 (a reproduction of Fig. 19 of Chapter 5), we see that our computer has the five basic functional components block-diagrammed in Fig. 6 of Chapter 5: Input, Output, Arithmetic Accumulator, Memory and Control. Input and Output functions are performed by punched-card-handling devices, the Accumulator consists of a binary adder (and subtractor) and control relays, and Memory is made up of 100 words**, addressed 00 through 99, each of which holds 3 decimal digits plus sign. Control in our machine is maintained by the Control Unit, consisting of two timed circuits and two register/cell selectors which function as rotary switches to connect specified devices and memory cells. There are also the two auxiliary registers for storing the current instruction and its address (or the address of the next instruction). Instructions, when "fetched" from memory to be copied signless into the instruction decoding circuits, are separated into a first "op code" digit (add,

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*A word derived from al-Khowarizmi, the name of a famous 9th-century Asian mathematician, who invented the "shift and add" algorithm for multiplying decimal numbers discussed in Chapter A-3.

**A word is defined as the contents of any memory location.

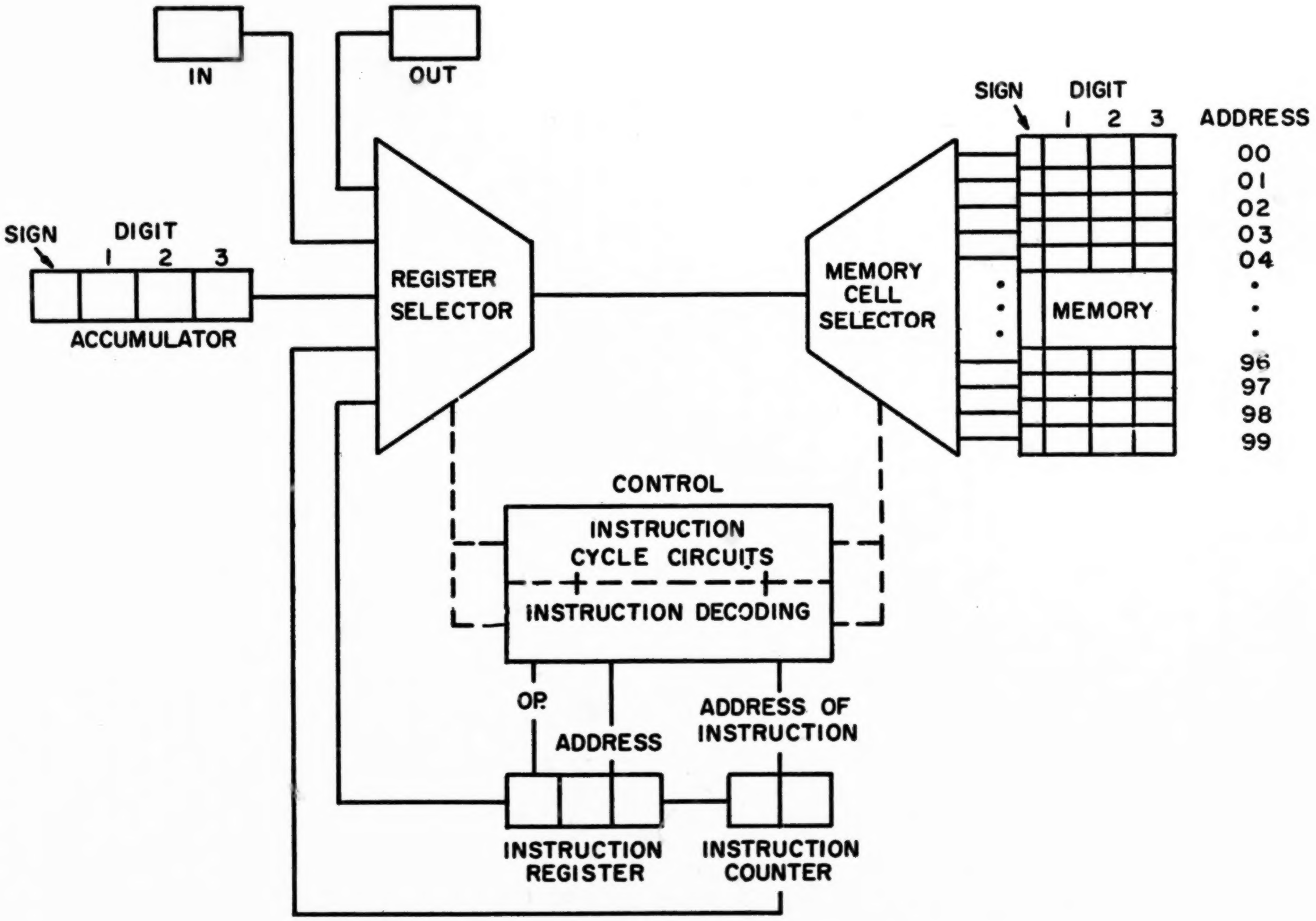

IN
OUT
SIGN
DIGIT
1
2
3
ACCUMULATOR
REGISTER SELECTOR
MEMORY CELL SELECTOR
SIGN
DIGIT
1
2
3
ADDRESS
00
01
02
03
04
96
97
98
99
MEMORY
CONTROL
INSTRUCTION CYCLE CIRCUITS
INSTRUCTION DECODING
OP.
ADDRESS
ADDRESS OF INSTRUCTION
INSTRUCTION REGISTER
INSTRUCTION COUNTER

Fig. 1 Block diagram of a 3-decimal-digit computer, with a 1-digit operation code (OP.), 2 digits address, one hundred 3-digit-plus-sign memory cells (addresses 00 through 99).

subtract, punch, store, jump, etc.) and a two-digit "address" whose contents are to be used in the operation. (For certain op codes (store, shift), the address specifies where the contents of the accumulator are to be copied, or the extent of a shift.)

Instructions and their data are stored together in Memory, and are indistinguishable, to the computer, from one another there. This common storage of data and instructions provides stored program capability and, as we shall see in section 11 of this chapter, accounts for the general-purpose flexibility of the computer. Meanwhile, how does the machine distinguish data from instructions? The answer is unambiguous: only those words whose addresses appear in the instruction counter, and which subsequently appear in the instruction register, are interpreted by the machine as instructions (during step III of the instruction cycle). It is therefore the programmer's responsibility to ensure that only those words which he puts into memory as instructions appear in the instruction register. Having seen in the snapshot sequence of section 9 of Chapter A-5 how the various units of the computer interact in the instruction cycle under the Control Unit's supervision to fetch (I), increment (II), execute (III) and test (IV), we will leave this level of circuit detail to concentrate on the effects of the individual instructions.

3. The Ten Basic Operations of our Computer

We start by defining the basic instruction set of our machine: those operations which the logical circuitry of the machine is capable of decoding and executing. We encountered most of these instructions in the last chapter, but we have added some new ones, the so-called branches (jumps), both conditional (op code = 3) and unconditional (op code = 8), and the conditional halt (op code = 9).

If the instruction is symbolized as a 3-digit word (XYZ), then the first digit (X) denotes the operation to be performed and the second third digits (YZ) denote data appropriate to the instruction. The ten possible values of X, and the operations each calls for, are indicated below in Table 1. In the following examples, the meaning of each of these instructions will become clear.

4. Adding Two Numbers

We are now ready to study the application of the operations of Table 1 to specific arithmetical and logical practice problems. We'll begin with the easy one of adding two unknown numbers p and q, which have been previously punched and placed on the top of the card reader stack. The flowchart for this problem (Fig. 2(a) of this chapter) is simple, but we include it for the sake of illustration.

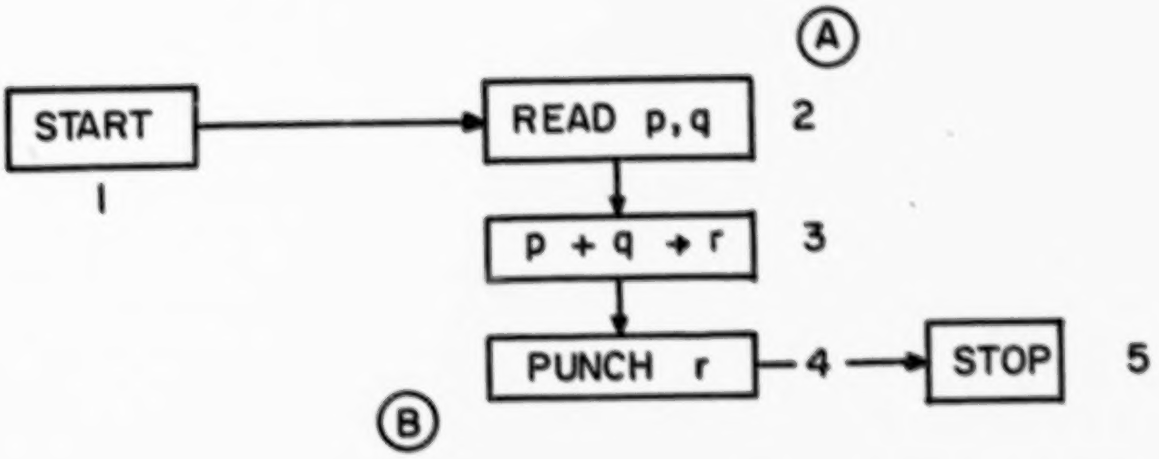


Fig. 2 (a) "Macro" flowchart for r = p + q

Table 1 The Basic Instruction Set

X = 0 (input):

a) copy into adress YZ the word on the top card of the input card stack thereby "erasing" the previous contents; then advance (i.e., remove) the top card of the stack.

b) If the input card is blank (or the stack is empty), reset the instruction counter to 00, set the "run-stop" switch to "stop" and advance (i.e., remove) the top card of the input stack.

X = 1 (clear and add):

a) Clear the accumulator to 000. Copy into the accumulator the word at address YZ. (Do not erase or change the word at address YZ.)

X = 2 (add):

a) Add to the accumulator the word at address YZ. (Do not erase or change the word at address YZ.)

X = 3 (test accumulator contents):

a) If the contents of the accumulator are zero or positive, go on to the next instruction. If the contents are negative, set the instruction counter to YZ.

X = 4 (shift):

a) Shift the number in the accumulator left Y places; then shift in right Z places.

X = 5 (output):

a) Copy the word at address YZ onto the blank output card on top. Advance (i.e., remove) this card. (Do not erase or change the word at address YZ.)

X = 6 (store):

a) Copy the contents of the accumulator into the address YZ, thereby erasing its previous contents. (Do not erase or change the word in the accumulator.)

X = 7 (subtract):

a) Subtract from the contents of the accumulator the word at address YZ. (Do not erase or change the word at that address.)

X = 8 (jump):

a) Replace the second and third digits (the address component) of the word stored at address 99 by the count in the instruction counter. (The first digit - the operation component - of the word stored in 99 is permanently set to the value "8".)

b) Then reset the instruction counter to YZ.

X = 9 (halt and reset):

a) If the address component (YZ) of the word in the instruction register is 00, set the instruction counter to 00, and set the "run-stop" switch to "stop".

b) If the address component of the word in the instruction is not 00, and if the "overflow indicator" is on, reset the instruction counter to YZ.

c) If the overflow indicator is not on, go on to the next instruction.

Notes:

1) For reasons to be explained later, the contents of address 00 are permanently wired to +001.

2) Also, the op code digit at address 99 is permanently wired to 8.

3) A blank card represents the "number" -000 which can therefore not be read in, nor produced by an arithmetic instruction. Thus (-A+A) and (A-A) both yield +000 as answers.

Box 1 specifies the start of the computations; box 2 specifies that two numbers are to be read and stored in two addresses symbolically identified by p and q. Note the similarity between referring to unknowns by letters in an algebraic equation (r = p + q) and by symbolic address, i.e., by letters corresponding to actual memory locations in the machine (p + q = r). Our machine operations will not be dependent on the specific contents of cells any more than algebraic equations are dependent on the specific values assigned to variables. You will see how this lack of dependence contributes to the general-purpose-ness of computer programs.

Box 3 is read as "take the contents of memory location p (whatever number that is), add the contents of memory location q (whatever number that is), and store the calculated result in temporary memory location r"; this is customarily abbreviated as "p plus q into r". Box 4 calls for output, and box 5 for the end of this computation. Note that we called for an addition of two numbers, an operation not implementable with a single machine instruction, since each machine add uses the accumulator contents as one of its operands. Using the technique learned in Section 6 of Chapter A-5, we must clear and add one number, then add the next to it. Thus we see that Fig. 2 (a) of this chapter was not a detailed or "micro" flowchart, which can be translated line for line into machine code. Hence we must break down the "macro" flowchart (i.e., translate it) into the "micro" flowchart (Fig. 2 (b)), after which we can code it, one line of code for each line of the micro flowchart. (Only the "start" instruction is done manually, by pushing the "run-stop" switch.)

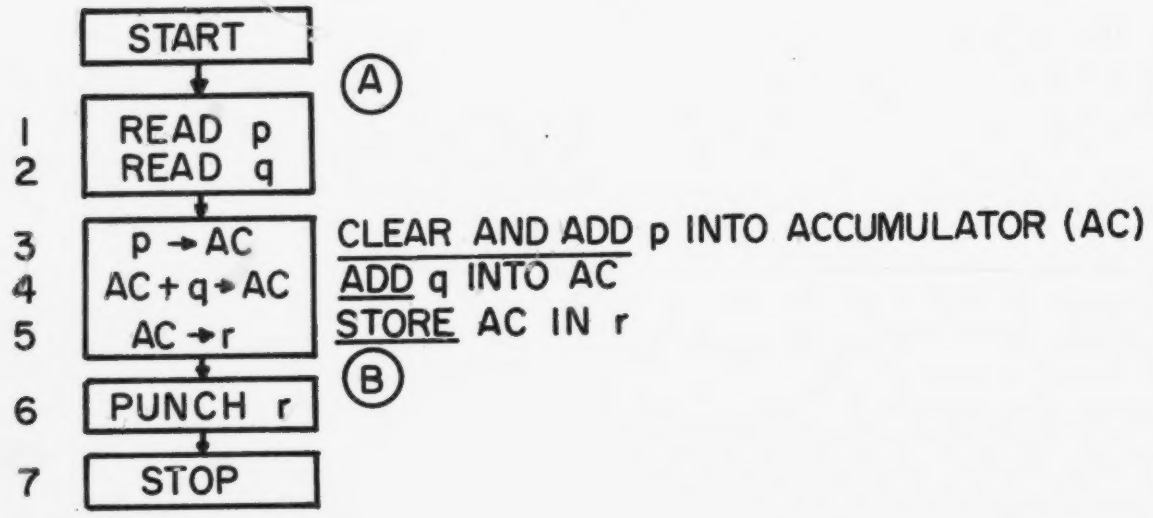

START
Ⓐ
1
READ p
2
READ q
3
p → AC
CLEAR AND ADD p INTO ACCUMULATOR (AC)
4
AC + q → AC
ADD q INTO AC
5
AC → r
STORE AC IN r
Ⓑ
6
PUNCH r
7
STOP

Fig. 2 (b) "Micro" flowchart for r = p + q

The coding process is obvious as soon as we decide which memory locations ought to represent the variables p, q and r, and where the instructions should be stored. Arbitrarily, we decide to store the required instructions in locations 10 to 16, and p, q, and r immediately following in 17, 18 and 19 respectively. We symbolize this memory assignment below:

address	contents
10	program
.	
.	
.	
16	
17	p
18	q
19	r

We can now uniquely translate the flowchart lines, since we know the op code digit X (see Table 1) and the YZ address digits for p, q or r for each line, as shown in Fig. 2 (c).

microflowchart step no.	microflowchart line	equivalent machine code	stored in location
1	Read p:	+ 0 ¦ 17	10
2	Read q:	+ 0 ¦ 18	11
3	p → AC:	+ 1 ¦ 17	12
4	AC + q → AC:	+ 2 ¦ 18	13
5	AC → r:	+ 6 ¦ 19	14
6	Punch r:	+ 5 ¦ 19	15
7	STOP:	+ 9 ¦ 00	16
variables	p	?	17
	q	?	18
	r	?	19

Fig. 2 (c) Coding for r = p + q

Having (1) composed our algorithm, and (2) coded it, we must now (3) load the program into the specified memory location and start the machine (4) executing it. While these four steps are typical of the normal flow of events, we shall omit details of the last two steps for a while, except to note that the machine is started by setting the instruction counter to 10 and setting the "run-stop" switch to "run". Fig. 2 (d) shows what happens to the accumulator and the three variable locations as each instruction is executed in turn. Assume that the two top cards held + 058 and + 003 respectively for this use of the program. The first instruction is 017. The digit 0 commands the machine to clear (erase) the word at the address indicated by the second and third digits and to copy into it the number on the top input card. Therefore, the computer copies the number 058 from the input and deposits it in memory address 17 (thereby erasing the previous word in 17). As each instruction is "fetched" from the memory to the instruction register, the count in the instruction counter increases by one. So, now, the control goes to address 11 and brings the contents (018) to the instruction register. This instruction says to copy the number 003 from the top (originally the second) card into address 18. Thus the two numbers are now stored in the memory and ready for addition.

Memory	Program Step							
Location	0	1	2	3	4	5	6	7
17	000	058	058	058	058	058	058	058
18	000	000	003	003	003	003	003	003
19	000	000	000	000	000	061	061	061
Accumulator	000	000	000	058	061	061	061	061

Fig. 2 (d) Sequence of actions caused by program for r = p + q

To carry out the operation of addition, instruction 117 says to copy into the accumulator the contents of address 17, namely 058; instruction 218 says to add to the accumulator the contents of address 18, namely 003. Thus the accumulator now contains the desired sum.

The remaining operations are devoted to extracting the sum and putting it on an output card. To execute instruction 619, the computer copies the accumulator (the desired sum) into address 19. Instruction 519 causes the word in address 19 to be copied onto an output card. Thus we have punched the solution to our problem. The final instruction, 900, tells the computer to reset the instruction counter to 00 and to set the "run-stop" switch to "stop". If we now wanted to add two new numbers we would merely punch them on cards, place them on the top of the input stack, reset the instruction counter to 10 and push the "run-stop" switch to "run". Thus, by writing the program using addresses of variables whose contents are read in at execution time, we have made the program general for adding any two such numbers, and therefore re-usable.

5. A Program which Triples Numbers: The "Unconditional Jump"

Each of the operations which our computer can perform could obviously also be performed by a human being. However, since each of its operations can be executed in as little as a thousandth of a second, it can easily out-perform the human being though it is merely doing what its program, and therefore its human programmer, tells it to do. For example, suppose a retail merchant's list price for items is three times the manufacturer's price. For each of a large set of manufacturer's prices, the merchant might have these prices punched on cards in a form which can be accepted as input by our computer, which will then punch the list prices on cards. Thus, the number + 298 written on an input card could be interpreted as $2.98. In order to simplify our program, we shall also assume that each of these prices is less than $3.34; otherwise a created list price might exceed $9.99, which is the largest number our computer can print on a single output card. Figures 3 (a) and (b) show macro and micro flowcharts for a program which reads a number, multiplies it by 3, punches it, and jumps to the beginning of the program for another round.

Note that the macro flowchart in these problems is really nothing more than a two-dimensional precise re-statement of the English language specification of the problem, and gives a machine-independent algorithm. The micro flowchart, once again, is machine-specific and is translatable line by line, one-for-one, into machine code instructions (Fig. 3 (c). Here are two specific points to note:

1) The two identical AC + x AC instructions have different effects, since the accumulator contains different numbers upon execution of the instructions.

2) Since we no longer need x, we can save a storage location by storing the final AC contents over x. Thus the micro flowchart is more efficient than the macro flowchart.

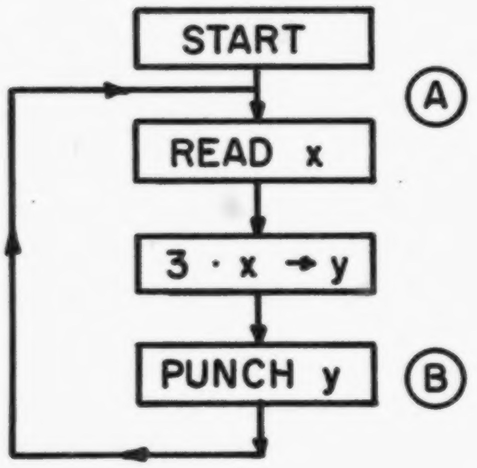


Fig. 3 (a) Macro flowchart for y = 3x

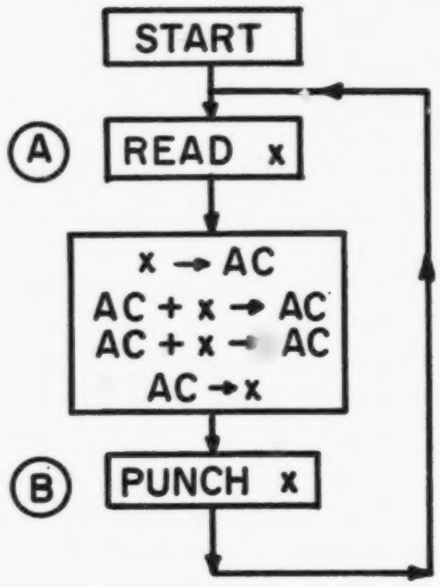


Fig. 3 (b) Micro flowchart for "x = 3x", i.e., 3x→x

Step	Memory Location	Instruction Word Stored	What Computer Does
1	10	017	Copies number from input card into address 17
2	11	117	Copies number from address 17 into accumulator
3	12	217	Adds number from address 17 into accumulator
4	13	217	Adds number from address 17 into accumulator
5	14	617	Erases number in address 17 and copies in contents of accumulator
6	15	517	Copies number in address 17 onto output card
7	16	810	Sends computer back to step 1 to repeat the sequence with a new input card

Fig. 3 (c) Coding for multiplying a number by 3

Assume that the code is stored beginning at location 10; the instructions shown in Fig. 3 (c) result. The only new instruction is 810: it loads 17 (the instruction counter's content) into location 99 for reasons not relevant to this use of the instruction and, what is relevant here, resets the instruction counter to 10. This forces step I of the next instruction cycle to fetch 017 (the first instruction in this program). so that the program is effectively rerun. The pattern is referred to as a simple loop, and we detect simple loops in coding as jumps back to previously executed instructions. Loops are fundamental to the general-purpose capability of computers. This particular type of simple loop is also known as a "read loop", since it causes a cycle of "read a card, do something", "read a card, do something", etc.

Since the program loops, how does the machine ever halt? The end of the list is signified by a blank card; when the 017 instruction is executed with a blank card, the 0 instruction (see its definition in Table 1) causes the "run-stop" switch to be set to "stop". We say that the loop is "terminated" by the reading of a blank card. As a matter of programming practice we will from now on cease to use the X = 9 halt instruction to terminate programs, but will loop back to the beginning of the program until the data cards are exhausted.

6. The Call-Up Problem: The "Conditional Jump"

As we stated in section 10 of Chapter A-5 (p. 57), the ability to make simple, two-way comparisons between numbers and to choose one of two alternative "next instructions" is essential to the general-purpose capability of a computer. In this section we will show how to use the two-way comparison (i.e., the conditional jump), how to combine conditional jumps with simple loops and how to program more complicated questions in terms of combinations of the simple conditional jump. As our first example, we will choose a very common and practical "data-processing" problem of the type found throughout business, industry and government. It concerns the maintenance of personnel records and various types of questions which can be asked about such files of people.

Assume, for instance, that a listing of Selective Service (SS) numbers has been punched on cards, with one SS number per four successive cards: district number on the first card, board number on the second, year of birth on the third, and a unique personal identifier on the fourth: e.g., 011 102 046 201. Signs are ignored. In preparation for a general "call-up", a listing of men born, say, in 1950 must be produced. We will scan the cards, checking the third card in each group of four for the year 050. The full SS number for each man born in 1950 will be punched. The macro flowchart for this program is shown in Fig. 4 (a). It is important to note that this program, like the previous one, contains a *loop*. Loops allow a *single* concise representation of instructions which are to be executed repeatedly. We could have written a separate group of instructions to read each SS number, etc., but this would make the program indefinitely long, since we do not know the exact length of the list to be inspected.

The only problem in the implementation of this macro flowchart (Fig. 4 (a)) is that of comparing the third card to 050. The conditional jump instruction allows only the testing of the accumulator sign. Hence we subtract 50 and test the sign; a negative sign obviously indicates no match, but a positive sign can mean a positive value of zero. So, to test for zero, we then subtract 1. Now

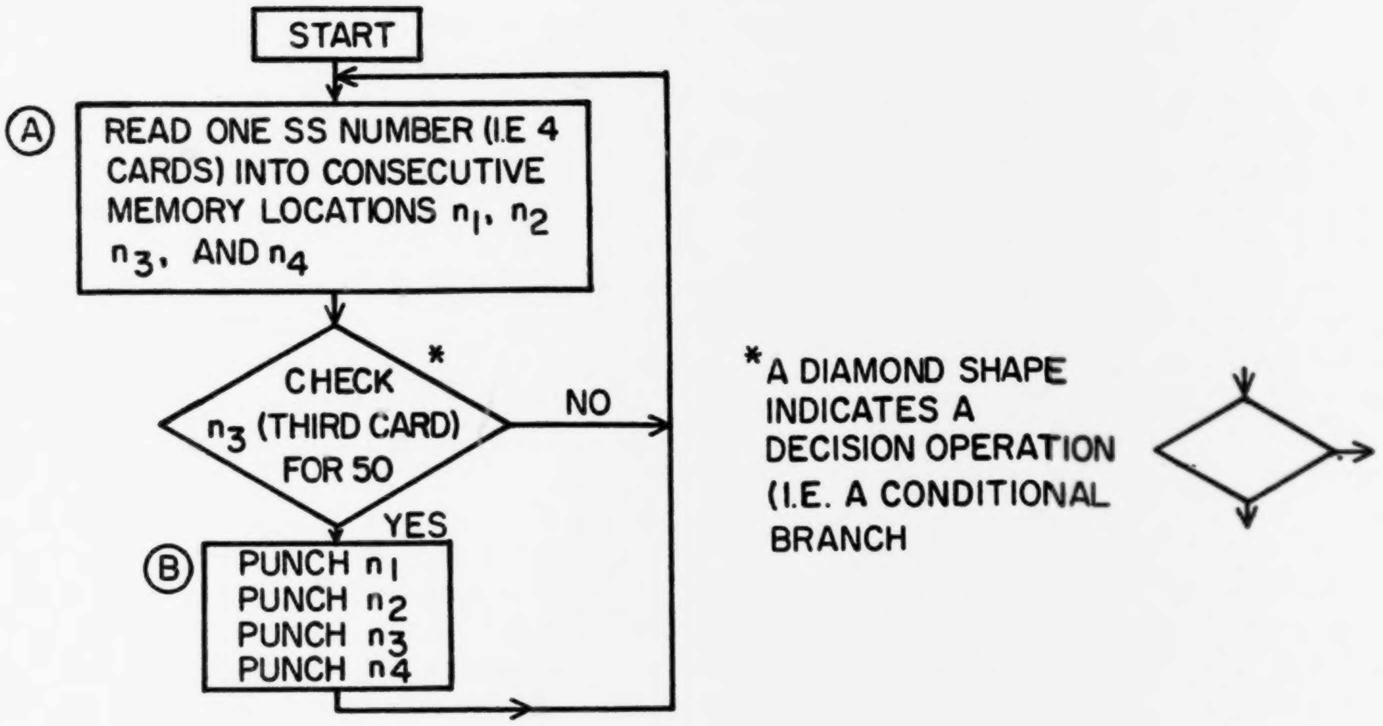

START
Ⓐ
READ ONE SS NUMBER (I.E 4 CARDS) INTO CONSECUTIVE MEMORY LOCATIONS n_1, n_2 n_3, AND n_4
CHECK n_3 (THIRD CARD) FOR 50 *
NO
YES
Ⓑ
PUNCH n_1
PUNCH n_2
PUNCH n_3
PUNCH n_4
*A DIAMOND SHAPE INDICATES A DECISION OPERATION (I.E. A CONDITIONAL BRANCH

Fig. 4 (a) Macro flowchart for Selective Service Call-Up Program if the sign is negative, we know that the original number* in the accumulator was 50 (e.g., $(50-50-1) < 0$). See Fig. 4 (b).

If we arbitrarily decided to store the four cards at locations 60 through 63, the constant 50 in 25, and 001 in 26, the actual coding would appear as in Fig. 4 (c). The constants 50 and 001 could have been read in by the program during execution, but in this case we stored them along with the instructions. Also, when coding the program, the address portions of the conditional branch instructions were left blank until the entire program was written and the addresses to which to branch were ascertained. How would you generalize this program to punch out a list for an arbitrary year, instead of 050?

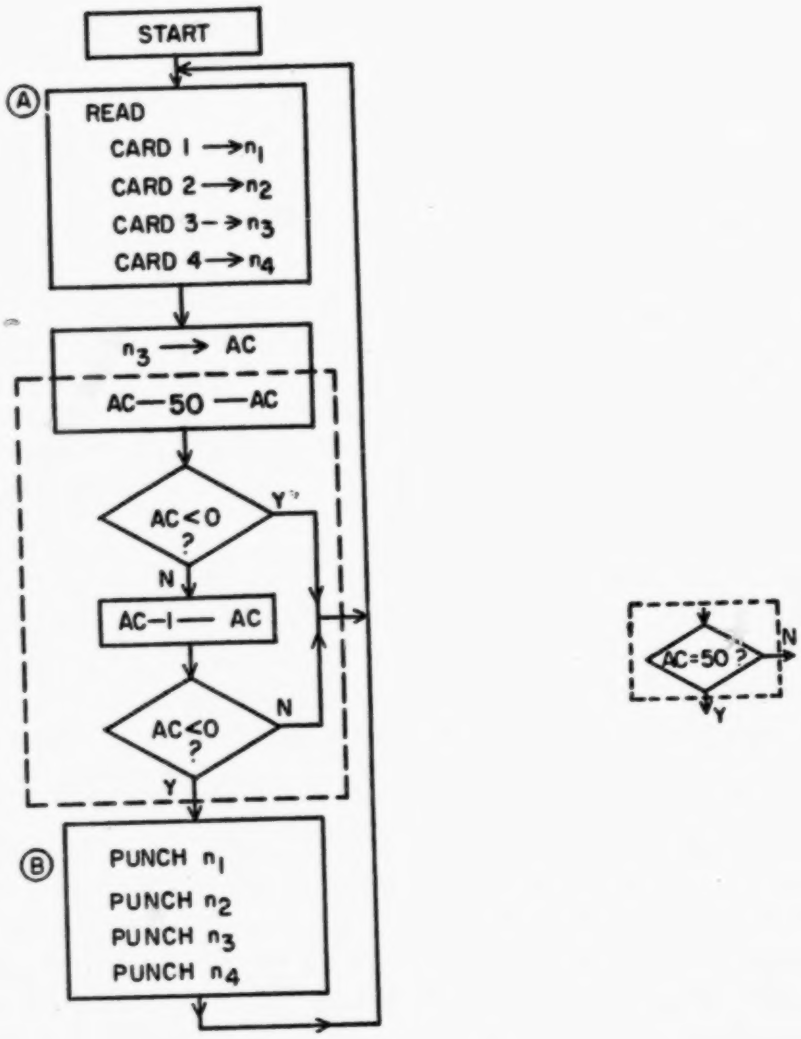

START
A
READ
CARD 1 → n1
CARD 2 → n2
CARD 3 → n3
CARD 4 → n4
n3 → AC
AC — 50 — AC
AC<0 ?
Y
N
AC—1— AC
AC<0 ?
N
Y
B
PUNCH n1
PUNCH n2
PUNCH n3
PUNCH n4
AC=50 ?
N
Y

Fig. 4 (b) Micro flowchart for Selective Service call-up program

*An easier to understand, though less efficient method for checking equality would be to ask you where two successive "no's" indicate equality of x and y.

Address	Instruction	
⋮		
10	060	Read SS number
11	061	
12	062	
13	063	
14	162	Load year to test against
15	725	If less than 050 read another SS number
16	310	
17	726	If 050, go to "punch this SS number..."
18	320	
19	810	If not, read another SS number
20	560	Punch this SS number, then begin again
21	561	
22	562	
23	563	
24	810	
25	050	Constants
26	001	
⋮		

Fig. 4 (c) Coding for Selective Service call-up program

7. Some Exercises Using Simple Loops and Tests

We now turn to three additional sample problems, each of which involves multiple decisions and a read loop. They have been programmed down to the level of micro flowcharts, but the purely mechanical jobs of memory assignment and coding is left as an exercise. The flowcharts are annotated when new ideas are presented.

Morse Code Problem: For our first exercise we shall assume that each input card contains a three-digit-plus-sign number that represents a single letter in Morse Code, where + stands for 'dot', - for 'dash', zero for 'dot' and one (1) for 'dash'. Furthermore, digit positions unused by Morse Code are filled with nines (the "letters" all start on the left with the sign). Thus the letter C is represented as $\overset{-}{-}\ \overset{\cdot}{0}\ \overset{-}{1}\ \overset{\cdot}{0}$, and the sequence of letters SOS would appear as $\overset{\cdot}{+}\ \overset{\cdot}{0}\ \overset{\cdot}{0}\ 9$, $\overset{-}{-}\ \overset{-}{1}\ \overset{-}{1}\ 9$, $\overset{\cdot}{+}\ \overset{\cdot}{0}\ \overset{\cdot}{0}\ 9$. To scan a message encoded on cards for the sequence "SOS", we must check every card until we find an "S" and then check the next card for "0". If the next card is not "0", we begin scanning for an "S" again; otherwise we look for the second "S", etc. For each card that is not part of an "SOS" sequence, we will punch 000; for an "S0S" sequence, we punch 999, and return to look for another sequence. (Thus for the sequence SOT we don't punch for the first two cards but punch three times in a row when the third letter is encountered to make up for the two missed ones.) You may wonder why we punch 000 at all - it is included only to make the problem slightly more interesting. Could you suggest something useful to punch instead?

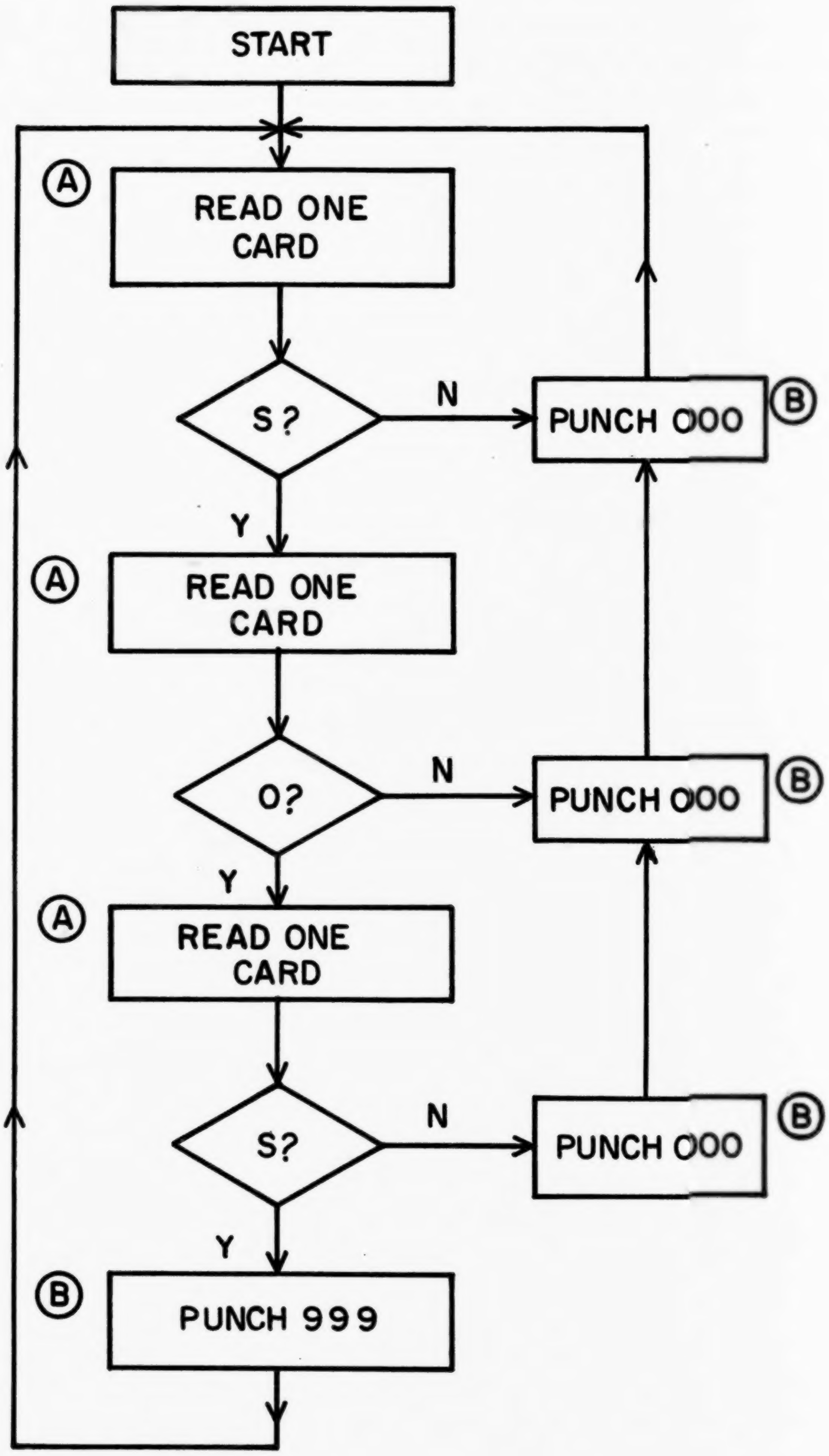
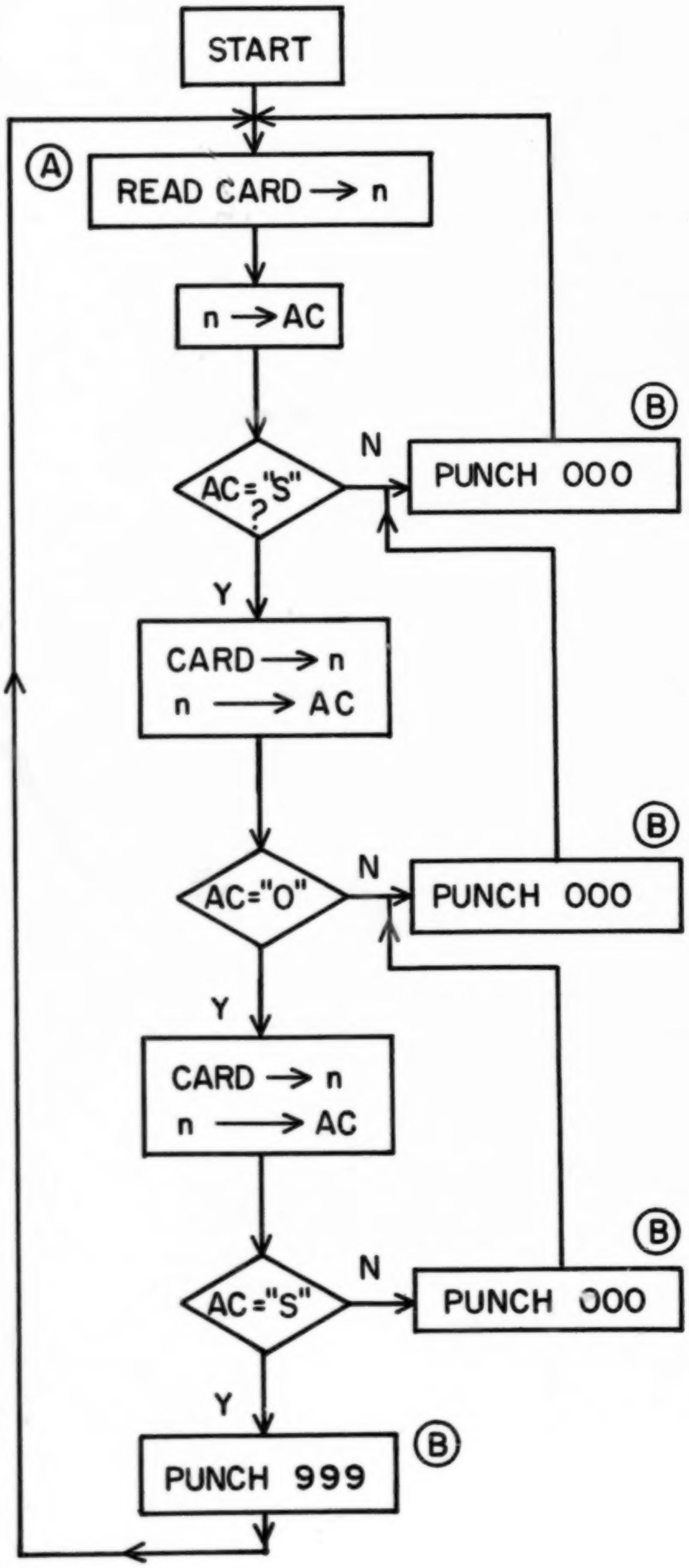
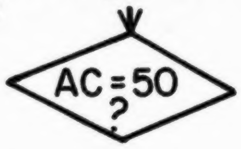

START
Ⓐ
READ ONE CARD
S?
N
PUNCH 000
Ⓑ
Y
Ⓐ
READ ONE CARD
O?
N
PUNCH 000
Ⓑ
Y
Ⓐ
READ ONE CARD
S?
N
PUNCH 000
Ⓑ
Y
Ⓑ
PUNCH 999

Fig. 5 (a) Macro flowchart for Morse Code recognizer program

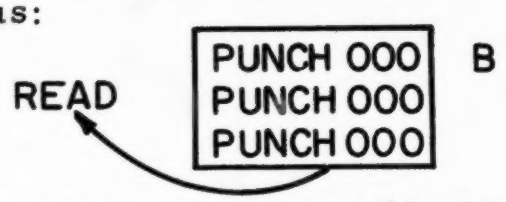


Note 1. While the decision boxes (the diamond-shaped boxes) are not strictly micro flowchart boxes, their micro expansions are similar to that of



in Figure 4 (b) of the call-up problem, and are omitted for the sake of brevity.

Note 2. There are four simple loops in this program, each back to the read. Notice that the "No" exit on both the second and third tests punches a card and jumps to another section of coding (already used at another part of the program). This section of coding, similarly, punches and jumps to yet another section of coding. We could have coded the third exit, for example, to punch three cards and jump directly to the read, thus:



However, the present "double-duty" use of instructions by different parts of the program saves instructions and is therefore more efficient.

Note 3. We have previously encountered simple read loops that terminate program execution upon reading a blank card or on an empty stack. In contrast, each of the three inner loops in this program (the sections of code that search for the first 'S', the 'O', and the second 'S' respectively) is terminated (completed) as a result of an arithmetic operation. However, program execution continues in this case (i.e., when the first 'S' is found by the first loop, the program exits to the second loop to check for an '0', etc.).

Fig. 5 (b) Microflowchart for Morse Code recognizer program

Largest of Three Numbers Problem: To determine the largest of three numbers we must, of course, arithmetically compare the three and discard the two smaller ones. Our straightforward method is to compare two (by subtracting), to choose the larger, and then to compare that one to the third number. We will simplify the problem by assuming that overflow (described in section 5b) will not occur (e.g., careful inspection of the process shows that for A-B where B is a large negative number, overflow may occur; this would increase the problem's complexity greatly). For each group of three numbers compared we will then punch out the largest.

Note: Even though we only test for one number to be less than another, the algorithm is valid if two numbers are equal. In this case, it does not matter which one we choose of two equal quantities.

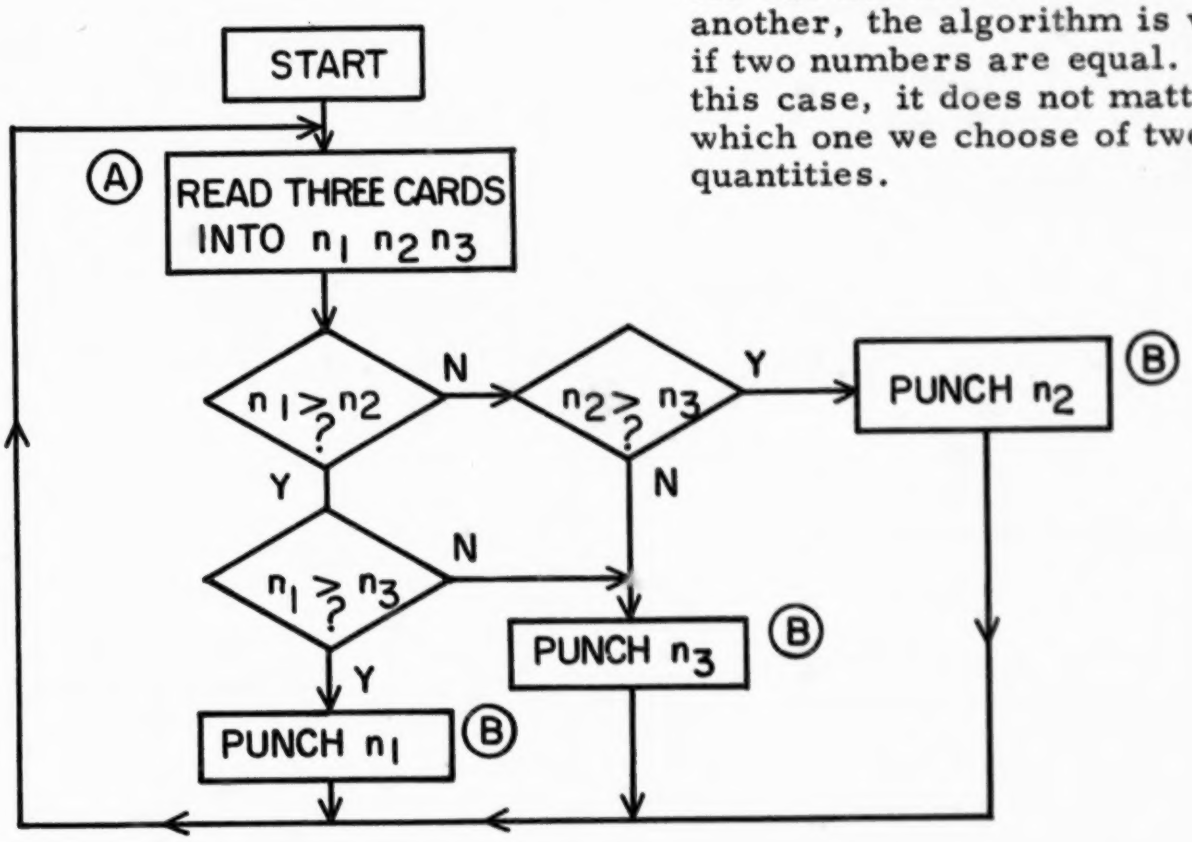


Fig. 6 (a) Macro flowchart for largest-of-three-numbers problem.

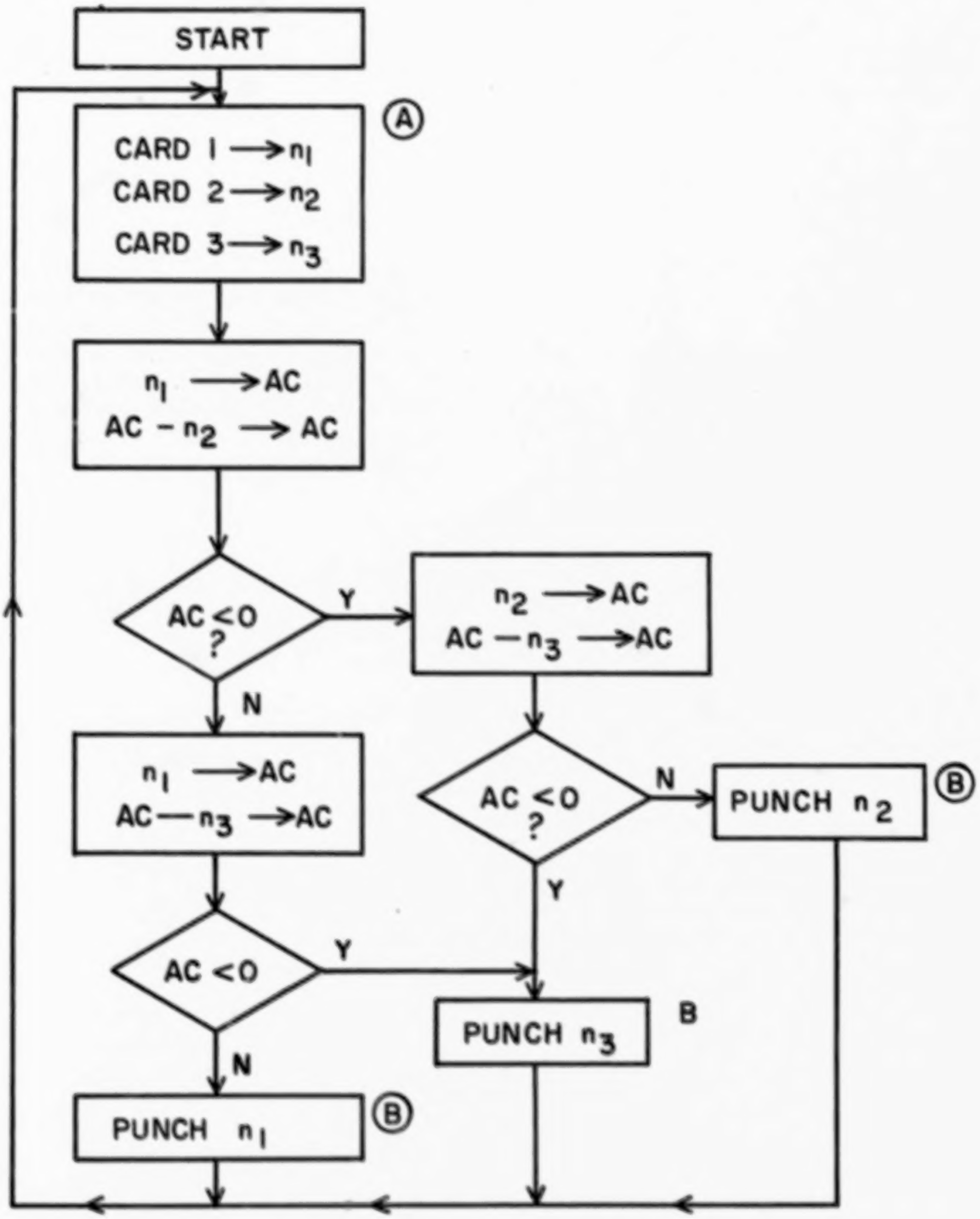

START
A
CARD 1 → n1
CARD 2 → n2
CARD 3 → n3
n1 → AC
AC − n2 → AC
AC < 0 ?
Y
N
n2 → AC
AC − n3 → AC
n1 → AC
AC − n3 → AC
AC < 0 ?
N
PUNCH n2
B
Y
AC < 0
Y
PUNCH n3
B
N
PUNCH n1
B

Fig. 6 (b) Micro flowchart of largest-of-three-numbers problem

Card game: In a card game called 21, a "player" and a "dealer" are originally dealt two cards. They then elect to receive more cards, aiming to get the sum of the values of the cards as close to 21 as possible, without exceeding 21. The player or dealer may receive as many cards as he wants but must report if his total exceeds 21. The player competes with the dealer and the one whose total is closer to 21 wins. In case of a tie, the dealer wins. Our last exercise for coding is to "simulate" the action of the dealer (not the entire game) on a computer. We must keep in mind the basic rules of the game, as well as make certain assumptions that will allow us to represent the necessary "props", e.g. playing cards, etc., in machine-readable form. As a simplification, we first assume that an ace, represented by 001, can be only a "one" card (and not an "eleven" card, as is usually allowed). Next, number cards are represented on punched cards, one playing "card" per card, as 002 through 010, and all face cards (jack, queen, king) are represented by 010. For this example, we will assume that the player has stopped receiving cards as he is content with his total which is under 21. Our program ("dealer") will read ("draw") cards, one at a time, adding as it goes. Of course, the dealer must know when to stop reading cards, so we will arbitrarily choose a total of 17 as our stopping point (e.g., when the dealer's total is equal to or greater than 17, no more cards will be drawn). We shall assume that the very next card on the input stack contains the player's total (our deck has been carefully stacked!). After our "dealer" stops "drawing", we will compare the dealer's total and player's total and punch 000 if the dealer wins and 999 if he loses. We shall then begin the game again.

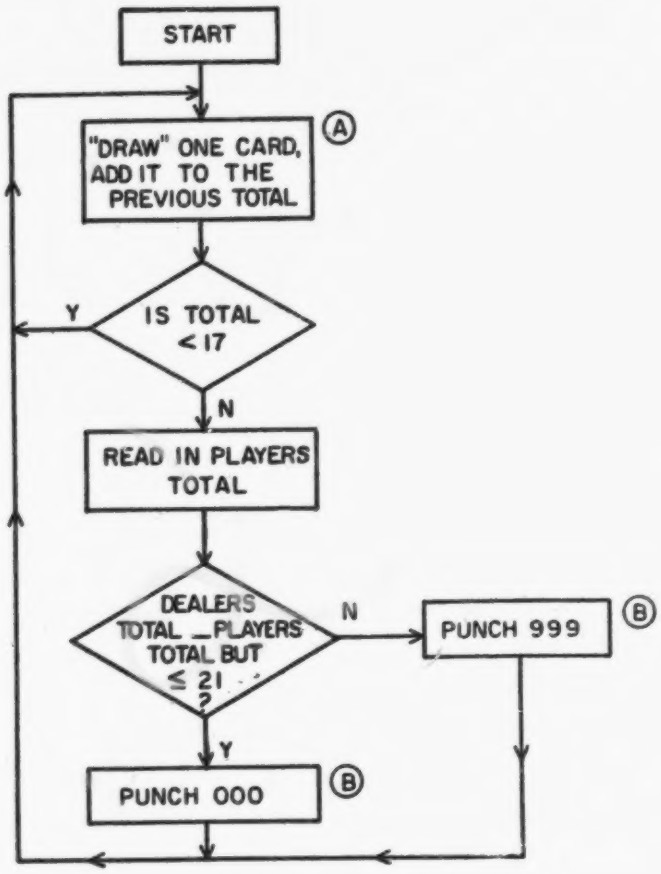
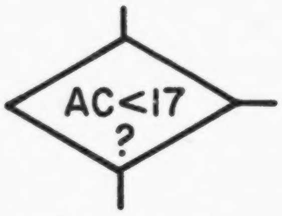

START
"DRAW" ONE CARD, ADD IT TO THE PREVIOUS TOTAL
Ⓐ
IS TOTAL < 17
Y
N
READ IN PLAYERS TOTAL
DEALERS TOTAL _ PLAYERS TOTAL BUT ≤ 21 ?
N
PUNCH 999
Ⓑ
Y
PUNCH 000
Ⓑ

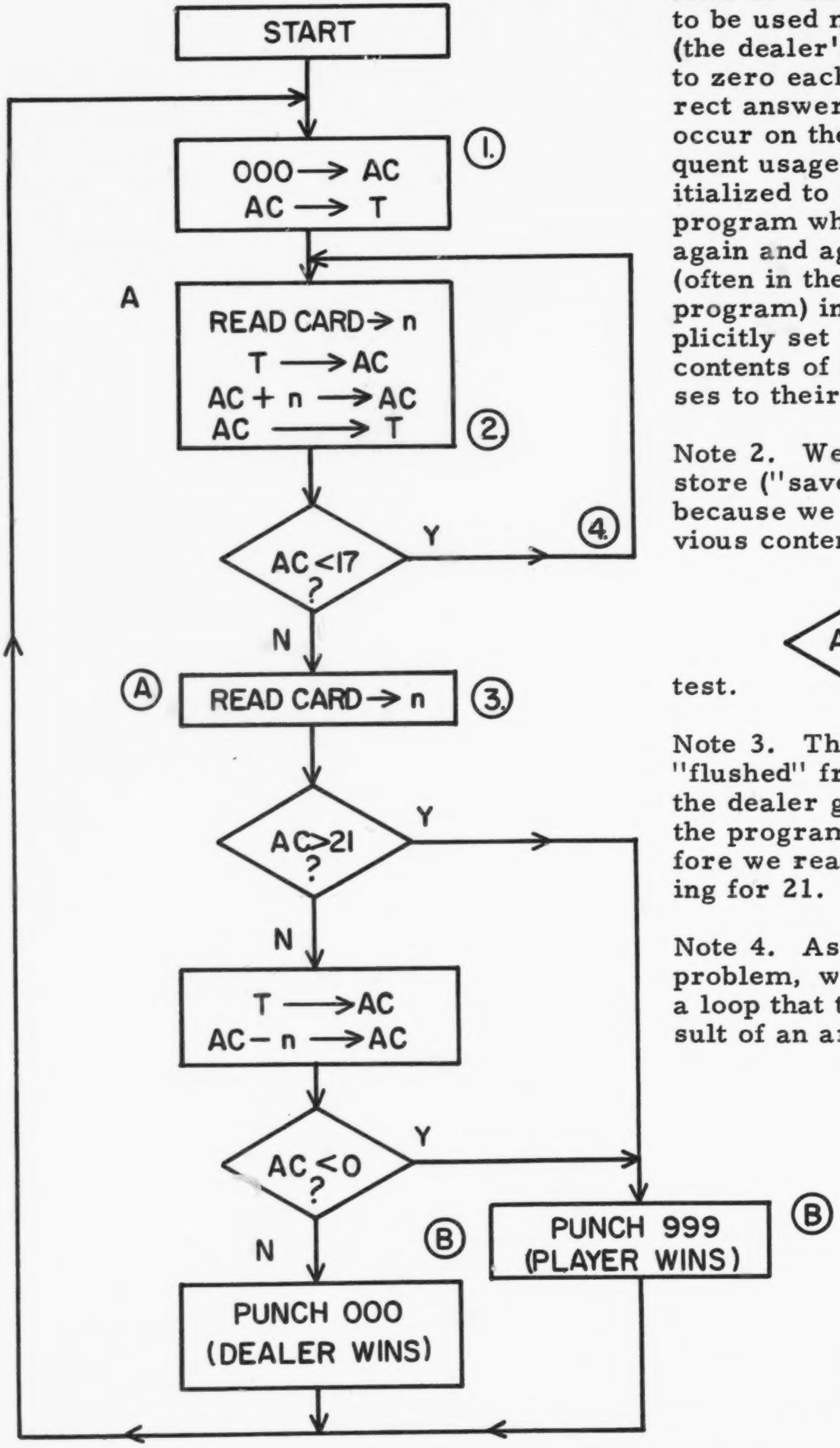
Fig. 7 (a) Macro flowchart for the dealer program

Note 1. Since the program is to be used more than once, T (the dealer's total) must be set to zero each time, since incorrect answers would obviously occur on the second and subsequent usages if T were not initialized to zero. In general, a program which is to be used again and again should contain (often in the early part of the program) instructions which explicitly set (or initialize) the contents of appropriate addresses to their desired values.

Note 2. We must temporarily store ("save") the accumulator because we will destroy its previous contents in performing the  test.

Note 3. This card must be "flushed" from the stack even if the dealer goes over 21, since the program must restart. Therefore we read it even before checking for 21.

Note 4. As in the Morse Code problem, we have an example of a loop that terminates as the result of an arithmetic test.



START
①
000 → AC
AC → T
A
READ CARD → n
T → AC
AC + n → AC
AC → T
②
AC<17
?
Y
④
N
Ⓐ
READ CARD → n
③
AC>21
?
Y
N
T → AC
AC − n → AC
AC<0
?
Y
N
Ⓑ
PUNCH 999
(PLAYER WINS)
Ⓑ
PUNCH 000
(DEALER WINS)

Fig. 7 (b) Micro flowchart for the Dealer program.

8. Indexed Loops: m x n

In section 5 we saw that the computer multiplies a number by three by adding the number to itself twice, i.e., it multiplies by repeated addition. We saw, too, that the computer could be programmed to go back and repeat the cycle using another number fed in from a new input card. The looping technique can be extended, for example to the problem of multiplying two arbitrary numbers, n and m, by repeated addition.

In studying this program, it may help to keep in mind some broad requirements on what the computer must do. The computer is required to add n to itself m-1 times. Following each cycle of operations, the partial sum is increased by n, while the number of further additions required is decreased by one. When the quantity m-1 becomes negative, the adding process is complete and the computer tests m-1 to see if it is positive or negative, each time through the loop.

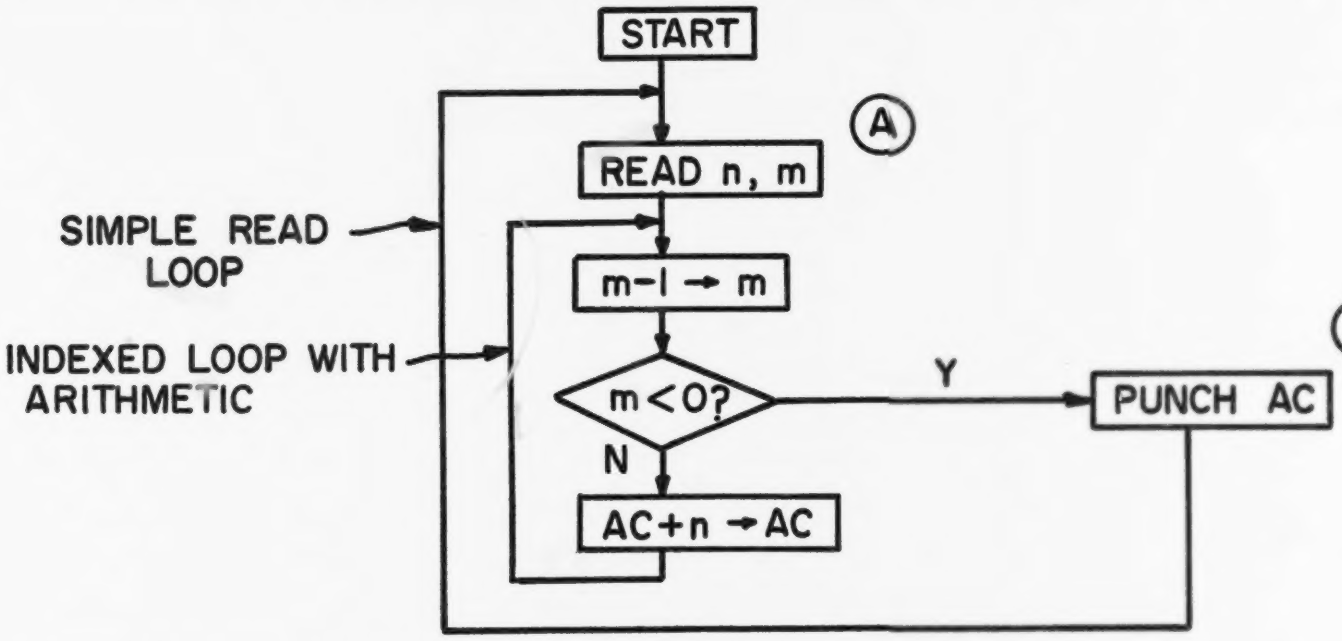

START
Ⓐ
READ n, m
SIMPLE READ LOOP
m-1 → m
Ⓑ
INDEXED LOOP WITH ARITHMETIC
m<0?
Y
PUNCH AC
N
AC+n → AC

Fig. 8 (a) Macro flowchart for program to multiply m x n

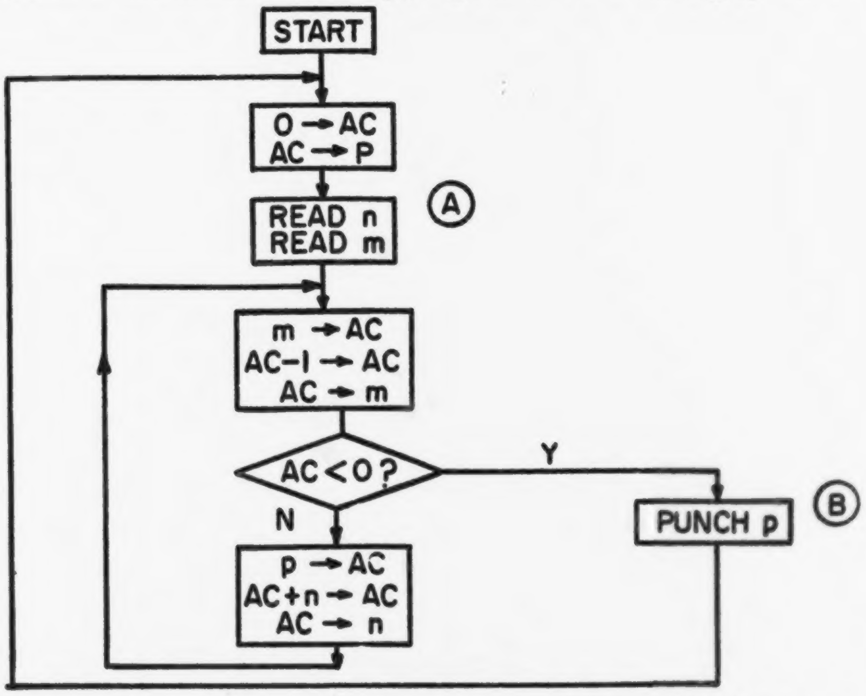

START
0 → AC
AC → P
READ n
READ m
Ⓐ
m → AC
AC-1 → AC
AC → m
AC<0?
Y
PUNCH p
Ⓑ
N
p → AC
AC+n → AC
AC → n

Fig. 8 (b) Micro flowchart for program to multiply m x n

	Memory Address Location	Stored Word	Comments
Initialize	07	125	Copy 000 in accumulator
	08	621	Copy 000 from accumulator into address 21 (p)
	09	022	Copy n from top card into address 22
	10	023	Copy m from next card into address 23
	11	123	Copy m from address 23 into accumulator
Update and test index	12	724	Subtract word 001 in address 24 from m in accumulator. This gives m-1.
	13	623	Copy m-1 from accumulator into address 23
	14	319	Test accumulator contents m-1. If it is 0 or positive, proceed to next instruction 15; if it is negative, jump out to address 19.
Do addition, computation; loop back	15	121	Copy word from address 21 into accumulator (000 initially)
	16	222	Add n from address 22 into accumulator
	17	621	Copy accumulator into address 21 to save partial sum prior to test
	18	811	Go back to address 11 and repeat (looping)
Clean up	19	521	m-1 was negative - therefore finish by copying word in address 21 on output card
	20	807	Loop back to beginning and repeat program for new input
	21	(p)	
Data and storage	22	(n)	Note: after each repeat operation, the number in address 21 increases by n; that in address 23 decreases by 1.
	23	(m)	
	24	001	
	25	000	

Fig. 8 (c) Coding for program to multiply m by n

(Questions: Is it necessary to restrict the value of m to be no larger than 9? Is the order of the two input cards with numbers n and m significant, or may the position of these two cards be interchanged?)

The variable m associated with the inner loop (i.e., not the read loop) is called an index (or often "counter" or "clock"), since it is changed each time the instructions in the loop are executed, and controls the termination of the loop. Thus, m, as one of the multipliers, was not used directly in an arithmetic operation, but indirectly as an index, controlling arithmetic operations.

Terminating an indexed loop (i.e., exitting from it) is usually done by testing the index to determine when it becomes larger or smaller than some specified value. In our program the loop was terminated when the index became negative. The position of the test in the program is usually flexible. Could we have interchanged the test and the addition of n? Would any other changes in the program be required?

The initialization of indices for indexed loops, and the problem of testing indices to determine exactly when exits should be made from loops are among the most vital topics in computer programming. We shall meet with them again. A symbolic representation of a generalized indexed loop process is demonstrated in the flowchart below (Fig. 9).

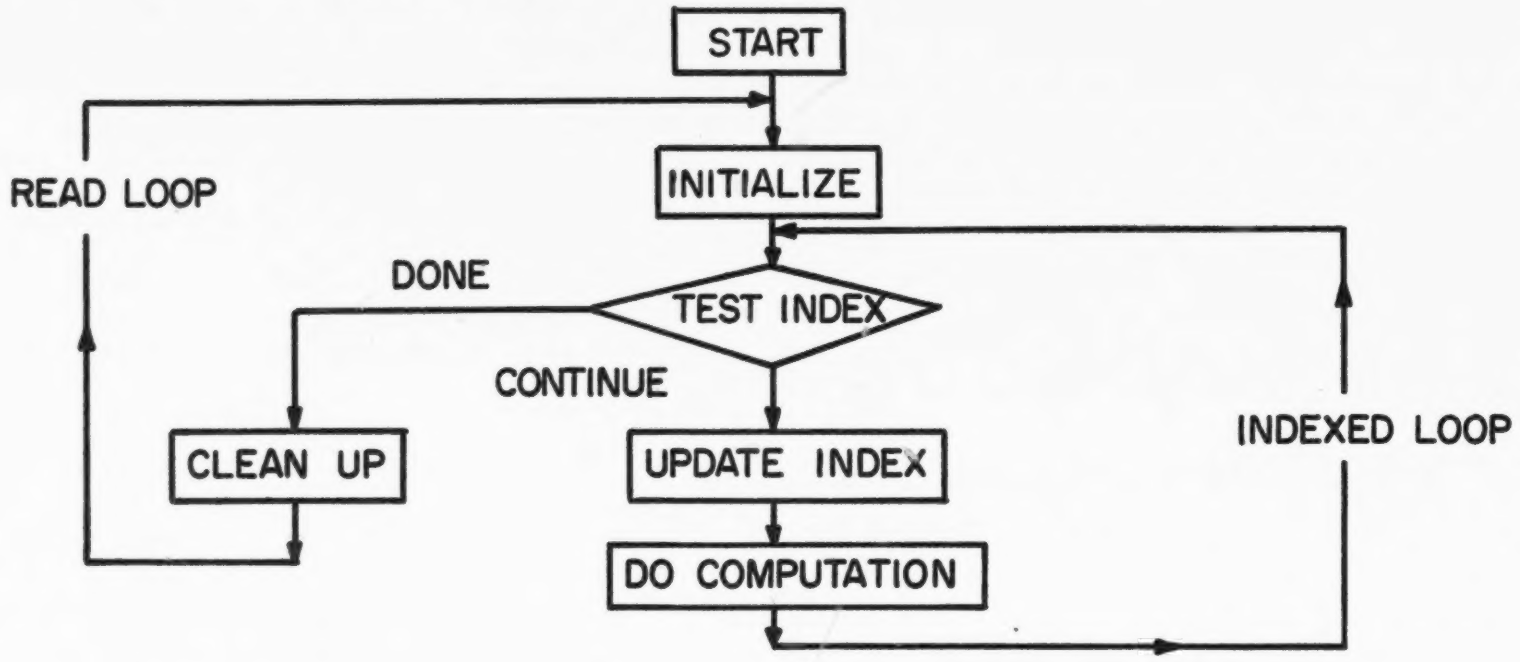


Fig. 9 Flowchart illustrating indexed looping

One other important principle shown in this multiplication example is that we can "simulate" through suitable programming any operation (in this case, multiplication) which is not built into our basic computer. We will come back to this crucial point also.

As an exercise, you should write a micro flowchart and code for finding the sum of the first n positive integers ($S = n + (n-1) + (n-2) + \ldots + 1$). Read n from a card and use an indexed loop in which the index is added to the partial sum.

9. Another Data-Processing Problem: Shifting

As we saw in several previous examples, computers need not be used to perform exclusively arithmetic tasks. One non-numerical job which then can do is the changing of the symbolic form of the representation of data. Take the problem of the principal of a school who has decided to change the grading system at that school. He has been using a three-digit numbering system for cards in his file. The first two decimal digits were a code which told which student the card referred to. The last digit represented a student's grade in a new experimental course on the programming of computers. Formerly, an A recorded as "4", B as "3", C as "2", D as "1" and E as "0". For instance, a card containing "793" meant that student "79" had received a B in the course. He wishes to changes his system of recording grades to one in which A is "5", B is "4", C is "3", D is "2" and E is "1".

At the same time that he makes these changes he wants to find out how many A students there are. Finally, he wishes to change the form of the data on the cards so that the first digit will be the new grade, and the last two digits will be the student code-number. For instance, the card which contained the number "793" should be changed to a card which is to contain the number "479"; this card should not be included in the count of those students who earned an A. (What should be done with a card containing, originally, "394"?)

A student in the programming course has offered to write a program which will take the original cards as input, and which will produce the new set of cards. The new cards which are produced are to be followed by one more card which will tell how many A's there are. This cannot be confused with the other cards because there are fewer than 100 students in all and the first digit on this last card will therefore have to be 0.

The set of original input cards will be folowed by a blank card so that the computer will know where the end of the set is. This blank card will be followed by one other control card containing the number 855. The purpose of the last two cards will become evident in the discussion below.

The Flowchart and Program: In Fig. 10 is a flowchart (somewhat a mixture of a macro and a micro flowchart) which will aid in the explanation of the student programmer's plan. He has called the three digits on a typical input card, j, k, and g. The first two are the student's code number and the last is the grade on the original numerical scale. He first sets the contents of a location called "count" to 0. After bringing the contents of the first card to the accumulator he shifts them so as to place the grade g in the leftmost position. (A detailed explanation of the shift instruction is given in Table 3). By adding 100 to the result he creates the new grade g + 1 (followed by two 0's), and stores it for future use. By subtracting the number 500 from this result he creates a difference which is negative if the student's grade was B or lower, and which is zero if the student's grade was A. By testing this difference he decides whether the count should be increased by 1.

Notice carefully that the arrow which bypasses the step "add 1 to count" does not represent a loop. It only indicates that certain instructions may be omitted under some circumstances.

By bringing the original input card contents to the accumulator again and shifting them he creates the number (0, j, k) which has the student code-number in the correct new position. This number, added to the previously stored (g+1, 0, 0), gives the desired data in the new format. This is copied onto an output card and the entire procedure is repeated.

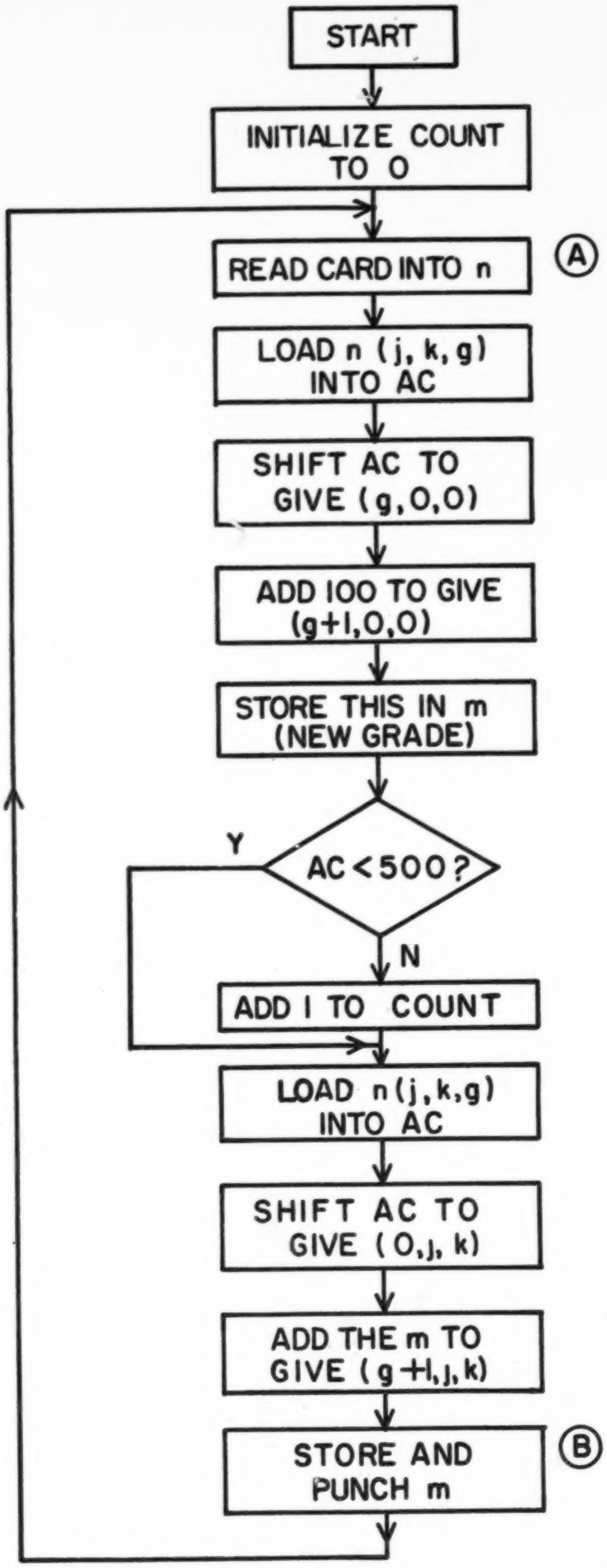

START
INITIALIZE COUNT TO 0
READ CARD INTO n
Ⓐ
LOAD n (j, k, g) INTO AC
SHIFT AC TO GIVE (g, 0, 0)
ADD 100 TO GIVE (g+1, 0, 0)
STORE THIS IN m (NEW GRADE)
AC < 500 ?
Y
N
ADD 1 TO COUNT
LOAD n (j, k, g) INTO AC
SHIFT AC TO GIVE (0, j, k)
ADD THE m TO GIVE (g+1, j, k)
STORE AND PUNCH m
Ⓑ

Fig. 10 A flow chart for modifying grade records

Instruction (XYZ)	Meaning of the Instruction	Final Contents of the Accumulator (symbolized as + abc initially)
400	no shift	± abc
401	Shift one to right	± 0ab
402	Shift two to right	± 00a
403	Shift three to right	± 000
410	Shift one to left	± bc0
420	Shift two to left	± c00
430	Shift three to left	± 000
411	Shift one to left and one to right	± 0bc
412	Shift one to left and two to right	± 00b
421	Shift two to left and one to right	± c00
422	Shift two to left and two to right	± 00c

Table 3 How shifting affects accumulator contents

Note 1: Digits shifted out of the accumulator are lost, and only zeros can be shifted in.

Note 2: 430 cards can be used to put ± 000 in the accumulator as an alternative to bringing it in from a storage location. (Assume that + 000 and - 000 are identically + 000.

In this program the indexed loop is not terminated by testing the value of an index against some other quantity, but rather by the reading of a blank card. By terminating the program in this manner, the student has created a problem, because the computer will stop before the number which he has called count has been printed out. We will see how this problem is solved.

The Purpose of the Control Cards: We have traced the execution of the program up to and including the point where the computer tried to read the blank input card which marked the end of the set of students' grade cards. At that point the computer stopped with its instruction set to 00. If the "run-stop" switch is again switched to "run" the first instruction which will then be executed will be the one at address 00.

Now remember that our computer has the property that the permanent contents at address 00 are + 001 (refer to Table 1). If the computer executes this instruction it will copy the top input card (the programming student has written the number 855 on this next card) into memory address 01. When, at the next step, the instruction at 01 is executed, it will be the "control" word 855 which

Address	Word	Comments
37	4 30	Initialize count to 0 by shifting accumulator right three places, leaving zero.
38	6 58	
39	0 57	Copy top input card and
40	1 57	bring it to the accumulator
41	4 20	Shift, to give (g, 0, 0)
42	2 60	Add 100 to it and
43	6 59	store as new grade (g + 1, 0, 0)
44	7 61	Subtract 500 and test result. Go
45	3 49	to address 49 if result is negative.
46	1 58	Add 1 to count
47	2 62	
48	6 58	
49	1 57	Bring card to the accumulator again
50	4 01	Shift, to give (0, j, k)
51	2 59	Add new grade, to give (g + 1,j,k) and
52	6 57	store as card
53	5 57	Print out card
54	8 39	Return to copy in another card
55	5 58	Print out count and
56	9 00	stop
57	0 00	(n: card)
58	0 00	(count)
59	0 00	(m: new grade)
60	1 00	constants
61	5 00	
62	0 01	

Fig. 11 The Program for modifying grade records

has just previously been put there. The instruction is interpreted as "jump to address 55 to find the next instruction to be executed". At addresses 55 and 56 we find instructions telling the computer to print out count, and to halt. Thus, with the price of having to restart the computer, the student has made good his offer. A more practical solution would have been to place a special control card, say a + 000, at the end, instead of a blank. Show how you would introduce a test for this last card in the read loop, so that you could then jump to 55 without an intermediate halt.

10. Loading a Program into Memory

We have seen how a program is written. How do we store a program in the memory? This is done by what is known as a loading program.

As an example, let us suppose that we wish to load the program in Fig. 11. The problem is to program the computer so that it will take each instruction word specified and store it in the address indicated. Thus, it must store 403 at address 37. Next it must store 658 at address 38 and so on until the entire program is stored in address 37 to 62, inclusively.

We do two things. We prepare the deck of input cards shown in Fig. 12. We program the computer to perform the necessary sequence of operations shown in Fig. 13. The basic operations are established by means of steps 0, 1 and 2 (address 00, 01 and 02). (We know that in this computer the instruction 001 at address 00 is permanently built into the circuitry of the computer and so cannot be modified.) When the run-stop button is pushed the computer automatically performs the instruction 001 which leads to the setting up of a program able to read each top card, one by one, then store it in the memory. At step 3 the computer begins the storage of the desired program; it keeps on repeating the instructions in addresses 00, 01 and 02 until it reaches address 62 after which the top card is blank so the computer stops. We observe that the preparation of the computer for the loading operation and the actual loading operation is conducted as one continuous sequence.

002
800
037
430
038
658
039
057
etc.

Fig. 12 Deck of input cards for loading the program shown in Fig. 11.

Program step		Memory Location	Instruction word	Action Resulting
0		00	001	Read top card into address 01. (Top card is 002)
1	the program	01	002	Read top card into address 02. (Top card is 800)
2		02	800	Jump back to address 00.
3	2nd time through	00	001	Read top card into address 01. (Top card is 037)
4	the loop	01	037	Read top card into address 37. (Top card is 403)
5		02	800	Jump back to address 00.
6	3rd time through	00	001	Read top card into address 01. (Top card is 038)
7	the loop	01	038	Read top card into address 38. (Top card is 658)

Fig. 13 Loading Sequence

11. Instruction Modification for Indexed Loops

Sum of N numbers: Until now, we have observed the use of simple read and indexed loops for executing sets of instructions repetitively until some special condition is met. We will now illustrate an important programming technique, often used in conjunction with loops, by coding a seemingly trivial problem twice. In the first version, we will read in N cards and sum the numbers contained on them; in the second, we will assume that the numbers to be summed are already stored in N consecutive locations beginning at 15 (we will read only one card in this version - it contains the number N). The distinction between the two versions, summing numbers contained on cards or numbers contained in memory, at first seems rather minor. However, on closer inspection we see that we have no direct method of summing the numbers stored in memory without using N consecutively stored add instructions. We shall resolve this difficulty after we illustrate the first version of the problem. (Both versions will assume, for simplicity, that overflow does not occur.)

For our first program, the first card will contain N and will be followed by N cards. We use the number on this first card to initialize a counter which is "decremented" (i.e., reduced by one) once for every card which is read and added to the previous total. When the counter becomes negative, we punch the total and begin again.

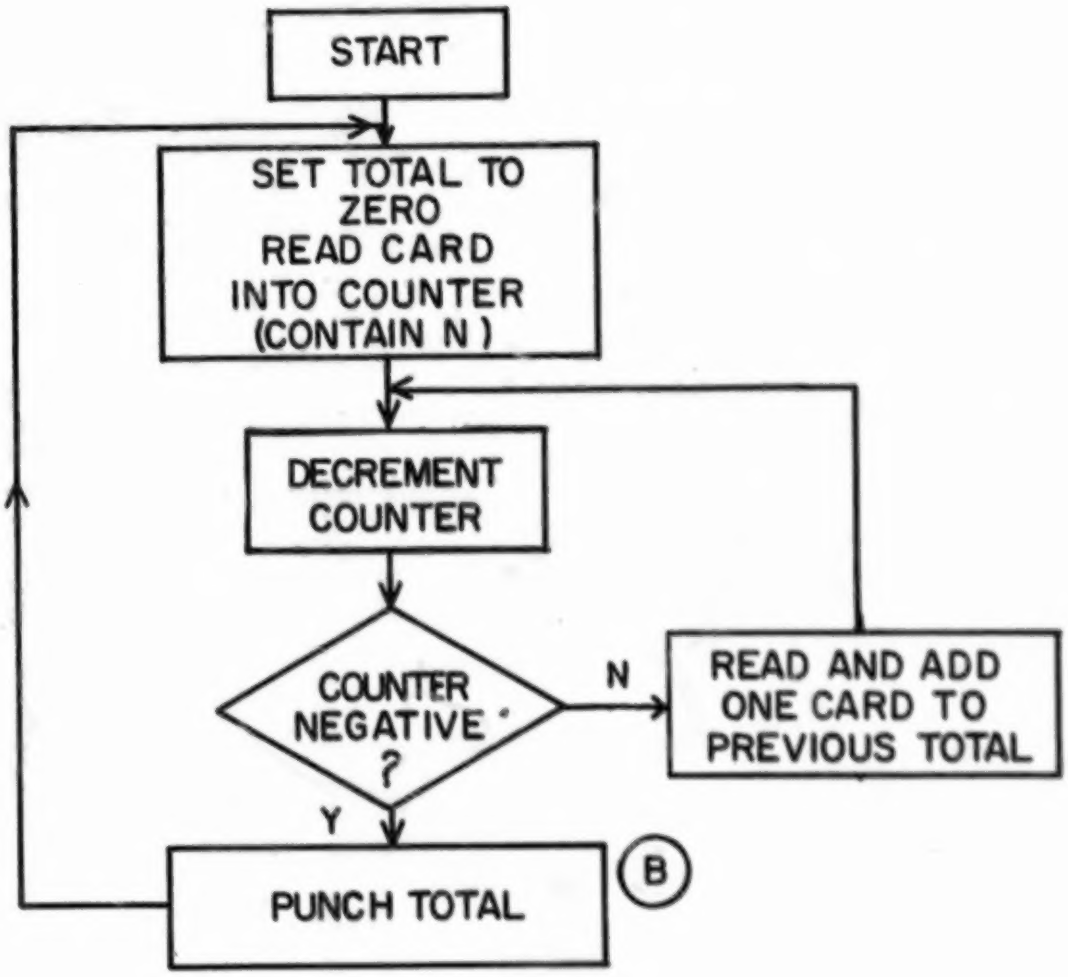


Fig. 14 (a) Macro flowchart for addition-of-N-numbers program

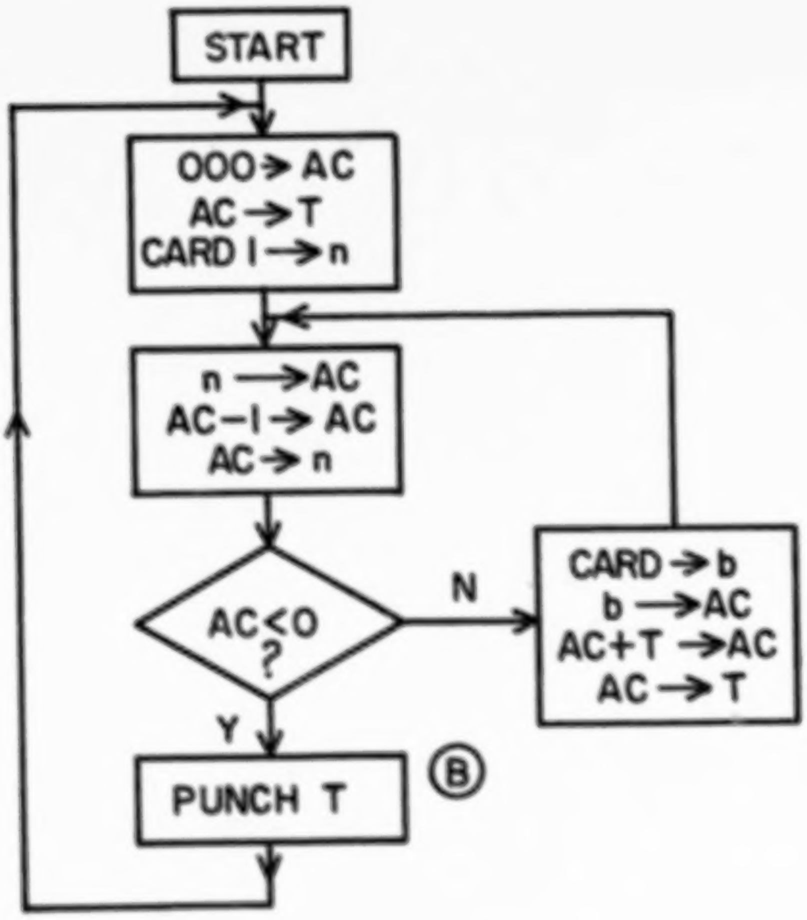


Fig. 14 (b) Micro flowchart for addition-of-N-numbers program

Address	Instruction		
10	430	-	initialize total T to 0
11	625		
12	026	-	read and load N into AC
13	126		
14	724	-	decrement and store n (counter)
15	626		
16	322	-	test n
17	027	-	if n not negative, read another card and add to total
18	127		
19	225		
20	625		
21	813	-	loop
22	525	-	If n negative, punch total; return to beginning
23	810		
24	001	-	constant
25	000		(T - total)
26	000		(n - counter)
27	000		(b)

Fig. 14 (c) Coding for addition-of-N-numbers program

In our second program, we assume that the first card contains N, the number of locations to be added. We see immediately that both brograms are quite similar, each employing an indexed loop (compare the two macro flowcharts).

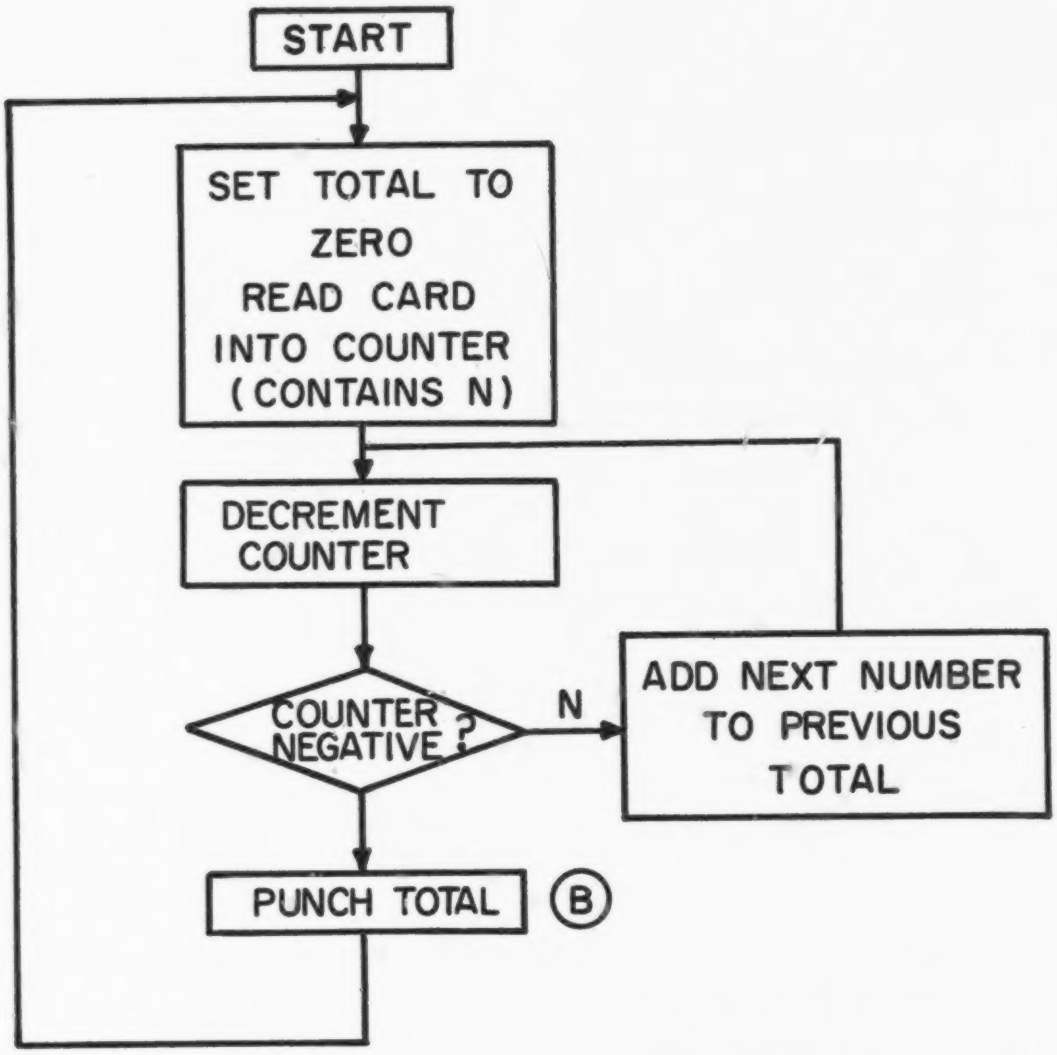
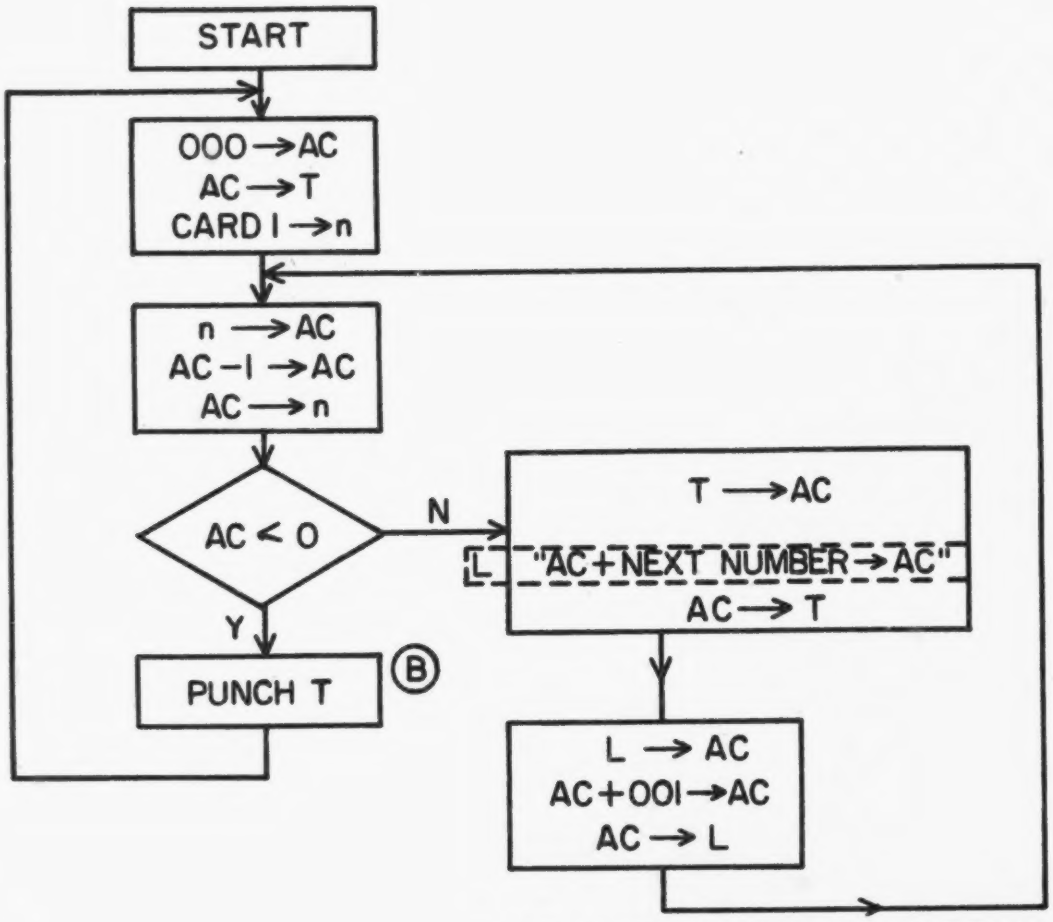


Fig. 15 (a) Macro flowchart for addition-of-N-numbers program using instruction modification

Now, when we try to convert the macro flowchart to the micro flowchart, we encounter the previously mentioned difficulty with the add-next-number instruction. If we wrote out the N consecutive add instructions they would like 215, 216, 217, ..., 2 (15 + N - 1). Note that all of these instructions are quite similar. In fact, each differs from its predecessor only by 001 in the address (i.e., 217 = 216 + 001). Now, remembering that our stored program computer cannot differentiate between data and instructions stored in memory (i.e., both are stored as 3-digit-plus-sign numbers). we can modify the original add instruction 215, in conjunction with an indexed loop, "manufacture" the rest of the add instructions. Our program first executes the 215 instruction, stores the intermediate sum from the AC, and increments and tests the counter. If the counter is not negative, we have more numbers to add, so we load the 215 instruction itself into the AC, add 001 to it to form 216 (the add for the next number); we then re-store this in the location which originally contained the 215, reload the intermediate sum into the AC and branch to the location that contained the 215 (which now has 216 init -- i.e., we now add the next number). Continuing in this manner, we can add all of our numbers. Note that this instruction modification technique enables the program to alter its instructions and stored data itself during execution, and thus adds significantly to the computer's flexibility.


START
000→AC
AC→T
CARD 1→n
n→AC
AC−1→AC
AC→n
AC < 0
N
Y
T→AC
L
"AC+NEXT NUMBER→AC"
AC→T
PUNCH T
B
L→AC
AC+001→AC
AC→L

N.B. L: THE INSTRUCTION BEING MODIFIED

Fig. 15 (b) Micro flowchart for addition-of-N-numbers program using instruction modification

Address	Instruction	
48	403	0→T
49	668	
50	067	read N
51	167	decrement and store counter (n)
52	766	
53	667	
54	362	test for negative counter
55	168	if not negative, load total T
56	215	L: add next number
57	668	and restore total T
58	156	modify instruction
59	266	
60	656	
61	851	loop
62	568	if n negative, punch total and reset add instruction
63	169	
64	656	
65	848	branch to beginning
66	001	Constants
67	000	n (counter)
68	000	T (total)
69	215	

Fig. 15(c) Coding for addition-of-N-numbers program using instruction modification

Factory Quota Problem: Another example of a program using instruction modification and shifting is one which checks the daily production efficiency of a factory with ten departments. Each day, a card is made up for each department in the form ONN, where NN is the expected number of pieces produced. These are read into the computer in ascending order (for departments 0 through 9) and are followed by cards of the form XNN, where NN is as before and X is the department number (0 - 9), indicating the actual output of the department. However, since all departments may not necessarily work on any given day and do not "quit" at the same time, the department cards may be in any order and total less than or equal to ten cards.

After reading and storing the ten quota cards, the program will read one department card at a time, use its department numbers to form a subtract instruction, compare the actual output to the quota, and punch the department number followed by 999 if the quota was met or exceeded or 000 if the quota was not met. Instruction modification will be needed, both for reading the ten quota cards (which also requires an indexed loop) and for comparing the production to the quota.

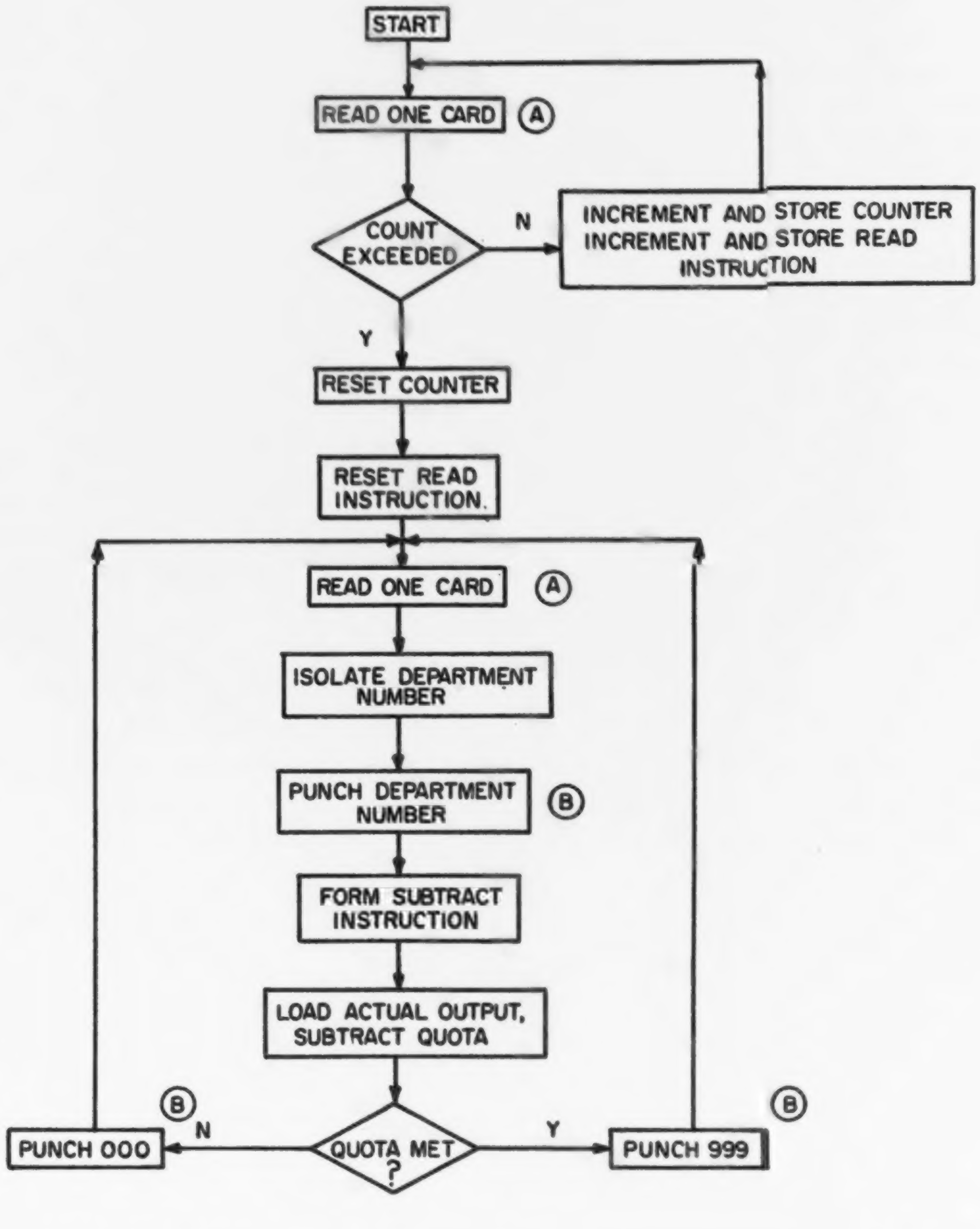
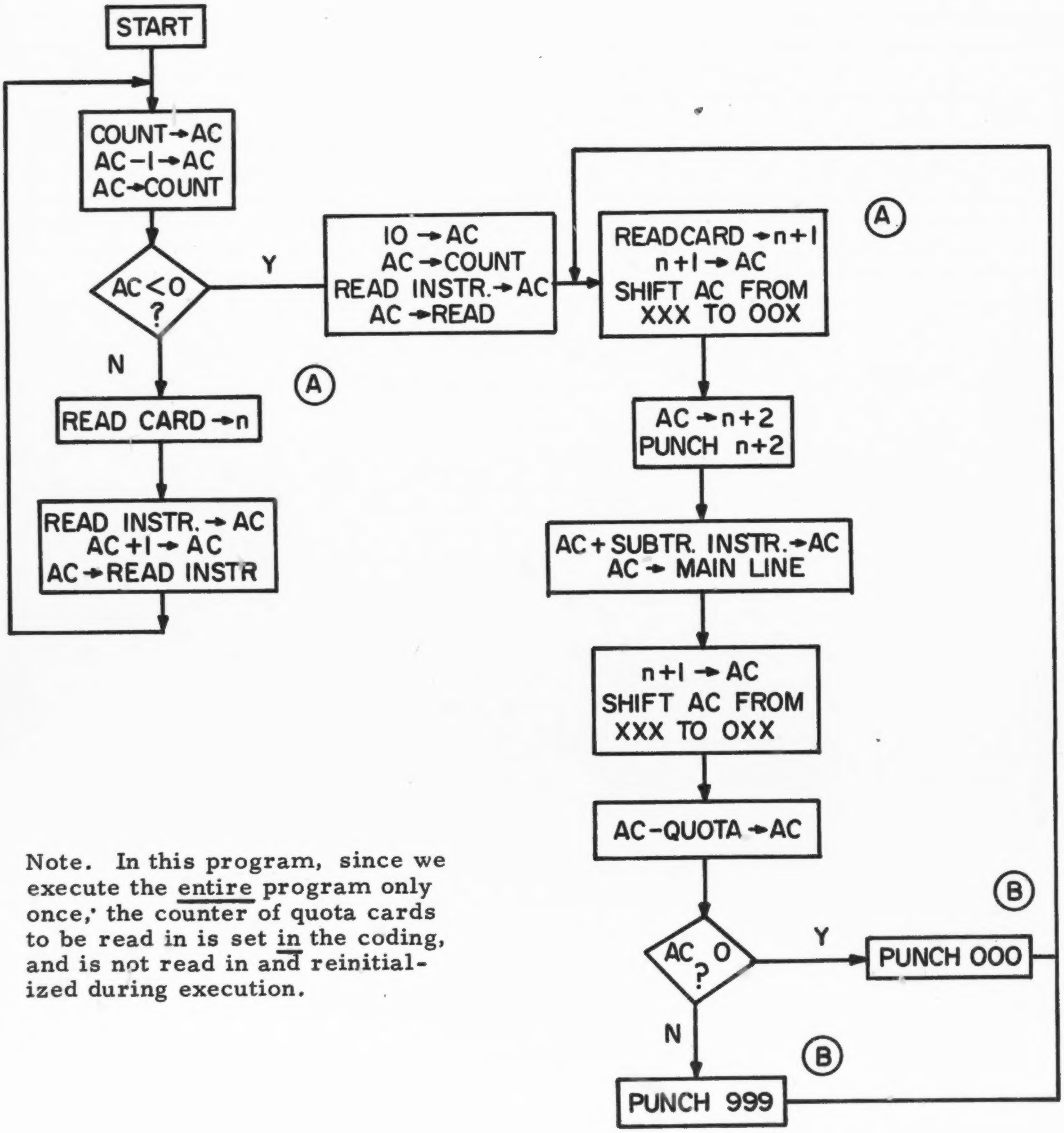

START
READ ONE CARD
Ⓐ
COUNT EXCEEDED
N
INCREMENT AND STORE COUNTER
INCREMENT AND STORE READ INSTRUCTION
Y
RESET COUNTER
RESET READ INSTRUCTION
READ ONE CARD
Ⓐ
ISOLATE DEPARTMENT NUMBER
PUNCH DEPARTMENT NUMBER
Ⓑ
FORM SUBTRACT INSTRUCTION
LOAD ACTUAL OUTPUT, SUBTRACT QUOTA
Ⓑ
N
PUNCH 000
QUOTA MET ?
Y
PUNCH 999
Ⓑ

Fig. 16 (a) Macro flowchart for factory quota problem


START
COUNT→AC
AC−1→AC
AC→COUNT
AC<0 ?
Y
N
10 → AC
AC → COUNT
READ INSTR. → AC
AC → READ
READCARD → n+1
n+1 → AC
SHIFT AC FROM
XXX TO 00X
Ⓐ
Ⓐ
READ CARD → n
READ INSTR. → AC
AC+1 → AC
AC → READ INSTR
AC → n+2
PUNCH n+2
AC + SUBTR. INSTR. → AC
AC → MAIN LINE
n+1 → AC
SHIFT AC FROM
XXX TO 0XX
AC−QUOTA → AC
AC O ?
Y
N
Ⓑ
PUNCH 000
Ⓑ
PUNCH 999

Note. In this program, since we execute the entire program only once, the counter of quota cards to be read in is set in the coding, and is not read in and reinitialized during execution.

Fig. 16 (b) Micro flowchart for factory quota problem

Address	Instruction	
10	148	test counter, decrement and store
11	749	
12	648	
13	319	
14	038	if not exceeded, read card, reset read instruction, return
15	114	
16	249	
17	614	
18	810	
19	151	when exceeded, reset counter and read instruction
20	648	
21	150	
22	614	
23	052	read next department card
24	152	
25	402	isolate department number and punch it
26	653	
27	553	
28	256	form subtract
29	632	
30	152	isolate production quantity
31	411	
32	700	test quota
33	336	
34	555	if met, punch 999
35	823	
36	554	if not met, punch 000
37	823	
38	000	quota storage
39	000	
40	000	
41	000	
42	000	
43	000	
44	000	
45	000	
46	000	
47	000	
48	010	counter
49	001	constant
50	038	read reset
51	010	counter reset
52	000	(dept. card storage)
53	000	(punch storage)
54	000	output
55	999	
56	738	subtract instruction

Fig. 16 (c) Coding for factory quota problem

12. Program Segmentation by Subroutines

Suppose you are writing a program that must perform a certain operation several times, but at different times during the processing; for example, taking (A x B) + C - (D x E) requires two separate m x n-type multiplication operations. Rather than writing the entire multiplication coding each time we need it, we could make our program shorter if we could make just one coding sequence perform "double duty" for us, as we did with the Punch instructions in the Morse Code problem. And, indeed, we can do this by utilizing our branch instructions. Note the programming sequence of Fig. 17 (a).

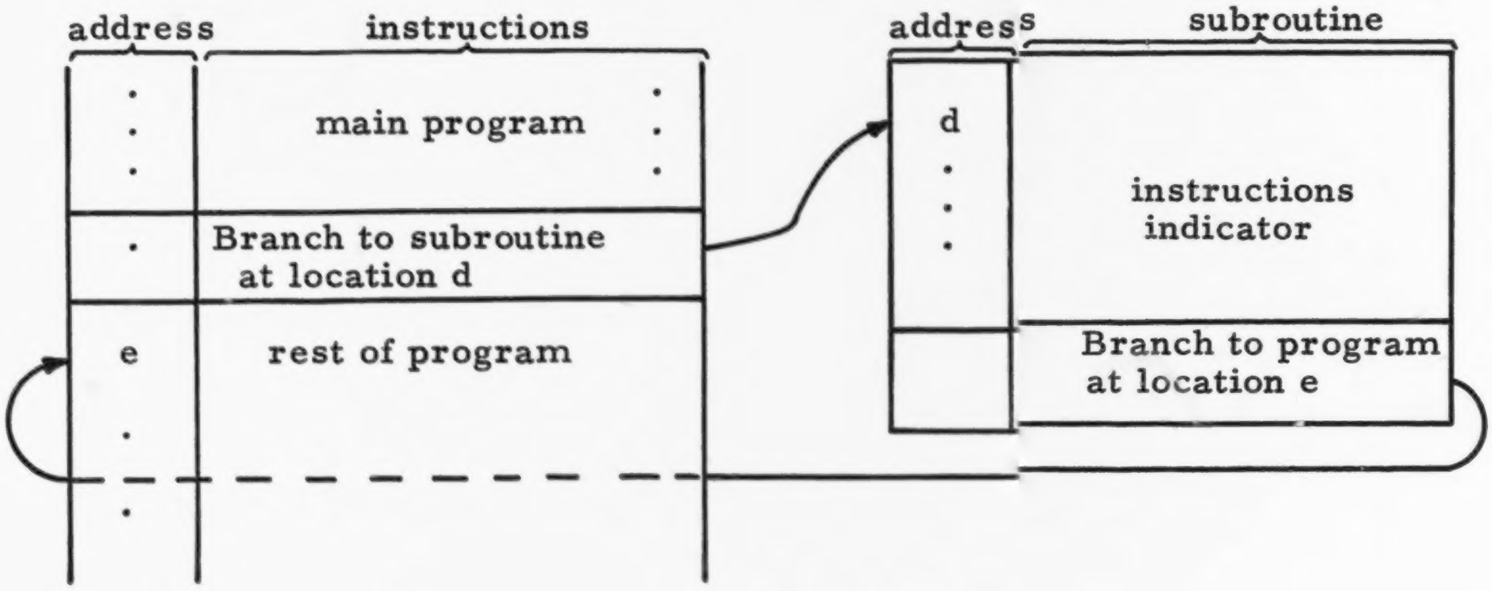


Fig. 17 (a) Programming sequence for subroutine use

Of course, we want our subroutine to multiply different pairs of numbers each time we use it, and we want to be able to branch to it from different locations in our main program. We thus have two problems: how to pass data to and from the subroutine, and how to specify the location in the program to which we want to branch after the multiplication is completed. The first difficulty is solved by using scratch areas within the subroutine. We write the subroutine so that it always multiplies the numbers stored at scratch locations a and b and places the result in scratch location c. Thus, before the main program branches to the subroutine, it must store the two numbers to be multiplied into a and b. After the subroutine branches back to the main program, the result can always be located by the main program - the subroutine always stores the result in address c.

The second problem is eliminated by the use of the 8YZ branch instruction at location PQ in the main program (see Fig. 17 (b)). As mentioned earlier, this instruction places the contents of the instruction counter (PQ + 1, i.e., the location of the next sequential instruction in the program) into digits 2 and 3 of location 99 (which always has an 8 in the first digit). It then reloads the instruction counter with YZ (causing a branch to location YZ). Thus, we enter our subroutine at YZ and we exit from the subroutine, to branch back to the main program, by branching to location 99. This cell contains an instruction of the form 8 (PQ + 1) in it, which will, in turn, cause a "return branch" to the correct location in the main program. In general, it is a good idea for the subroutine to copy the contents of location 99 within itself, since the subroutine might want to branch to a subroutine of its own (and would thus need location 99 for its own return branch). The saved copy of the contents of 99 is thus used as a last instruction in the subroutine in place of a jump to 99. (Cell 99 is therefore only used as a quick method for forming a return branch for later use). A sample subroutine sequence appears in Fig. 17 (b).

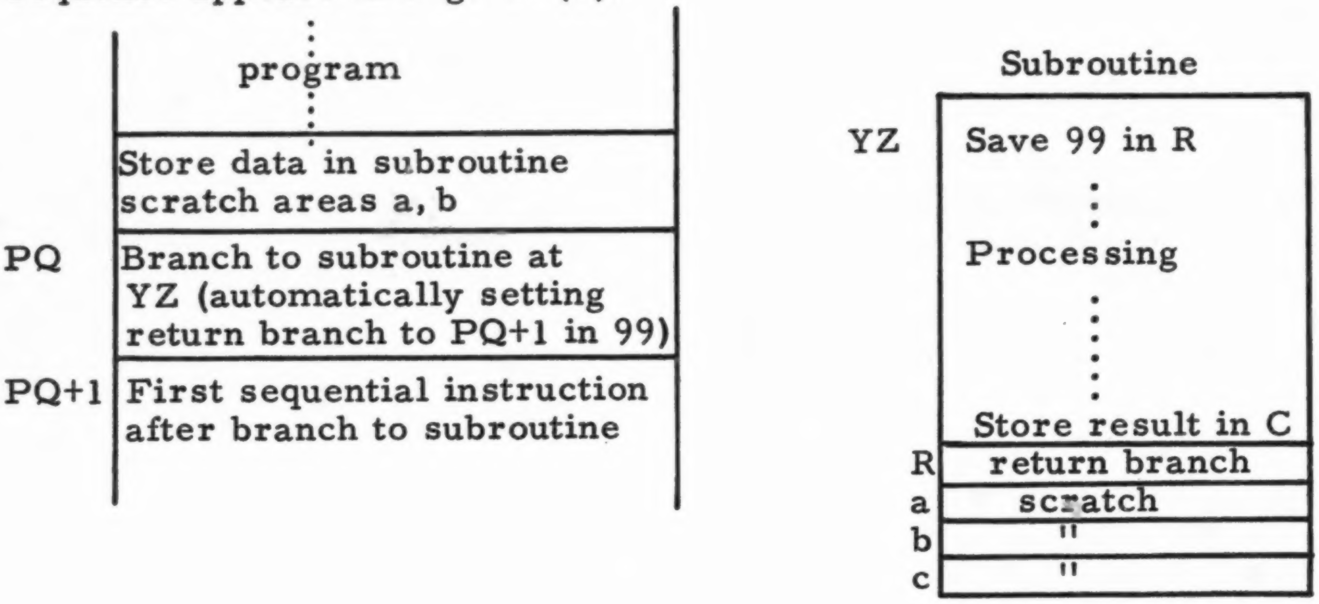

program
Store data in subroutine scratch areas a, b
PQ
Branch to subroutine at YZ (automatically setting return branch to PQ+1 in 99)
PQ+1
First sequential instruction after branch to subroutine
Subroutine
YZ
Save 99 in R
Processing
Store result in C
R
return branch
a
scratch
b
"
c
"

Fig. 17 (b) Sample subroutine sequence

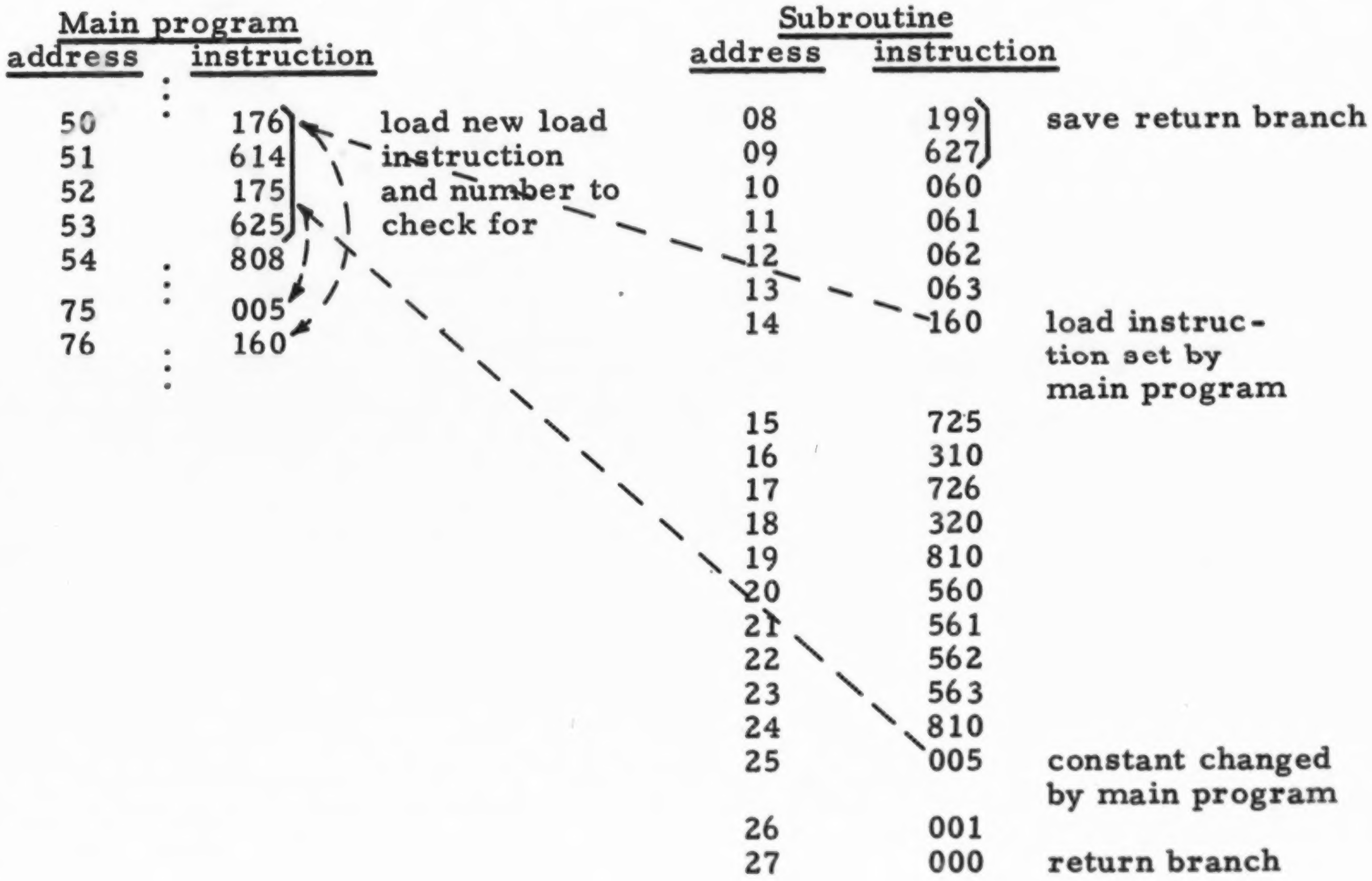
We will now illustrate the use of this technique, called subroutining, by recording as subroutines two previous problems and one new problem.

In the m x n multiplication program of section 8 (see Fig. 8 (c)), the read instructions at locations 9 and 10 are replaced by a "load from 99" instruction and a "store into 20" instruction (this saves the return branch set up by the 8YZ instruction used to branch to the subroutine)(see Fig. 17 (c)). Hence, instead of stopping with a 900 instruction at location 20, the subroutine returns to the instruction following the subroutine branch in the main program (PQ + 1). In addition, we put a "load 21" instruction at location 19 instead of the punch instruction. Thus, the results of the multiplication operation are in the accumulator when processing resumes in the main program. Before branching to the subroutine, the two numbers to be multiplied are stored in locations 22 and 23 by the main program.

Similarly, the Call-Up program of section 7 can easily be converted to a subroutine that will check any of the four fields for any desired number. First, we must add two instructions at 08 and 09 to save the return branch (since the subroutine inputs SS cards from the stack, we will use the device of a blank card to signify the end of the data, and will use the trick of the student in section 9 to flip the "run-stop" switch). If we store the return branch in location 27, then the control card following the blank card will by 827. Each time the main program branches to the subroutine, it first stores a different load instruction at 14 (depending upon which of the four cards is to be inspected) and also stores a different constant at 25 (e.g., if the district number of 5 was to be searched for, 160 would be stored at 14 and 005 would be stored at 25).

```
     Main program                                  |       Subroutine
address    instruction                             |  address    instruction
   :            }     processing                   |     :
  60          179 }                                |    07          403
  61          622 }  - store multi-                |    08          621
  62          176 }    plicands in                 |    09          199 }  save return
  63          623 }    subroutine                  |    10          620 }
  64          807    - branch to                   |    11          123 }
                       subroutine                  |    12          724 }
  65          675    - first instruc-              |    13          623 }
                       tion executed               |    14          319 }  multiply, etc.
                       following                   |    15          121 }
                       return from                 |    16          222 }
   :                   subroutine                  |    17          621 }
                                                   |    18          811 }
  75          m1                                   |    19          121 }  place result in
  76          n1                                   |    20          000 }  AC and return
  77          m2                                   |    21          000
  78          n2                                   |    22          (m) }
   :                                               |    23          (n) }  scratch areas
                                                   |    24          001 }
```

Fig. 17 (c) Coding for m x n program as a subroutine



```
     Main program                                        Subroutine
address    instruction                             address    instruction
   :
  50          176 }  load new load                   08          199 }  save return branch
  51          614 }  instruction                     09          627 }
  52          175 }  and number to                   10          060
  53          625 }  check for                       11          061
  54          808                                    12          062
   :                                                 13          063
  75          005                                    14          160    load instruc-
  76          160                                                       tion set by
   :                                                                    main program
                                                     15          725
                                                     16          310
                                                     17          726
                                                     18          320
                                                     19          810
                                                     20          560
                                                     21          561
                                                     22          562
                                                     23          563
                                                     24          810
                                                     25          005    constant changed
                                                                        by main program
                                                     26          001
                                                     27          000    return branch
```

Fig. 17 (d) Coding for Selective Service call-up program as a subroutine

6-digit precision addition as a subroutine: In most realistic applications, the 3-digit maximum number size of our machine would clearly be inadequate. The capacity to handle 10 or more digits is desirable, and possible, but quite complicated. We will demonstrate the process which is necessary by writing a program to do 6-digit addition, making use of the concept of subroutining just introduced. A 6-digit number is assumed to be two consecutive 3-digit numbers with both having the same sign. The main program stores the two 6-digit numbers to be added into locations 27-30 in the subroutine and then branches to the subroutine. The subroutine performs the addition by first adding the last 3 digits of each 6-digit number. If overflow occurs, 001 is added to the first 3 digits of the first number; then the first 3 digits of the second are added. The 6-digit result (i.e., the two 3-digit results) is placed in locations 31-32 by the subroutine, which then returns to the main program. To encode the occurrence of overflow on the final addition, we will set the AC to 001; otherwise we set it to 000.

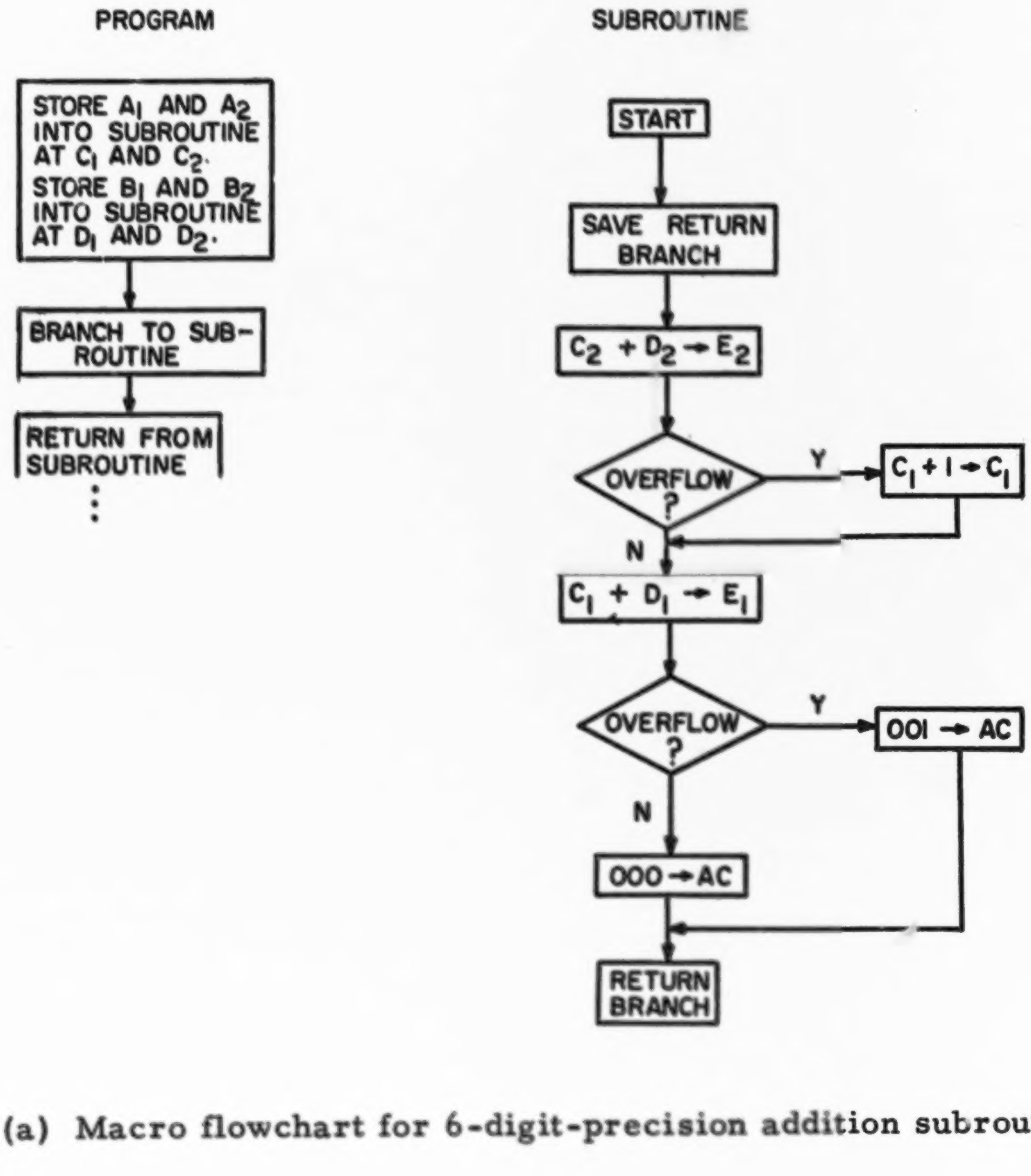

PROGRAM
STORE A_1 AND A_2 INTO SUBROUTINE AT C_1 AND C_2.
STORE B_1 AND B_2 INTO SUBROUTINE AT D_1 AND D_2.
BRANCH TO SUBROUTINE
RETURN FROM SUBROUTINE
SUBROUTINE
START
SAVE RETURN BRANCH
$C_2 + D_2 \rightarrow E_2$
OVERFLOW ?
Y
$C_1 + 1 \rightarrow C_1$
N
$C_1 + D_1 \rightarrow E_1$
OVERFLOW ?
Y
001 → AC
N
000 → AC
RETURN BRANCH

Fig. 18 (a) Macro flowchart for 6-digit-precision addition subroutine

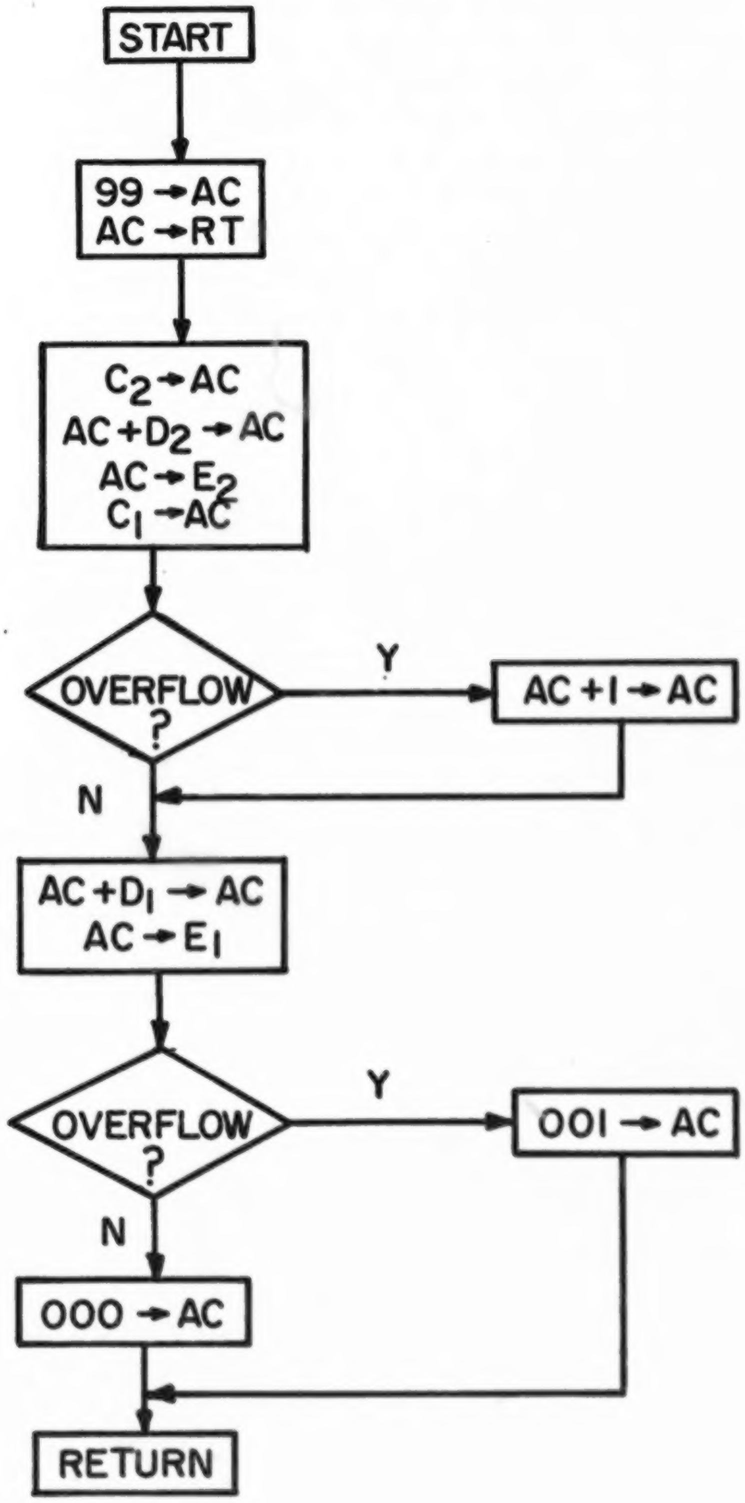

START
99 → AC
AC → RT
C2 → AC
AC + D2 → AC
AC → E2
C1 → AC
OVERFLOW ?
Y
AC + 1 → AC
N
AC + D1 → AC
AC → E1
OVERFLOW ?
Y
001 → AC
N
000 → AC
RETURN

Fig. 18 (b) Micro flowchart for 6-digit-precision addition subroutine

Address	Code	
10	199	save return
11	623	
12	128	
13	230	$C_2 + D_2 \rightarrow E_2$
14	632	
15	127	$C_1 \rightarrow AC$
16	918	
17	726	overflow compensation
18	226	
19	229	$C_1 + D_1 \rightarrow E_1$
20	631	
21	403	
22	924	
23	800	set AC to zero or
24	126	one and return
25	823	
26	001	constant
27	000	
28	000	two 6-digit numbers
29	000	
30	000	
31	000	6-digit result
32	000	

Note. Assume that the two 6-digit numbers are already stored into locations 27 through 30 by the program.

Fig. 18 (c) Coding for 6-digit-precision addition subroutine

13. Billiard Table Simulation

The following problem again illustrates instruction modification. It also illustrates how computer programs may be set up to simulate (i.e., to make a model of) real situations. Let us construct a simplified model of the path of a ball on a billiard table (Fig. 19 (c)). The ball is projected at 45° from lower left-hand corner of the table, which is 11 units wide and 15 units long. We will assume that the ball travels back and forth across the table, always rebounding at 45° from each edge. This program determines the position of the ball on the table at the end of a specified time.

With horizontal and vertical directions represented by the coordinates x and y, the position of the ball at a particular time is defined by the values of x and y. What are the values of x and y at the end of, say, 50 intervals of time?

Let us follow the path of the ball in terms of x and y, starting at 0. At the end of the first time interval, $x = 1$ and $y = 1$. At the end of the second interval, $x = 2$ and $y = 2$. Thus, in each interval we add 1 to x and 1 to y.

At the end of 11 intervals $x = 11$ and $y = 11$: the ball strikes the edge of the table and rebounds. The ball now begins to travel back across the table in a

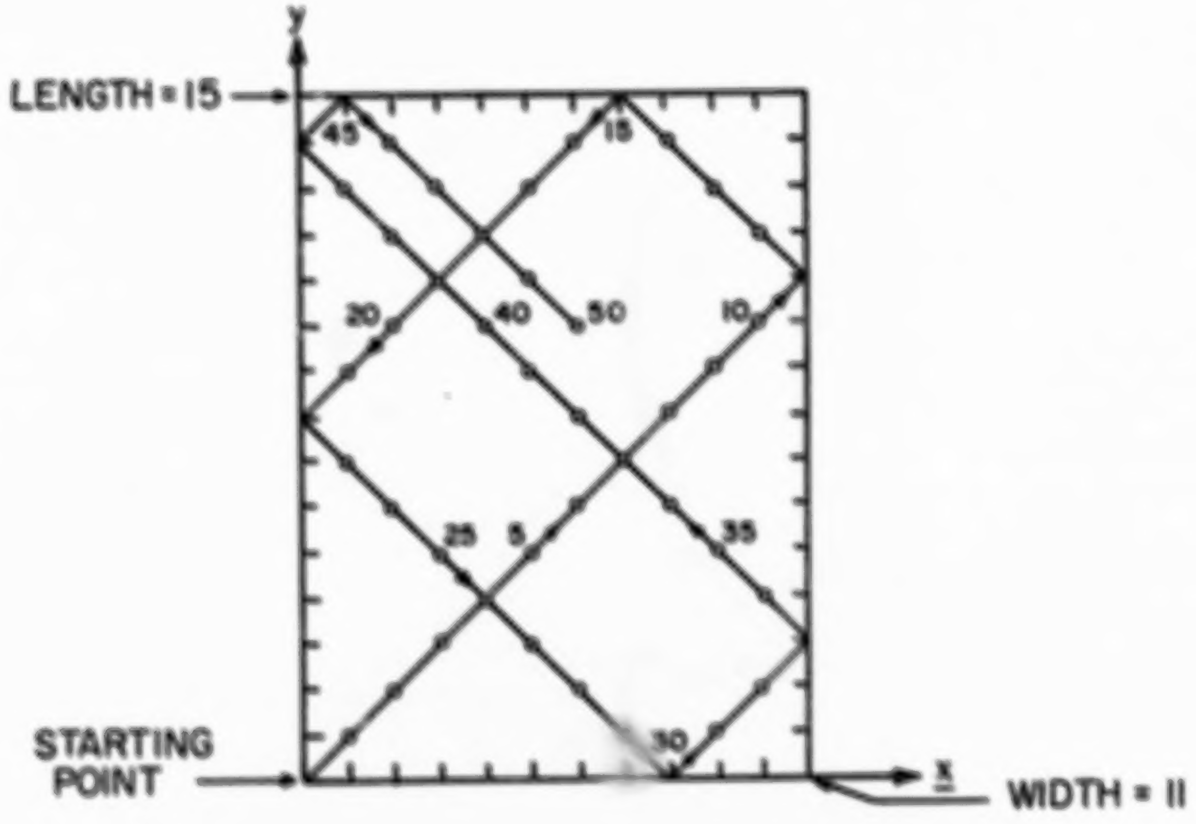


Fig. 19 (a) Idealized journey of a ball on a billiard table. Ball is projected at 45° from the lower left corner. Circles and associated numbers denote time intervals.

direction that makes x smaller (y continues to increase as before). Later (when y = 15) the ball strikes, and rebounds from, the top of the table. Thereafter, the ball's direction is such that both x and y decrease with the passage of time.

We must program the computer to detect when x becomes equal to the width of the table and then to modify an instruction to subtract 1 from the value of x instead of adding. When y becomes equal to the length of the table, the computer must then modify an instruction to subtract from the value of y instead of adding to it. A flow chart for accomplishing this billiard table calculation is shown in Fig. 19 (b).

We first read in the table length (15 units), the width (11 units) and the total time (50 intervals), and we initialize x, y and the interval counter to 0. We set two instructions, A and B (shown toward the bottom of the flow chart) to "add 1". These are the instructions that are modifiable to "subtract" as the values of x and/or y require.

Looking at the flow chart, let us see what happens when x = 5 and y = 5. A test of x at the first test point shows that x is greater than 0. We therefore by-pass the instruction modification and go to the second test point, which shows that x is less than the width. Therefore we again by-pass to a similar pair of tests for y, which discloses that y is also greater than 0 and that it is less than the length. Thus, the two instructions A and B add 1 to x and y, respectively.

Next let us examine the situation where x = 11 and y = 11 - the first point of rebound from a table edge. Now, the first test point shows that x is again greater than 0, so we again take the by-pass to the second test point. Here, since x is now equal to the width, we do not by-pass the instruction modification, and thus change the instruction at A so that it "subtracts 1" from x. Testing the y-

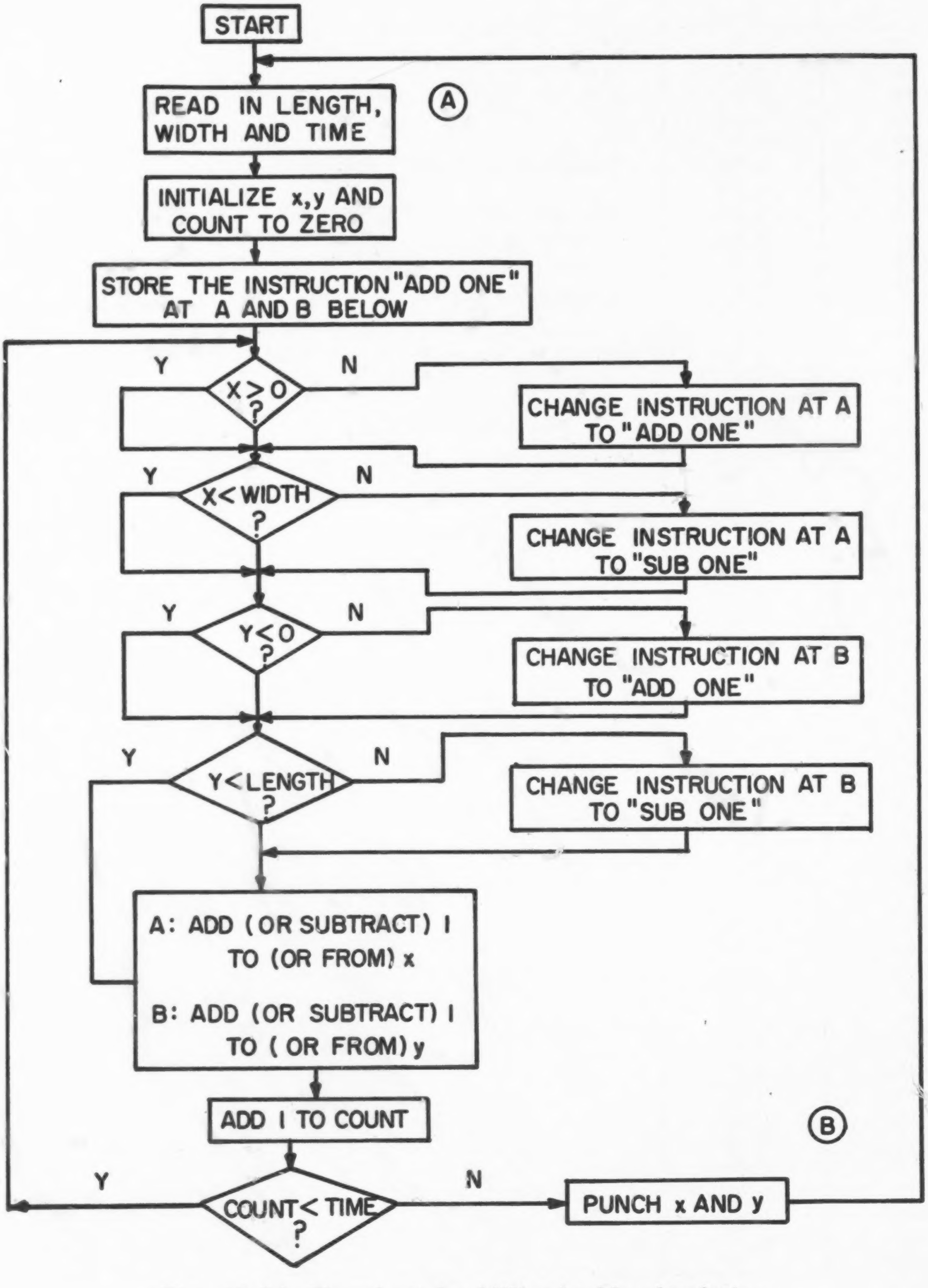

START
READ IN LENGTH, WIDTH AND TIME
Ⓐ
INITIALIZE x,y AND COUNT TO ZERO
STORE THE INSTRUCTION "ADD ONE" AT A AND B BELOW
Y
N
X > 0 ?
CHANGE INSTRUCTION AT A TO "ADD ONE"
Y
N
X< WIDTH ?
CHANGE INSTRUCTION AT A TO "SUB ONE"
Y
N
Y< 0 ?
CHANGE INSTRUCTION AT B TO "ADD ONE"
Y
N
Y< LENGTH ?
CHANGE INSTRUCTION AT B TO "SUB ONE"
A: ADD (OR SUBTRACT) 1 TO (OR FROM) x
B: ADD (OR SUBTRACT) 1 TO (OR FROM) y
ADD 1 TO COUNT
Ⓑ
Y
N
COUNT< TIME ?
PUNCH x AND y

Fig. 19 (b) Flowchart for billiard table simulation

Address	Word	Comment
05	0 49	Read in length
06	0 50	Read in width
07	0 51	Read in time
08	4 03	
09	6 52	
10	6 53	Initialize x, y and count to 0
11	6 54	
12	1 55	Get the instruction "ADD ONE",
13	6 36	store it at address 36, and also
14	6 39	store it at address 39.
15	4 03	Create 0 - x and
16	7 52	
17	3 20	test it; if 0 < x go to address 20.
18	1 55	If 0 ≥ x, get instruction "ADD ONE" and
19	6 36	store it at address 36.
20	1 52	Create x-width and
21	7 50	
22	3 25	test it; if x < width go to address 25.
23	1 56	If x ≥ width, get instruction "SUB ONE" and
24	6 36	store it at address 36.
25	4 03	Create 0 - y and
26	7 53	
27	3 30	test it; if 0 < y go to address 30.
28	1 55	If 0 ≥ y, get instruction "ADD ONE" and
29	6 39	store it at address 39.
30	1 53	Create y-length and
31	7 49	
32	3 35	test it; if y < length go to address 35.
33	1 56	If y ≥ length, get instruction "SUB ONE" and
34	6 39	store it at address 39.
35	1 52	Get x and
(A) 36	0 00	either add or subtract 1 from it, and
37	6 52	store the result as the new value of x.
38	1 53	Get y and
(B) 39	0 00	either add or subtract 1 from it, and
40	6 53	store the result as the new value of y.
41	1 54	
42	2 00	Add 1 to count
43	6 54	
44	7 51	Subtract time from count and
45	3 15	test it; if count < time go back to address 15.
46	5 52	If count ≥ time, print out the value of x
47	5 53	and of y,
48	9 00	and halt.
49	0 00	(length)
50	0 00	(width)
51	0 00	(time)
52	0 00	(x)
53	0 00	(y)
54	0 00	(count)
55	2 00	(instruction "ADD ONE")
56	7 00	(instruction "SUB ONE")

Fig. 19 (c) A program for billiard table simulation value, we find that no instruction modification is necessary, and that y is incremented by one.

Continuing to follow the ball's upward journey, we reach the top edge where the ball's rebound causes x and y to enter a phase in which both coordinates decrease. Thus, step by step, we make the flow chart picture the ball's journey. In Fig. 19 (c) we show the problem coded as a computer program.

In this program we note that instruction 36 (A) covering x and 39 (B) covering y cause the computer to add 1 to the values of x and y in the following way. Instruction at address 12, namely, the word 200. Instruction at address 13, namely, word 636, copies into address 36 the contents of the accumulator, namely, the word 200. Instruction at address 14, namely word 639, copies into address 39 the contents of the accumulator namely the word 200.

Now the word 200 says to add to the accumulator the contents of address 00. And in our computer the contents of address 00 is 001. Thus we add 1 to the accumulator contents, that is, to x or y.

In contrast if the operation calls for "subtract 1", instruction 23, namely word 156, brings to the accumulator the word 700, which calls for "subtract 1". This modified instruction is now copied into the addresses 36 and 39 which previously called for "add 1".

The foregoing program can be varied in many ways to match different conditions. For example, it may be varied to follow a ball shot from the lower right corner, or to take into account the effect of "english" on the ball's path and recoil pattern, or the energy-sapping effects of friction. Since these physical factors are represented by numerical quantities in the memory, we can make the computer "model" or "simulate" as many possible physical conditions as we please. And since a computer can speedily execute the instructions, computer simulation programs can be made to represent a speeded-up, advance-view of a real world situation before it happens. Such high-speed, advance simulations - made possible in part by stored programs and modifiable instructions - are indispensable in space flight experiments.

Suppose, for instance, that the space coordinates of the orbital motion of a space capsule are computed by rapidly solving equations in an appropriate computer program. If the computation proceeds quickly enough the computer will know where the capsule will be long before it actually gets there. Computation which proceeds this rapidly is called real-time computation, although perhaps a better name would be "in time" computation. Computation in real time is important in a variety of applications, such as the guidance of interplanetary probes and the diagnosis of disease. (Can you think of other applications in which real time computation would be vital?)

Problems

Relative difficulty of questions found in Chapter A-4:

EASY	MODERATE	DIFFICULT
* 1	* 4 11, 12	* 8
* 2	5 14, 15	10
* 3	7	13
6	9	

*Key Problems to be completed by all students.

1. What single machine code instruction would you write in order to have the computer do each of the following?
(a) Read the top input card and put its contents into address (memory location) 34.
(b) Add to the accumulator a copy of the contents in address 52.
(c) Clear the accumulator and bring to the accumulator a copy of the contents in address 95.
(d) Jump to the instruction given at address 24.
(e) Copy the contents of the accumulator into address 42.
(f) Substract from the contents of the accumulator a copy of the contents in address 33.
(g) Shift the contents of the accumulator first one place to the left and then two places to the right.
(h) Halt and reset the instruction counter to instruction at address 00.
(i) Test the contents of the accumulator. If the contents are negative go to the instruction at address 13.
(j) Print onto an output card the contents at address 19.

2. What is the meaning of each of the following instructions written in machine code? Write out the meaning of each in a complete English sentence?
(a) 042

(b) 403

(c) 171

(d) 410
(e) 672
(f) 819

(g) 713

(h) 215

(i) 341

(j) 516
(k) 900
(l) 309

3. If the top input card has the number 473 printed on it, and the second card has the number 052, what will each of the following programs do with these two numbers? (Assume that the top instruction is executed first.)

Memory Address	Word Stored
56	063
57	064
58	163
59	264
60	664
61	564
62	900
63	—
64	—

(a)

Memory Address	Word Stored
28	036
29	136
30	036
31	736
32	736
33	636
34	536
35	900
36	—

(b)

4. The following program is one that might be used to find out if a number A is larger than another B or not. The top input card contains A, the second input card contains B. The answer "yes" is printed out as 001, the answer "no" is printed out as 000.

(a) How many tests are required to determine if A > B or not? Why?

(b) If the question was "is A $\geq$ B or not", how could this program be made shorter?
(c) If the result of the test at instruction 22 is positive what is the next instruction?
(d) What does this program do if the number A is a negative number?

5. The contents of the accumulator are changing most of the time during any calculation. These changes in the accumulator are important. In each of the short programs below tell what is in the accumulator after the execution of each instruction.

Memory Address	Word Stored	Contents of Accumulator
55	162	
56	263	
57	324	
58	430	
59	664	
60	564	
61	900	
62	008	
63	003	
64	—	

(a)

Memory Address	Word Stored	Contents of Accumulator
27	134	
28	735	
29	735	
30	326	
31	636	
32	536	
33	900	
34	329	
35	127	
36	—	

(b)

6. Write as brief a program as you can (in machine code, starting at address 53) which will find and print out the value of M-N where M> N and M is positive.

7. Write a machine code program for finding the value of (M-5N). Start your program with a flow chart.

8. Write a machine code program that will put any three numbers, A, B, and C, copied from input cards in descending order.

9. Below you will find some parts of real program. An arrow indicated the instruction that is presently being executed. There are some memory locations that are left blank; determine what should go into each blank memory location and write what would be in the accumulator. The arrow shows the initial instruction in each case.

Memory Location	Word Stored
25	154
26	755
→ 27	656
.	.
.	.
.	.
54	329
55	312
56	017

Memory Location	Word Stored
19	430
20	642
→21	141
.	
.	
.	
41	937
42	—

Memory Location	Word Stored
10	029
11	129
12	728
→ 13	630
.	
.	
28	001
29	289
30	577

10. What does the following program do?

Memory Address	Stored Word
21	0 36
22	1 36
23	4 20
24	6 38
25	1 36
26	4 12
27	4 10
28	6 37
29	1 36
30	4 02
31	2 37
32	2 38
33	6 36
34	5 36
35	8 21
36	---
37	---
38	---

11. Write a flow chart and corresponding machine program which will examine an arbitrarily large set of numbers on input cards (a blank card marks the end of the set) and which will print out only those cards which have on them non-zero integers.

12. Write a flow chart and machine program which will print out only those input cards which have on them odd (and positive) integers. (Suggestion: By a "422" instruction delete all but the right-most digits of the numbers.) A blank card makes the end of the set of input integers.

13. A set of input cards (terminated by a blank card) contains numbers in which the right-most two digits specify an address, YZ. The left-most digit (X) is to be ignored in this problem. Write a machine program which will print out the contents at these addresses in memory. (For instance, if the input card reads 056, 156, 256, ..., 856 or 956 the corresponding output card should print out the contents at address 56.) This problem is most easily done by generating an instruction equivalent to "5YZ" which is executed later in the program.

14. Analyze what the program given below does.

```
20  403
21  628
22  028
23  129
24  228
25  628
26  528
27  822
28  000
29  000
```

15. Analyze what the program given below does.

```
20  029
21  030
22  130
23  729
24  631
25  531
26  130
27  629
28  821
29  000
30  000
31  000
```

Chapter B-1

DECISION MAKING

1. THE ELEMENTS OF DECISION-MAKING

> You have just been given permission to drive the family car, but your father keeps the key to the locked gas cap, and tells you that he will fill the tank on Friday night and you can have the car all day Saturday, but drive slowly because the slower you drive the better will be your gas mileage. You have a single decision to make: drive slowly and get maximum mileage or drive fast and get minimum mileage. Or is it that simple?
>
> A boy's father indicates that five dollars is sufficient to spend for a Saturday football game and the dance which follows, while his girl friend has been discussing plans which if carried out will cost more than ten. Both indicate that the decision is quite simple and that the answer is obvious.
>
> The Board of Education of a school district asks the principal of the local high school why they should not ban all student cars from the school parking lot. At the same time the Student Council asks the same principal why he will not allow all students to drive to school. Both groups indicate that the decision is really quite simple.
>
> The "Committee for Clean Air" of a large city meets with the mayor and demands that all incinerators used for burning garbage within the city limits be shut down immediately to cut down air pollution. The mayor promises to discuss the problem with his commissioners of traffic, sanitation rivers and harbors, and report back to the committee. The committee chairman snorts, "More Bureacuracy," and marches out of the meeting to a press conference where he reports that the mayor is stalling on a simple question which has a very simple answer. The mayor in return replies, "For every complicated question there is an answer which is forthright, simple, direct, and wrong!"

One activity which is fundamental to the creation of the man-made world (indeed, fundamental to all human activity) is the making of decisions. The engineer basically is a decision-maker. The chemical engineer in designing a petroleum-processing plant decides how many stages to use in the distillation column, which products (100-octane gasoline, motor oil, kerosene, etc.) the plant should manufacture, and the proportions of each product. The electrical engineer, in designing a television receiver, decides what size picture tube to use, what picture- and sound-quality should be achieved, how many sets are to be manufactured and the arrangement of the manufacturing facilities to produce these sets. In the design of a new building the civil engineer must decide whether to use steel girders or poured concrete, the extent to which internal columns are permissible, and whether to construct a ten-story building with a given area per floor or a five-story building with twice the area per floor.

In each of these cases, there are many decisions to be made--some insignificant and others crucial. Each of these decisions is based on definite criteria. In this chapter, we look at the nature of decision making and at some of the techniques which can be used to arrive at intelligent decisions.

> As an example, let us look at the first "simple" situation described at the beginning of this chapter. This situation contains elements common to all decision-making processes. The answer to the problem of an appropriate speed at which to drive an automobile, while relatively simple, does not become "drive slowly and get maximum mileage or drive fast and get minimum mileage". We will examine the problem of selecting an appropriate speed at which to drive an automobile by looking at all the factors involved.

Figure 1 shows the relationship between automobile speed and gasoline economy (the number of miles we travel per gallon of fuel consumed). At standstill, with the engine running, the efficiency is zero--no matter how much fuel we use, the automobile doesn't get anywhere. As the speed increases, the

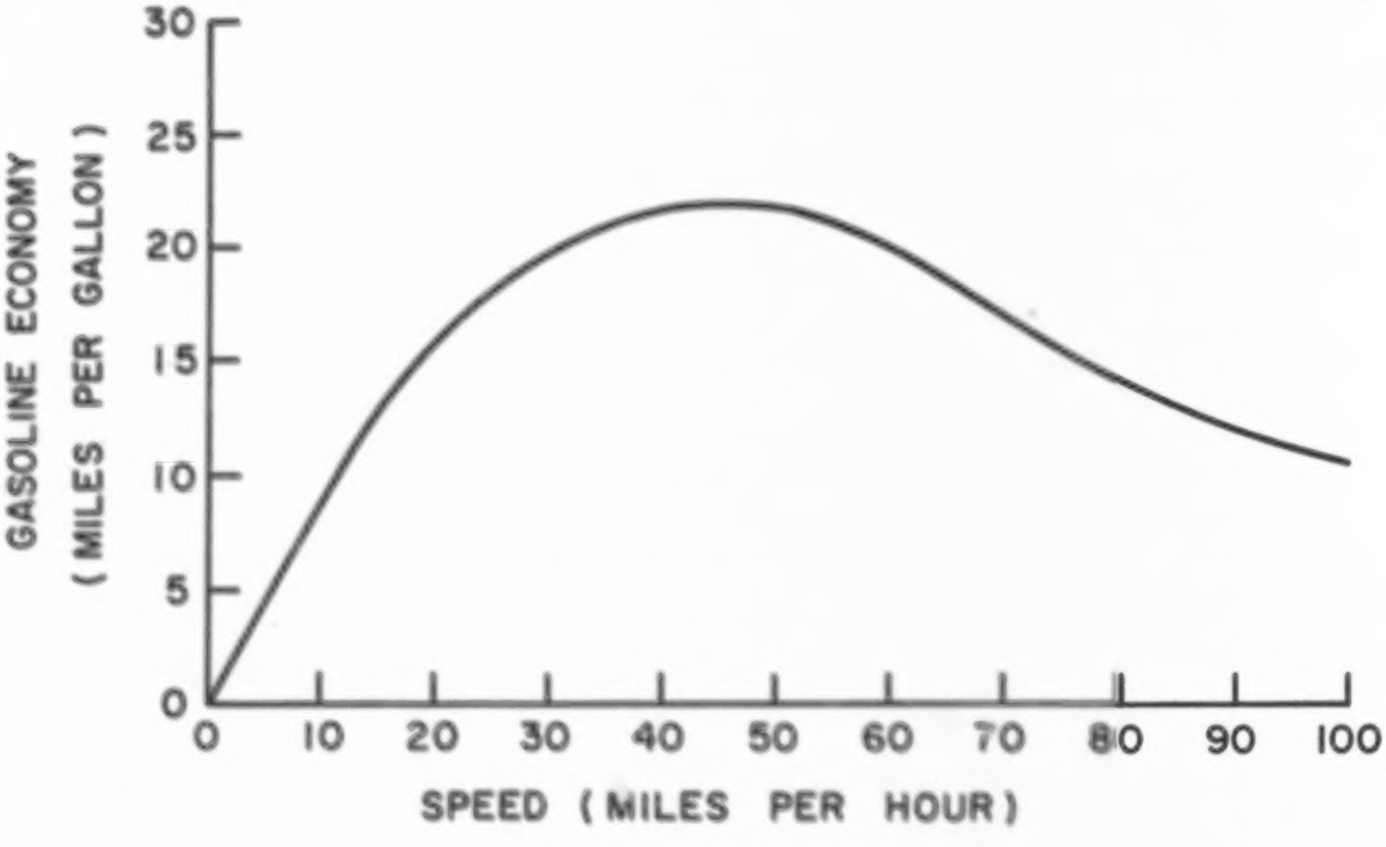


Fig. 1 Gasoline economy as a function of automobile speed.

gasoline economy improves, until, at 45 miles per hour, it reaches a maximum of twenty-two miles per gallon. Beyond 45 miles per hour, the economy falls off, because of friction in the engine, in the wheel bearings, and in the air through which the car moves, as well as because of a decrease of engine efficiency at high speeds. It is obvious that the most economical operation of this car, when travelling between two points, is realized when we drive at a constant speed of 45 miles per hour.

It is rarely possible, in any realistic situation, to travel at a constant speed, because of traffic conditions, curves, hills, etc. Fortunately, in this example, small departures from 45 mph do not cause a serious problem. At all speeds between 35 and 55 mph, the gasoline economy is within 5% of the maximum. As long as we stay reasonably close to 45 mph, we are operating at an essentially optimum condition.

We base our decision to drive at 45 mph on our desire to maximize the gasoline economy (or, conversely, to minimize the cost of gasoline). In the real world, decisions seldom are so easily made, because of the constraints imposed by circumstance, as well as the conflicting criteria imposed upon the decision maker.

If we were to drive through the center of a city, where the speed limit might be set at 25 mph, the police would take a very unsympathetic attitude toward a speed of 45 mph. Our decision-making becomes somewhat more complex because of the imposition of this constraint. Its effect is shown in Fig. 2. The legally-

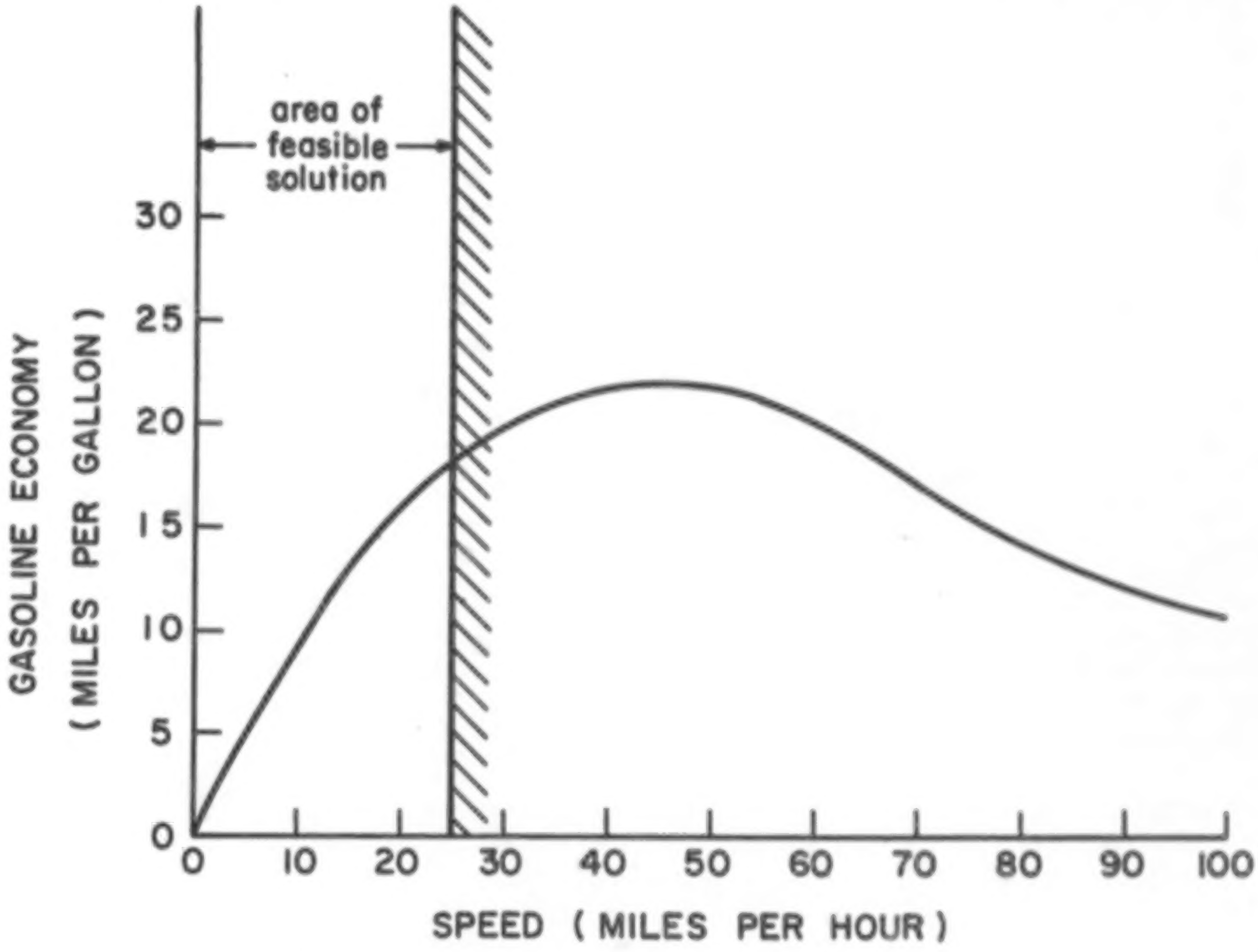


Fig. 2 Imposition of speed-limit constraint.

imposed constraint permits operation at any speed between zero and 25 mph. If we wish to maintain maximum gasoline economy under this constraint, we must operate at 25 mph, because this is the highest point on the curve of economy versus speed within the region in which feasible solutions to our problem exist.

If, on the other hand, we drive along a limited-access road such as the New York Thruway or the Pennsylvania Turnpike, a different set of constraints may be imposed. One such set is shown in Fig. 3. Here, we are not permitted to travel faster than 65 mph, nor slower than 50 mph. In this case, if our criterion is gasoline economy, we must drive at 50 mph. We should note here that, if our minimum-speed constraint had been 40 mph rather than 50 mph, the decision would have been to drive at 45 mph rather than at either constraining limit, because the criterion for choice is maximum possible economy.

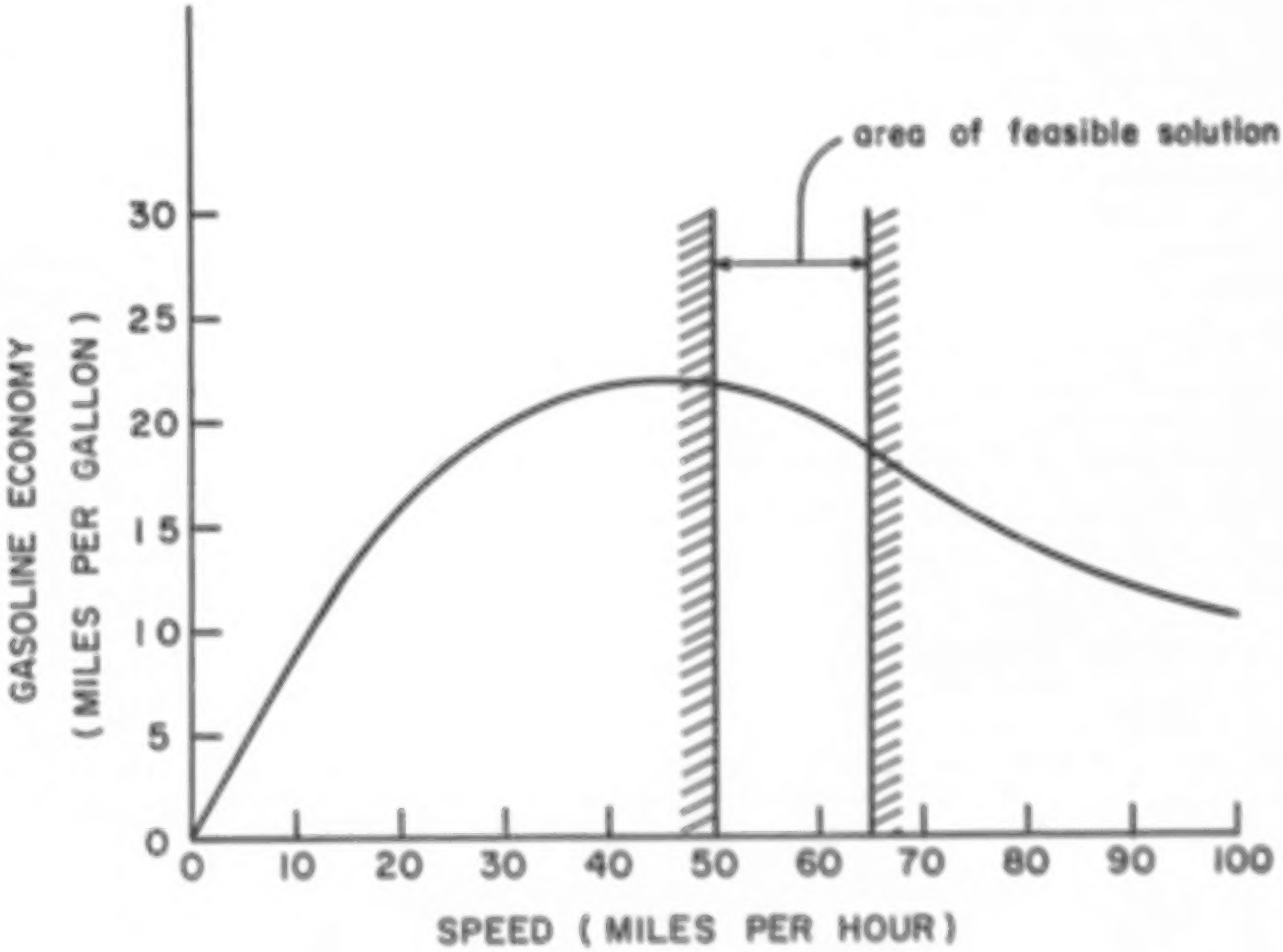


Fig. 3 Constraints imposed on limited-access road.

Frequently, we are required to make decisions on the basis of conflicting criteria. In our limited-access-road example of Fig. 3, we may be required to travel between two points in as short a time as possible (perhaps we have a date with a girl who becomes angry if we arrive late), and, in addition, we may wish to use as little fuel as possible. We see, from Fig. 3, that we cannot simultaneously drive at maximum speed (within the speed-limit, of course) and maximum economy. These are mutually-exclusive decision criteria. Such conflicts are typical of engineering decision-making.

We may decide to drive at 50 mph for economy, and to risk the wrath of our girl friend, or, if we don't wish to assume this risk, we may drive at 65 mph and arrive on time to find a happy young lady.

As an alternative, we may compromise and choose some speed between 50 mph and 65 mph--perhaps 58 mph--which will save us some money and permit an arrival only slightly tardy. Such a choice between two conflicting criteria actually involves the selection of a new criterion which seeks a partial satisfaction of each of the original criteria. In this particular example, at 50 mph the gasoline required is one gallon for every 21.5 miles, while at 60 mph we realise only 20 miles on a gallon. The higher speed, however, requires less time; this time saving may be more valuable to the driver than the extra cost of the gasoline.

The degree to which each of the various criteria is satisfied is determined by the decision maker. In other words, in the real world we are often confronted by decisions in which the selection criterion involves a compromise choice among several, conflicting goals.

This gasoline-economy example contains all of the fundamental elements of decision-making problems. Let us review these so that we may focus our attention on them later. They are four in number.

(1) Model. The model is the mathematical or quantitative description of the problem we are concerned with. In our example, we are interested in obtaining the maximum gasoline economy; hence the model is the curve of Fig. 1, which shows how gasoline economy depends on the speed which we are to select. In subsequent chapters we consider the various forms of models in much greater detail, since the model is the item which changes the problem from one of intuition or common sense into a quantitative problem which we can hope to solve precisely.

(2) Criteria. The decision problem also includes a criterion or set of criteria which must be satisfied. If there is a set of conflicting criteria, there must be some assignment of importance to each criterion--i.e., a weighting according to the importance of the several criteria. In our example, the criteria were to achieve maximum gasoline economy and to arrive somewhere in as short a time as possible, or some compromise between these. The civil engineer must design a building which costs as little as possible, while simultaneously making it esthetically pleasing and strong enough to withstand the loads to be expected. In most engineering cases, one criterion usually is financial (either maximization of profit, as in a commercial organization, or minimization of cost, as in government-equipment design).

(3) Constraints. The imposition of constraints, or the definition of the area

of feasible solutions brings the problem into the real world by imposing conditions which must be satisfied if the solution is to be useful. In our example, specification of maximum and minimum speeds defines a region within which we may look for a solution. Other constraints which may be imposed in our example are unsafe road conditions and driver fatigue (speed may be limited by road conditions, and the time involved in long trips may have to include rest periods).

(4) Optimization. Once the problem is formulated (the model), we decide what we really want (the criteria), and statements exist as to what is permissible (the constraints), we are ready to attempt to find the best or optimum solution. In our example, solution is possible merely by examination of the model and consideration of the constraints. In more complex problems, it may be necessary to find special engineering or mathematical techniques; in many practical cases, we have to adopt a trial-and-error approach.

Thus, in this section we consider a very simple example of a decision problem (the selection of the optimum speed at which to drive a car). In this example, we find all four elements of the typical decision-making problem:

Model	Constraints
Criteria	Optimization

In the remainder of this chapter, we consider a small group of other decision problems in order to illustrate the forms taken by each of these four elements, and in order to present a broad picture of how modern technology attempts to find solutions to problems of this nature. In these sections, attention is focussed on different optimization techniques, with the examples illustrating incidentally a few of the forms taken by the model, criteria, and constraints.

Problem: 1-1 The curve of Fig. 1 for gasoline economy versus speed is assumed valid for this problem. We have rented a car from the No. 1 rental agency and will have to pay a fixed 9¢ per mile for the distance travelled. We wish to drive from the airport (where we have picked up the car) to a research laboratory 80 miles distant. Half of the trip distance is on an expressway with a minimum speed of 40 and a maximum of 70; the other 40 miles of the trip are on a country road with a maximum speed of 45 and a minimum of 20. Since we are stockholders of the No. 2 rental agency (who unfortunately have no cars available at this airport today), we are anxious to maximize our fuel consumption during the trip. How should we drive? How many more gallons of gas shall we use by following this optimum schedule rather than following the schedule which would bring us to our destination with minimum fuel consumption? How many more gallons do we use than if we drove at maximum speed at all times? How much longer is required for our "optimum" trip than for the shortest trip (in other words, how much time do we have to waste to do maximum damage to rental agency No. 1. ?).

Let us look at the example of a complex problem referred to on page 1-1. It is a problem for which we can produce a descriptive model, one for which we can set up criteria (mostly subjective) and for which many constraints will be found. With all the necessary elements it would seem that an optimized solution would be merely a matter of working at the mathematics as in the previous problems. But life is not always that simple. The problem to which we refer is the urban system problem. In 100 years we have moved from a society in which about eighty percent of the population lived on farms to a society in which sixty percent of the population lives in a metropolitan urban area.

The unusual aspect of the urban problem is that the system is so complex that it makes our missile control system seem trivial.

The urban system is one with many input and output signals, a host of constraints (social, economic, and political) and a multi-dimensional, dynamic mathematical model.

We can illustrate the complexity of the urban system by just a few thoughts on the mundane problem of solid waste disposal. In the United States we are currently generating about four pounds of solid garbage per person per day. If the eight million residents of New York City maintain this average, we need to dispose of about 16,000 tons of solid refuse each day within the city.

While the problems of New York City may seem unique because of the size and population density of that area, it is estimated that by the year 2000 there will be more than a dozen cities in the United States, each as large (in population) as New York City is today.

At the present time, much of this garbage is burned in private and municipal incinerators, thereby contributing in a major way to the air pollution problem. Proposals to help the air pollution situation by forcing multiple dwellings to package garbage for truck collection are discouraged by the fact that every available garbage truck in the United States would then be needed to transport the material to locations sufficiently remote from the city--with the obvious impact on the traffic flow problems of the city. Proposals to enforce stringent specifications on incinerators promise to raise rents and thereby encourage the further exodus from the city of the already vanishing middle class. A proposal to transport the solid rubbish on the subway system during the early morning hours obviously requires major changes in the mass transportation system. Finally, a proposal to give tax advantages to companies which package products in dissolvable containers clearly requires political and popular understanding of the nature of the overall problem.

The above, very abbreviated discussion emphasizes the complexity of merely one small facet of the problem of the urban environment. The urban system is technologically complex primarily because of this interdependence of the various sub-systems (including the models for the urban behavioral sciences, the urban geography, and the urban history).

The Difficulties of the System Problem

In very recent years, there have been several major attempts to apply system engineering to social and urban problems. The four studies commissioned from the aerospace industry by ex-Governor Brown of California are perhaps the most publicized of these efforts. Both New York City and New York State have subsequently attempted to develop further along these lines, and specific studies are contemplated or underway in most other cities around the country. Throughout meetings of mayors and federal agencies runs the question of what technology can offer to the urban community. In these attempts, at least three factors seem to represent major blocks to progress.

The first difficulty is the obvious one of communication between the technologist and the political leader. Most city leaders tend to be fully occupied with day-by-day crises under a government organization originally developed in a time of small population and relatively simple problems; the major studies required for effective action never progress beyond the initial, conception stage--except possibly in a city department (e.g., traffic or air pollution) where it is not possible to consider the full impact of a change on the total urban environment. The communication problem is severely complicated by over-zealous technologists, who feel that major progress can be effected by single steps which all too often duplicate work already done by other groups. If our past experience with other complex system problems is any guide, progress in an urban community can only be achieved if responsible government leaders are able to work with a responsible technological and social science group over a period of several years.

The second problem impeding progress is the extreme difficulty of adequate modelling. Modern system engineering starts with the determination and evaluation of at least approximate models of the existing system and the selection of appropriate performance criteria.' In most of our desired models for urban problems, the basic data do not exist or, at least, are not readily available. For example, in New York City it is difficult to obtain detailed data on the population density: the average for the city is about 85,000 per square mile (incidentally far below the density of Rome in 100 A.D., which was over 200,000 in spite of the absence of high-rise buildings), but system planning required detailed two-dimensional data not only on density, but as well on the living habits, ethnic and educational backgrounds, and so forth.

In other areas, data are equally difficult to determine. The current administration in New York City has made major attempts to reverse the out-flow of small businesses (particularly those employing labor forces which are not highly skilled). In many cases, the reasons for businesses emigrating in the past are not well documented. Construction of an appropriate model requires evaluation of the effects of water and electricity rates, the influences of wage scales and union-management relations, and the interdependencies with the quality of the public transportation system (e.g., if a factory can expect one hour per day additional work from employees by moving from the city, the labor force can be reduced by 14).

The third significant impediment to progress is the difficulty of defining experimentation on the urban environment. Science and engineering progress through controlled experiments; new automatic control techniques are evolved by experimentation to measure process characteristics, experiments to attempt control of process simulations, and finally experimental development of the final system. Analysis and theory are used to suggest and interpret the experiments.

In the urban problem, however, it is difficult to determine what constitutes a meaningful experiment. The model-cities program of the federal government will (if the program is sufficiently innovative) indicate certain system characteristics, although it is not clear how the results for a new city of 100,000 can be extrapolated to an existing city of a million or more. Even in the model cities, however, significant experiments require a receptiveness of the developer and the government to attempt major modifications in the system.

In an existing city, the problem of how to experiment at reasonable cost is of even greater difficulty. In New York City (as an example), a significant transportation-system experiment would be a one-year abolition of fares on the Long Island Railroad, which serves about 200,000 commuters daily. Such an experiment (costing perhaps $300 million, with the essential increase in parking facilities at stations and expansion in railroad service) would yield useful evaluations of the transportation and traffic problems only if it were coupled with extensive studies in urban behavioral science.

Such an experiment might well eventually result in a lower cost to the State and City, particularly if it were combined with dynamically scheduled busses running from the railroad terminals and management of the tolls collected from private automobiles entering the city from the region served by the railroad. Clearly, however, the magnitude of such an experiment is beyond the scope of our past thinking on urban problems; initiation of such an experiment would require a monumental effort in the political and social arenas. Yet it is difficult to envision major improvements in the urban environment resulting from experiments of smaller magnitude.

The preceding description outlines a series of problems in which you might well be involved sometime in the future. For the present let us look at a series of problems which can be solved in a reasonable time.

2. ALGORITHMS

Throughout the remainder of this chapter, we discuss a series of decision-making problems and attempt to find optimal solutions. As these examples illustrate, in most cases the optimum solution is not obvious merely by inspection of the problem (as it is in the simple example of Sec. 1). Rather, we often must seek a logical, step-by-step procedure to move toward the answer.

Such a list of instructions for a sequence of operations which leads to the answers of all problems of a particular type is called an algorithm.*

Example

In order to illustrate what is meant by the above definition, we consider first one of the best known algorithms - the Euclidian algorithm for finding the greatest common divisor of two positive integers. (This particular example is not related to optimization, but we use it to emphasize the character of an algorithm.)

The greatest common divisor of two positive integers is the largest number which will divide into each with no remainder. For example, if the two positive integers are 6 and 9, the greatest common divisor is 3 (i. e., we could divide both 6 and 9 by 3 to reduce 6/9 to 2/3).

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*The name algorithm is derived from a Uzbek mathematician, al-Khowārizmā, who developed such sets of rules in the ninth century. (The Uzbek Soviet Socialist Republic is a portion of the Soviet Union north of Afghanistan).

Before stating the general method of solution (the algorithm), we consider the particular example of the two integers, 1740 and 2436. What is the largest number which divides evenly into both of these integers? In order to answer this question, we form a vertical column, with the larger integer listed first:

2436

1740

The next entry is the remainder when 1740 is divided into 2436:

$$\begin{array}{r} 2436 \\ 1740 \\ 696 \end{array} \qquad\qquad \begin{array}{r} 1 \\ 1740\,\overline{)2436} \\ \underline{1740} \\ 696 \end{array}$$

We now proceed with the last two number (1740 and 696) exactly as we did with the original two. The next entry is the remainder when 696 is divided into 1740:

$$\begin{array}{r} 2436 \\ 1740 \\ 696 \\ 348 \end{array} \qquad\qquad \begin{array}{r} 2 \\ 696\,\overline{)1740} \\ \underline{1392} \\ 348 \\ \\ 2 \\ 348\,\overline{)696} \\ \underline{696} \\ 0 \end{array}$$

When 348 is divided into 696, the remainder is zero, so we stop the column; the last entry (348) is the greatest common divisor of 2436 and 1740 (indeed, 2436 = 348 x 7 and 1740 = 348 x 5).

The algorithm for all problems of this type then is described as follows.

Step (1) Make the two given numbers the first two entries in a column with the first entry greater than or equal to the second.

Step (2) As the next entry in the column, insert the remainder when the number above is divided into the number one place higher.

Step (3) Repeat step (2) until the remainder obtained is zero.

Step (4) The greatest common divisor is the last column entry before the zero.

Several comments on the algorithm are important. First, the procedure must work for any given pair of positive integers; for this reason (we might be given two equal integers), the phrase "or equal to" must be included in step (1). Second, the form used above for the algorithm is by no means the only possible statement. Since division is accomplished by repeated subtraction, it is possible to phrase the algorithm purely in terms of a sequence of subtractions (as indicated in the problem at the end of this section). Finally, the algorithm is essentially a form of computer program for the solution: a sequence of steps which can not be misinterpreted and which leads logically to the problem answer.

A second example

In the example which follows this one, we wish to solve a problem of routing a police car through the city streets. Before considering that problem, however, we describe first a simpler version, which happens also to be a famous problem historically. In 1736, Euler* considered the question of whether it was possible to walk once across each of the seven bridges over the Pregel River in Königsberg, Germany (i.e., to walk over all bridges without crossing any bridge twice). One could start at any desired point and end up at any point.

A crude map of Königsberg is shown in Fig. 4. There is an island (marked region C), two sides B and D of the main river, and a region A between the two sections of the river after it divides.

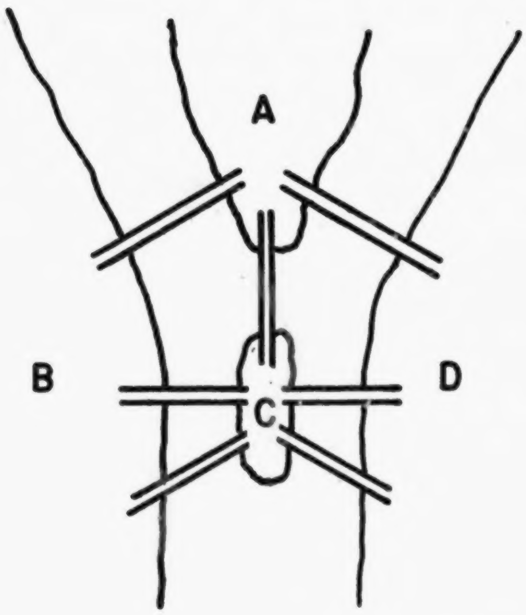

Fig. 4 Seven bridges of Königsberg

The solution of the problem is obvious if we draw a graph as a model for the actual problem (Fig. 5). In this graph, each vertex represents a region (A, B, C, or D) and the lines represent the bridges. Travel over a bridge once is represented by one traversal of a line in the graph.

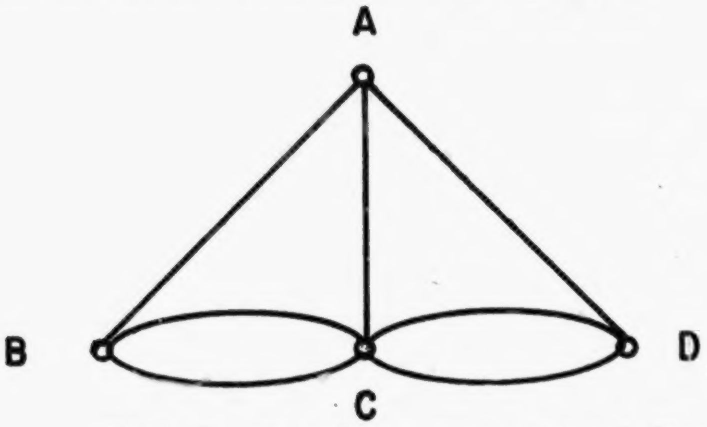

Fig. 5 Graph as a model of Fig. 4

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*The most famous of Swiss mathematicians, who published an enormous number of articles on all phases of mathematics and physics.

Thus, Fig. 5 is simply another way of showing the seven bridges (with each line in the graph indicating one bridge).

We now wish to determine what property this graph must have if we can indeed travel over each bridge once and only once. Let us suppose there is such a path. If it starts from vertex A, we might go first to vertex B. We would enter B on one line and leave on another line. Thus, each time we pass through a vertex, there must be exactly two lines connected to that vertex.

After the total path is drawn, we can say: every vertex has an even number of lines connected to it. The only possible exceptions are the vertex from which we start and the vertex at which we end (e. g., if we start at A and go to B and so on, we leave A along one line; thereafter, we may pass through A again and two lines to A are added). *

Thus, a closed path covering all bridges is possible only if:

(1) Every vertex has an even number of lines (then we start and end up at the same point), or

(2) Exactly two vertices have an odd number of lines (then we start at one of these and end up at the other).

Inspection of Fig. 5 reveals that the vertices have the following numbers of lines:

A	3	C	5
B	3	D	3

All four numbers are odd; hence there is no hope of solving the Königsberg bridge problem.

If one additional bridge were built (or we were permitted to swim across a river once), as in Fig. 6, the problem could be solved (and Euler also showed this part of the solution). Now only vertices B and D are odd (as shown by the numbers in the figure); hence we can start from B, traverse all bridges, and end up at D.

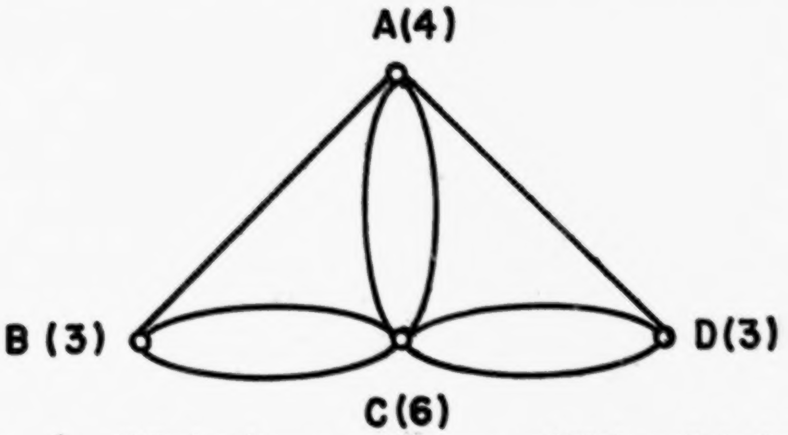

Fig. 6 Königsberg with one additional bridge.

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*If we start and end anywhere between vertices, every vertex must possess an even number of lines.

While the present case is so simple we can see a solution by inspection, we really want an algorithm which works in all cases. Euler's solution is as follows:

(1) Follow any path from B to D (e.g., BAD).

(2) Redraw the graph and omit the path already traced (Fig. 7).

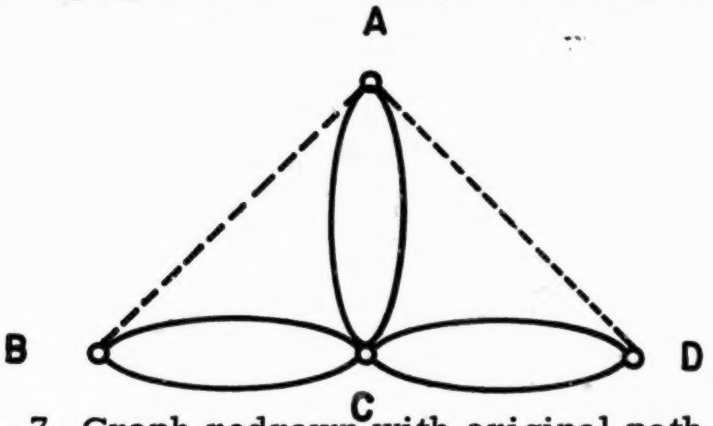

Fig. 7 Graph redrawn with original path omitted.

(3) From any vertex along this omitted (BAD) path, trace a closed loop (e.g., BCB).

(4) Redraw the graph and omit this loop of (3) (Fig. 8).

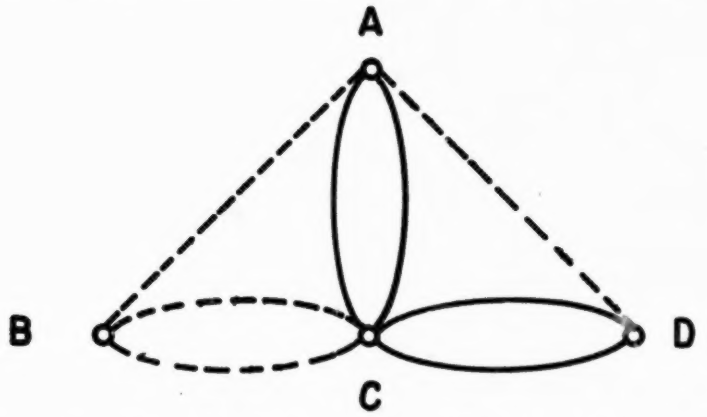

Fig. 8 Graph redrawn with first loop omitted.

(5) Continue this process until all the original graph has been covered (e.g., we might next trace the ACDCA loop).

(6) A solution now consists of the following steps. We start at B. We first follow all loops out of B (in our example, BCB only in Step 3 above). We then travel along the original path (BAD) to the next vertex (A). Here we stop to trace out all loops starting from A (the loop ACDCA in Step 5 above). We then proceed to the next vertex on the main path, and continue this process until the entire diagram is covered and we have reached the terminal point. At each stage, we can redraw the graph as indicated in the above example, in order to keep track of the positions which have been traversed.

In other words, our solution says we should travel in the sequence

$$\underbrace{B\ C\ B}_{\text{First loop}} \quad \underbrace{A\ C\ D\ C\ A\ \ D}_{\text{Second loop}}$$

in order to cross all eight bridges with no retracing of steps

Thus, for this example we have essentially derived two algorithms: one for determination of whether a solution is possible, and the second for determination of a route to be followed.

Problem 2-1: Test the steps of the algorithm which can be used to solve any problem in the above category. We can assume we start with a graph as a model for the system; what the solution desires is either a statement that the problem is not solvable or a systematic procedure for determination of a path which traverses every line once and only once.

Routing police cars

A town precinct covers the grid of streets shown in Fig. 9. A police car starts at A, and we wish to route this car so that all streets will be patrolled once, preferably in a minimum time (so that the second patrolling can be started as soon as possible).* In order to determine a desirable route for the car, we can apply directly the methods of the preceding example; for this case, every

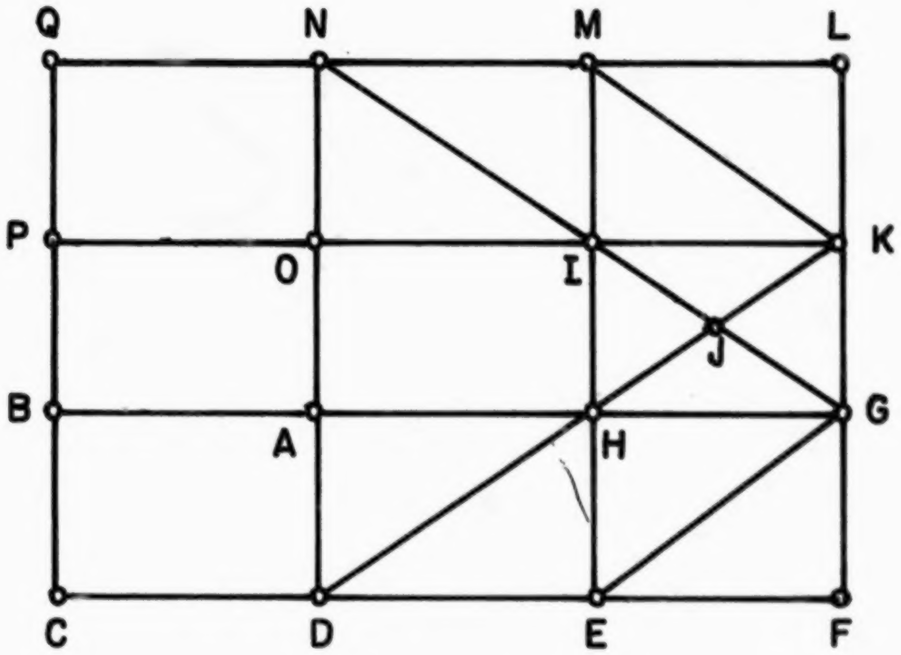

Fig. 9 Precinct streets

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*Covering the precinct in minimum time is obviously achieved if we can find a route which covers each block once only. If this is not possible (as in this problem), the route must retrace as few blocks as possible (or select those blocks in which travel time is minimum).

intersection (marked by a letter in Fig. 9) is a vertex.

Ideally, we should like to start from A, traverse all streets once, and end up at A. Inspection of the figure reveals that this is impossible because the following vertices have an odd number of lines:

B	3	K	5
G	5	P	3

In order to permit a path from A back to A and through all streets, we must add paths which change these odd numbers to even (and leave the others even). In other words, we can make a total closed path possible by inserting an extra path from P to B and from G to K (meaning that the car will traverse the P-B and K-G blocks twice during the patrol). This change is shown in Fig. 10.

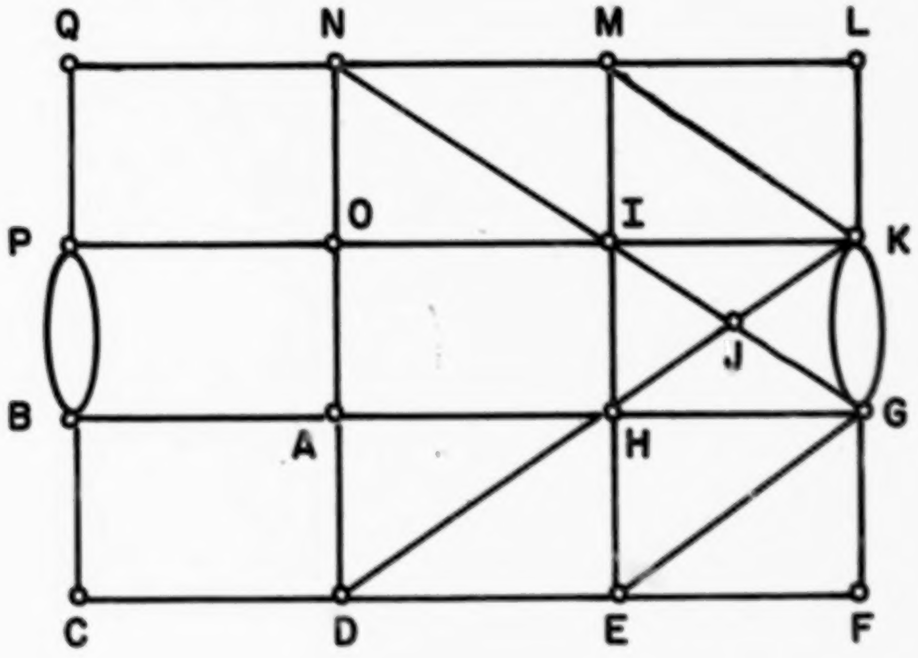

Fig. 10 Street map with added paths from P to B, K to G (corresponding to two traversals of these two blocks).

Once this change in the diagram is made, the route-selection algorithm of the preceding example can be used to determine an appropriate route for the patrol car.

Two comments are important in this particular example:

(1) In a more complex problem, there may be numerous widely separated vertices with an odd number of lines. In such a case, there are many different ways to insert extra lines (retracing of certain streets) to make all vertices even. If we wish a patrol requiring minimum time, we must evaluate the extra time required for each of these possible retracings.

(2) We normally wish to vary the patrol pattern of the police car (so that the patrol car does not pass a particular point at regular intervals of time). This variation can be achieved if we find different solutions from the route-selection algorithm.

Problem 2-2 For the system of Fig. 10 with the lines from P-B and K-G added, determine an optimum path on the basis of the route-selection algorithm. Start with the A-to-A path made up of

A B C D E F G H I J K L M N O A

and then add loops from the diagram remaining after this path is deleted. If each block requires one minute for traversal, what is the minimum time for a complete patrol? How much time is saved compared to the common back-and-forth patrol represented by the path

A O N Q P O A B P B C D A H E D H I O N M I N M L K M I K J I H J K G J H G E F G H A.

(For evaluation of the time required here, the original street diagram of Fig. 9 should be used since this drawing represents the actual problem).

Problem 2-3 The algorithm for the determination of the greatest common divisor of two positive integers can be rephrased in terms of subtraction rather than division. To go from entries 1 and 2 in the column to entry 3, we used the remainder resulting from division of 2 into 1. Alternatively, we can subtract entry 2 from 1; if the result is less than entry 2, we use it as entry 3; if it is greater than entry 2, the difference becomes entry 2 and the previous entry 2 becomes 3. The process is now repeated with entries 2 and 3, etc., until the difference obtained is zero. The last preceding entry is the desired answer. Show that this subtraction algorithm works when we start with 1740 and 2436. Compare the effort involved in the two forms of the algorithm when the original integers are 42 and 9.

Problem 2-4 As a final example of an algorithm, we consider the following game. There are 27 matches on a table and two players, A and B. The players alternately pick up and retain 1, 2, 3, or 4 matches. Each player knows how many matches the other has at all times. The winner is the player who has an even number of matches at the end of the game. A desirable algorithm for A consists of the following steps:

(1) On the first move, A picks up 2 matches.

(2) At each move thereafter, A proceeds as follows:

(a) If B has an even number: divide number of matches still on the table by 6 and find the remainder of this division. Take one less than the remainder.

(b) If B has an odd number: take one more than the remainder unless the remainder is 4, in which case take four (as soon as there are 1 or 3 matches on the table, A should take them all).

Take several examples and show that A wins by this strategy. As an example, let B take successively 1, 3, 3, 1, 2, and 1 at his turns. This example of an algorithm demonstrates that (just as in a computer program) there may be alternate paths which we have to follow to obtain a solution. In this case, the strategy to be used by player A depends on whether player B has an odd or an even number of matches.

3. CRITERION

In Sec. 1, the four elements of the decision-making problem are listed as

Model	Constraints
Criterion	Optimization

In the preceding section, we discuss the optimization part, particularly when it is possible to find an algorithm or logical, complete procedure to a solution. In this section, we consider (via another example) the element or factor called the criterion .

In many decision problems, the choice of criterion is far from obvious. Probably the most significant decision made by most men throughout their lives is the selection of a wife. In this case, if one were rational and logical, he might attempt to formulate the problem in the following way:

(1) Model: quantitative description of the characteristics of different girls (appearance, beauty, intelligence, personality, character, aesthetic preferences, sense of values, attitude toward marriage, etc.).

(2) Criterion: the relative values given to each of the above characteristics by the young man.

(3) Constraints: represented by the limited number of girls who would accept a proposal (if this number is precisely one, the decision problem simply reduces to the question of whether to marry at all).

(4) Optimization: selection of the girl to maximize the criterion subject to the constraints.

While this particular problem perhaps best illustrates the fact that most decision-making in life is carried out intuitively and on the basis of poorly defined criteria, it is also clear that the two critical parts of an optimization problem are the model and the criterion--particularly the latter. While two young men might agree fairly well on the model (the different characteristics to be included and the relative quantitative ratings of particular girls), they would probably disagree strongly on the relative importance of these various characteristics. The criterion would vary markedly from one young man to the next.*

---

*This is, of course, most fortunate; otherwise every man would want to marry the same type of girl.

In a much simpler example (simpler because we can all agree on the statement of the criterion), we can illustrate another aspect of the criterion: the fact that we often have to change the criterion as the problem is being solved. In many cases, we discover that no solution can be found unless the criterion is simplified; in other problems, we can sharpen up the criterion as we move toward a solution. The latter situation is illustrated in the following example.

Figure 11 shows a model of a system of corridors in a building. There are three corridors: one from a to b, one from a to c, and one from b to c. These three corridors connect the points a, b, and c. During the time in which we are interested (e.g., the time during class changes in a school), the flow of people is from a to b in that corridor; hence, it is not convenient to try to walk from b toward a. We show this property by placing an arrow from a toward b.

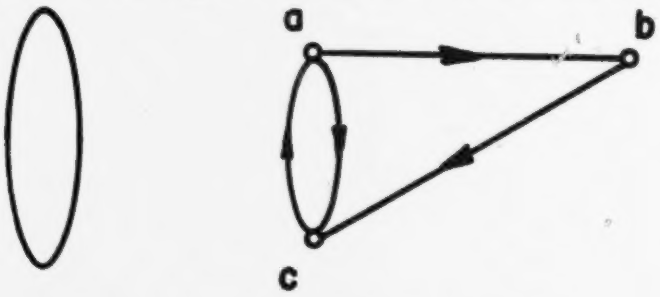

Fig. 11 A set of corridors

The corridor from a to c is similarly essentially one-way from a to c, however, the corridor is wide, and we can move rapidly in either direction (this situation is represented by a double line connecting a and c, with an arrow in each direction). Thus, Fig. 11 represents the convenient pattern of travel throughout the system of corridors.

Now we turn to a statement of the problem we wish to consider in this section. Our task is to station one or more men at the intersections (a, b, and c) in Fig. 11 in such a way that every intersection is covered by a man at most one "block" away. In other words, we might use three men: one at a, one at b, and one at c; each man would then be responsible for monitoring his intersection.

Alternatively, we might place a man at b. He is able to cover both b and c, since he can travel to c rapidly in case he is needed there (there is a one-way path from b to c directly). Our man at b cannot, however, monitor intersection a: to reach a, he would have to travel two "blocks", from b to c and then from c to a. Hence, if we place one man at b, we must place another man at either a or c to cover intersection a.

The problem is where to locate the men in order to minimize the number required. The system of Fig. 11 is so simple that we can see by inspection that one man stationed at a could cover all three intersections (a, b, and c). If our only purpose is to solve this single problem, we are now finished. If is not difficult, however, to visualize a much more complicated pattern of corridors in which the answer is not obvious. In the hope of being able to solve such problems, we consider this simple example and look for an algorithm which works for all problems of this type.

The algorithm

We must station our men to "cover" or monitor each intersection. We first consider intersection a. This can be covered by a man stationed at a or at c. Hence, we represent coverage of a by the expression

$$(a + c)$$

Similarly, intersection b can be covered by a man at a or at b, represented by

$$(a + b)$$

Now to cover both intersection a and intersection b, we can represent the stationing of men by the product

$$\underset{\text{of a}}{\underset{\text{coverage}}{(a + c)}} \quad \underset{\text{of b}}{\underset{\text{coverage}}{(a + b)}} = a^2 + ab + ca + cb$$

The first term, a^2, means a man at $\underline{a}$ covers both intersections; the second term, ab, means the man at a covers a, the man at b covers b; ca means the man at c covers a, at a covers b; and cb means the man at c covers a, at b covers b. In other words, the product

$$(a + c)\ (a + b)$$

represents all possible ways of covering intersections a and b.

To cover all three intersections, we consider

$$\underset{\text{of a}}{\underset{\text{Coverage}}{(a + c)}} \quad \underset{\text{of b}}{\underset{\text{Coverage}}{(a + b)}} \quad \underset{\text{c}}{\underset{\text{Coverage of}}{(a + b + c)}}$$

If we multiply out this product, we find one term (a^3) involving only one letter. In other words, there is just one location (a) at which we can place a man who can cover all three of the intersections.

A more complex example

The problem of Fig. 11 is, as we noted, too simple really to indicate the power of this method. In order to consider a more interesting case, we turn to the corridor arrangement shown in Fig. 12, with six intersections -- a, b, c, d,

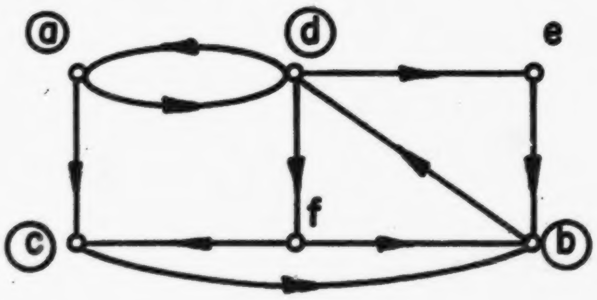

Fig. 12 Corridor map

e, and f. Because of the history of activity in this system, we desire to place men in such a way as to cover the key intersections a, b, c, and d. (Intersections e and f are of secondary importance and can be less accessible to monitoring).

Obviously a suitable disposition of men is to station one at each of the four intersections. In order to decrease the number of men required, however, we desire to provide the required coverage with a minimum number of men.

The statement of the problem is now complete, and we can turn to the solution. We must "cover", in the sense used above, each of the four intersections: a, b, c, and d. Inspection of Fig. 12 reveals that intersection a can be covered by a man stationed at intersection a or d, intersection b can be covered by a man stationed at b or c or e or f, and so on. If our short-hand description of the problem is expanded to cover the four essential intersections, we obtain the following expression for the stationing of men:

$$\underbrace{(a+d)}_{\substack{\text{To cover}\\ a}}\;\underbrace{(b+c+e+f)}_{\substack{\text{To cover}\\ b}}\;\underbrace{(a+c+f)}_{\substack{\text{To cover}\\ c}}\;\underbrace{(a+b+d)}_{\substack{\text{To cover}\\ d}}$$

After multiplication of this, we obtain 72 terms (2x4x3x3), each of which represents a suitable location of men to provide the required coverage.*

In this horrendous product, typical terms are, for example,

$$dcab \qquad a^2bc \qquad a^2b^2$$

The term, dcab, means a man at d, one at c, one at a, and one at b. The term a^2bc means three men at a, b, and c, with the man at a covering two intersections. Since there are terms with only two letters, we can provide the required coverage with two men only; we can forget about all terms in the product with more than two letters.

The product can then be written**

$$(a+d)(b+c+e+f)(a+c+f)(a+b+d)=a^3b+a^3c+a^3f+a^3e+a^2b^2+c^2d^2+a^2c^2+a^2f^2+d^2f^2+\text{ terms with more than two letters.}$$

---

*A few of the terms are identical, so there are slightly less than 72 different possibilities.

**These terms can be written without the tedium of writing out all 72 terms. Each product term contains one element from each factor; we merely look for combinations involving two letters only. For example, a appears in the first, third, and fourth term. Hence a^3 goes with any element in the second term. If we were to follow the occasional pattern of textbook authors, at this point we would leave the listing of all 72 factors as "an exercise for the student." The dominant characteristic of applied science and engineering is, however, that it is fun; hence we attempt to find those terms we need merely by inspection of the product.

We have now nine solutions to our original problem, nine different ways to station two men to provide the coverage required. The meaning of the terms is illustrated by the following two specific examples:

a^3b The man at a covers three intersections, that at b covers one;

c^2d^2 The man at c covers two intersections, that a d covers two.

The existence of nine equally valid solutions is an unexpected delight. In practice, however, we must choose one of these. Apparently we did not ask for enough in the original choice of a criterion. Instead of just asking for the minimum number of men, we can add additional criteria (i.e., we can change our original criterion).

For example, we might require that each man be responsible for only two intersections. In other words, we ask for as even a division of the work load as feasible. In our example, this added criterion rules out the terms $a^3(b+c+f+e)$ above, and leaves us with only the five possible solutions:

$$a^2b^2 + c^2d^2 + a^2c^2 + a^2f^2 + d^2f^2$$

Even with five equivalent solutions we are unusually wealthy and we can add another criterion. We ask that each man be responsible for the intersection at which he is stationed plus one other (in order to minimize his travel time). This condition rules out a^2f^2 and d^2f^2, since in both cases the man at f is responsible for intersections b and c. We then have three possible solutions:

$$a^2b^2 + c^2d^2 + a^2c^2$$

Finally, we might select one of these three merely on the basis of the probable location of crime or possibly the personal desires of the man involved.

General comments

The interesting part of this problem is the gradual evolution of the criterion during the process of solution. We started off with the simple criterion of minimizing the number of men. We found that we could add additional parts to the criterion: evening of work load, minimizing required travel, and finally catering to personal preferences.

In most optimization problems, the difficulty of finding a solution requires that the criterion initially be chosen as simple as possible. Once solutions are found, additional criteria can be used to select among these.

The entire question of what constitutes a satisfactory criterion underlies a large part of optimization work and decision-making. The continuing debate in the United States throughout the 1960's on the need for new weapon systems (for example, anti-missile missiles) stems from different criteria of evaluation. The proponents of a new system weightheavily the military tactical and strategic advantages; the opponents frequently tend to place major weight on the economic needs of the country for urban rehabilitation (for example). The difficulty of decision arises because of the constraints imposed by the need to limit government spending in order to control the national economic picture. When radically different criteria are used in the decision-making process, it is not at all surprising that grossly different decisions are reached.

In this section, we attempt to emphasize the key role played by the criterion in the typical decision problem. Of our four parts,

Model	Constraints
Criterion	Optimization

the model represents the system with which we are working, the constraints represent the limitations imposed on the permissible solutions, the criterion represents a statement of system objectives, and the optimization is the algorithm yielding a solution. In applied science, as well as in the realm of personal decisions, the criterion is that element of the problem which is most difficult to describe precisely.

4. THE SEARCH FOR ALGORITHMS

Throughout the first three sections of this chapter, a series of optimization examples are discussed:

The selection of a car speed to maximize gas economy

The routing of police patrol cars to minimize time required

The stationing of men to minimize the number needed to cover a set of intersections and corridors.

Although these case studies have been included to emphasize the different parts of a decision or optimization problem, we should like to be able to begin to develop an understanding of the method of solution of such problems. After all, once these problems are phrased (model, criterion, and constraints listed), they are mathematical problems, and one of the most attractive features of mathematics is that general rules can be developed for the solution of an entire class of problems.

For example, one class of mathematical problems is concerned with determination of the average of two different numbers a and b. The average is simply the sum divided by the number of terms, or in the case of only two given quantities

$$\text{average} = \frac{a + b}{2}$$

The average height of the boys in a group can be found by adding all heights and dividing by the number of boys. Thus, for finding the average of a set of numbers, we have a straightforward algorithm or set of steps to follow.

Is there an algorithm (or at least a small number of algorithms) which suffices to determine the solution of decision or optimization problems?

Unfortunately, the answer is no.

There are, of course, algorithms and approaches which are useful for a group or class of optimization problems. The remaining chapters of this text consider a few of the more important methods of solution; in the rest of this chapter and the next we focus exclusively on these methods. The fascination of applied science and engineering lies, however, in the challenge provided to us as we search through our arsenal of mathematical techniques for methods which are useful in the particular problem under consideration.

Thus, it is not useful to attempt to list different categories of optimization problems and then to write beside each the appropriate mathematical approach. While this inability to classify problems in a clean and neat list is unfortunate for the individual meeting the subject for the first time, it does add to the excitement of the field. Hardly anyone feels much of a thrill when he calculates

$$\frac{a + b}{2}$$

The solution of our stationing problem in the last section is exalting: when we started, we really didn't know what form the solution would take or even if it would be possible to find a solution with less than four men.* In this sort of mathematics and applied science, suspense is a common element.

A second feature of optimization problems is that common sense (or intuition) often is not very much help. Simple mathematical "illusions" are as common as optical illusions, and one is often deceived by common sense. In the most familiar example of such a mathematical illusion, an honest coin is tossed repeatedly, with a sequence of heads and tails. As soon as four heads in a row come up, the tossing pauses while bets are placed on the single, next toss. Most people would prefer to bet on a tail, and occasionally we can even find an "educated" individual who will give us favorable odds if we allow him to bet on a tail coming up next. Actually, the probability of a head coming up in the next toss is exactly 1/2, since preceding tosses are irrelevant, past history.

The examples of such breakdowns of common sense are legion. As another example, we consider a system in which a series of men check their hats as they enter a restaurant. Unfortunately, the checkroom attendant is hopelessly unintelligent and simply gives each man, as he leaves, a hat selected at random from her collection. What is the probability that no man receives his own hat?

The interesting feature of this problem is not the particular answer, but two aspects of the answer:

(1) The probability of no one receiving his own hat is essentially the same if there are 8, 80, 800, or 8000 men. Once the number exceeds 8, the actual number is largely immaterial (37% of the time this experiment is tried, no one receives his own hat).

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*Incidentally, the author was not sure either as he started to write that section. He knew only that the approach described there was often useful. If the problem had not worked out successfully, he would have had to select a different corridor map and start again.

(2) The probability is higher for an even number than for either adjacent odd number. In other words, the event is more likely to occur with four men than with either three or five, more likely with six than with five or seven.

Both these results are rather startling and counter to the intuition of most people.

This same feature frequently characterizes optimization problems. It is for precisely this reason that a logical, algorithmic solution is often most desirable. In order to illustrate this feature, we consider one final example in this section.

Stockton's famous story, The Lady and the Tiger, describes the plight of a common man in love with a princess. The king discovers the romance of his daughter and condemns the man to open either of two doors in the arena. Behind one door is a tiger who will devour the man; behind the other door is the loveliest maiden in the land whom the common man will marry if he opens that door. The princess discovers which door conceals the tiger and which the maiden. As her lover stands in the arena making his decision, the princess points to the right-hand door.

The man is then confronted with a decision: should he follow the suggestion of the princess or not? Has she pointed to the tiger (to prevent him from marrying the maiden) or to the maiden (to save his life)?

In order to make an optimum decision, the man must assign some appropriate,numerical values to the different outcomes. For example, he might decide being eaten by the tiger has a value -10 (the actual number is not particularly significant, but the value is certainly negative since this is a highly undesirable outcome). If he opens the door for the tiger, it makes no difference whether the princess lied or not.

If he opens the door to the maiden, however, the value to him depends on whether the princess lied. If she lied (and was trying to kill him), he will be extremely happy with the maiden; if she told the truth (and was anxious to save his life), he will be much less happily married to the maiden while he realizes the depth of the princess' love. We might assign +20 and +10 to these two outcomes, respectively.*

Actually, the problem we have phrased is termed in mathematics a game played between two sides, the man and the princess. It is described by the table shown below:

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*The actual numbers here must, of course, be determined by the man as he stands before the two doors and mentally solves this optimization problem. Since he is playing this "game," he alone can decide on the relative value of different outcomes. We are merely guessing his thoughts (in the actual story, the author does not indicate the outcome).

		Princess (B)	
		Points to lady	Points to tiger
Man (A) chooses	Door to which princess points	10	-10
	Other door	-10	20

The princess has two possible strategies: she can point to the lady or to the tiger. The man likewise has two options: to select the door to which the princess points or to choose the other door.

If we assume the princess is trying to kill the man and the man wants to maximize his return, we can solve this problem (although we shall not discuss the details of solution here). The best strategy for the man to follow is for him to select the door to which the princess is pointing 3/5 of the time. Since he is only going to play the game once, he should make a random decision, with 3/5 probability of the door to which the princess points. One easy way to do this is to look at the second hand of his watch as he approaches the doors. If this hand points anywhere between 0 and 36 seconds, he selects the door indicated by the princess. If the hand is between 36 and 60, he opens the other door. This strategy maximizes his expected benefit from the game.*

The interesting feature of this optimization problem is that the man should favor the move which promises him a smaller payoff (or gain): 60% of the time he is playing in such a way he can win only +10; the other 40% he plays when he can win +20. His optimum strategy is counter to that which he might well follow by intuition.

The details of this particular example are not important for the basic objectives of this chapter. The example is included only as an illustration of the common failure of intuition in optimization problems--and the importance of an algorithmic solution. The example, itself, is certainly open to argument: e.g., why did we select +10 and +20 for our positive values in the table? Fortunately, the actual numbers chosen are not very important; we obtain very nearly the same answer if we use +5 and +10, or +20 and +40.

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*We continue to call this a game because of the mathematical acceptance of that term. For our young man, it is obviously a game of life and death. We might formulate a similar gambling game. The two strategies could be the holding forth of one or two fingers simultaneously by the two players; the numbers in the table then show the dollar payoff to A after each play. If this game is played, repeatedly, A can expect to win an average of $2 each play, and he should select one finger 60% of the time, two 40% chosen at random. Similarly, the same sort of game arises in military tactics (e.g., the deployment of opposing forces).

While this section emphasizes that there is no possibility of establishing a list of strict rules to follow in solving optimization problems (and each type of problem must be considered from an understanding of the system), we do wish to consider, in this and the following chapter, a few basic types of problems for which we can discuss systematic solution methods. In the next section, we consider problems with a reasonably small number of alternatives; in the last section of this chapter, we consider problems solvable by a method called dynamic programming.

5. OPTIMIZATION WITH FEW ALTERNATIVES

When optimization involves only the choice of one from a few alternatives, the selection can often be made by a straightforward comparison of the criterion function evaluated for each possibility. In this section we consider two examples of such an approach, the first involving only two alternatives and the second requiring a selection from among a manageably small set.

Choice from two alternatives

If only two feasible designs or plans are found, selection of the better of the two is usually simple if a quantitative performance criterion is agreed upon by those responsible for the evaluation of the design.

A simple route-planning problem illustrates this class of problems. We consider the situation shown in Fig. 13. Bill is late for a date with Jean. He is anxious to drive from his house to Jean's house in the minimum time. By rural road, which is 6 miles long, he can average 30 miles per hour. Driving over a winding feeder road, he can reach a parkway on which he is permitted to drive at the rate of 60 miles per hour; he then takes another feed road to Jean's house. He can drive at 30 mph on the feeder roads which are each one mile long. The parkway *section is six miles long, so the total route is* $6 + 2 = 8$ *miles, as* compared to the six miles for the rural road. Which road should Bill take?

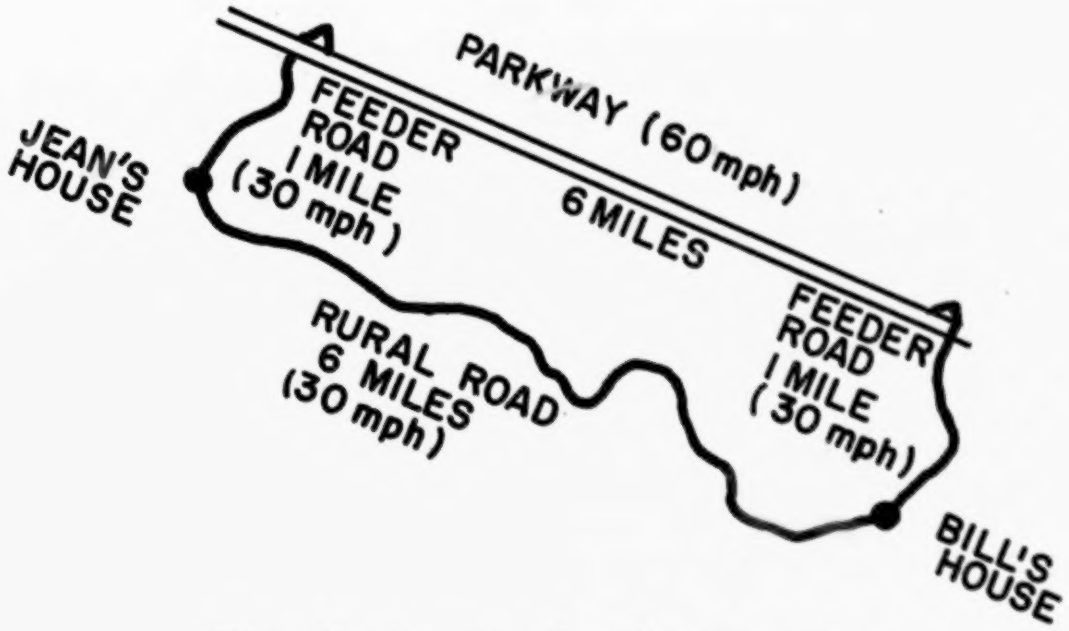
PARKWAY (60 mph)
6 MILES
FEEDER ROAD 1 MILE (30 mph)
JEAN'S HOUSE
RURAL ROAD 6 MILES (30 mph)
FEEDER ROAD 1 MILE (30 mph)
BILL'S HOUSE

Fig. 13 A route planning program.

The rural route can be driven in 6/30 = 1/5 hour, or in 12 minutes. On the other route, the two-mile trip along the feeder road can be completed in 2/30 = 1/15 hour, or 4 minutes, and the 6 miles of parkway require 6/60 = 1/10 hour, or 6 minutes: a total of 10 minutes. Thus the longer route is two minutes shorter in time. Bill should take the parkway route because he is late and must use the faster route. If he were not late, he might choose the rural road because it requires less gasoline, or because he enjoys the scenery. Clearly the selected route depends on the performance criterion that is used.

The problem above may seem rather unexciting: one can hardly care very much about a time difference of a few minutes. Obviously, however, the numbers can be selected so that the difference is more significant. Furthermore, if we are interested in dispatching fire equipment from a station house to a fire, the saving of a few minutes by optimum routing may be of critical importance.

The type of analysis involved in this problem is also useful in designing an improved urban transportation system. One of the astonishing features of traffic systems in our cities today is the poor information which they communicate to the driver. For example, the motorist driving toward the center of the city from the suburbs typically passes several points at which he can take alternative roads. These are the places at which clearly visible signs should indicate the current travel time to the city center along parallel routes so that he could make a reasonable choice. Instead, he normally has no help at all in the decision process (except possibly for a radio reporter in a helicopter who is giving very qualitative and general impressions of the overall traffic situation). While major expressways in some city areas cost $20,000,000 per mile or more, the trivial expense of installing sensors to measure traffic flow and communication equipment to inform the motorist is neglected altogether.

Choice from a few alternatives

If a small number of feasible designs or plans is available, it is possible to study each plan carefully and to select that plan which is best. This is sometimes referred to as "the brute force approach". To illustrate this method, we consider a very common replacement problem: namely, when should one "trade-in" his car to reduce the average operating cost per year to a minimum?

The problem may be simplified by considering trade-ins only after 1, 2, 3, ... years of use. We must also estimate the cost of operation of the car for each year of use. This cost must include the cost of fuel, oil, tires, batteries, maintenance, insurance, registration fees, taxes, and probable repair. The cost of operation depends on the number of miles the car is to be driven each year, since this use determines depreciation, repairs, and so forth.

In the table below we list the data (columns 2 and 3) and the computations (columns 4 to 7) necessary to answer our question. In column 1 we list the age of the car, and in column 2 the corresponding estimate of the sale value of the car.*

---

*The car is assumed to depreciate only $600 during the first year. The actual value depends on the make of car, the market for second-hand cars, the mileage driven, and the quality of the care given to the car.

Column 3 lists estimates of the operating expense for each year from the year of purchase.*

Year (1)	Sale Value (2)	Oper. Cost (3)	Depreciation (4)	Cumu. Oper. Cost (5)	Total Cost (6)	Av. Cost Per Year (7)
0	$3000	-	-	-	-	-
1	2400	$ 800	$ 600	$ 800	$ 1400	$1400
2	1920	850	1080	1650	2730	1365
3	1540	900	1460	2550	4010	1337
4	1230	950	1770	3500	5270	1318
5	980	1000	2020	4500	6520	1304
6	740	1050	2260	5550	7810	1302 ←SELL
7	520	1100	2480	6650	9130	1304
8	320	1150	2680	7800	10480	1310
9	150	1200	2850	9000	11850	1317
10	50	1250	2950	10250	13200	1320

Column 4 is the depreciation, which represents the initial cost ($3000) minus the sale value in column 2. Column 5 is the cumulative operating cost, which is simply the total of the operating costs in column 2 up to and including the present year. Column 6 is the total cost which is the sum of the depreciation and the cumulative operating cost (sum of columns 4 and 5). The last column is the average cost per year, which is simply the total cost divided by the number of years (this is the cost per year if we trade in after 1, 2, 3, ... years).

The minimum average operating cost per year is apparent from the last column. We select that figure which is minimum here. Hence, the best operating cost per year is achieved if we trade-in after six years. It is important to note that this minimum is very "flat"; it costs only $35 per year more (about 3% more) to trade in the car after the third year. If the factors of reliability of operation or of status which results from driving a late model car are given consideration, the car should be traded-in before the sixth year (remember that performance criteria are expressions of subjective evaluations!)

The conclusion that has been reached is, of course, no better than the estimates that have been made of the sale value and of the operating costs. The collection of the necessary data, upon which realistic estimates can be made, represents the major difficulty in the solution of this problem. Furthermore, the solution of the problem does not take into account at all the probable future changes in the price of cars and the cost of operation or the changing value of the dollar to the owner of the car. As an extreme illustration of this latter factor, if severe runaway inflation should occur (as in Germany in the 1920's) money becomes worthless, costs rise at uncontrolled rates, and any economic analysis is worthless days after it is made. Even a slight inflationary trend, however, can modify markedly the optimum trade-in time.

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*The operating costs listed here are low and imply the car is driven only about 10,000 miles per year. They do increase slightly as the years go by because of more repairs and poorer performance.

The two examples described above are among the simplest cases of optimization. In both cases, once a model is constructed (the data are collected), determination of the optimum is a straightforward matter. In the next section, we consider a more complex problem in which there are so many possibilities they cannot all be enumerated and evaluated one-by-one.

Problem 5.1 One of the obstacles to an optimization solution is that it is based on data which are often open to suspicion. In the above problem, for example, we can only guess at the operating costs in the future or even at the mileage driven per year. If the car is demolished in an accident not covered by insurance, operating costs may soar and sale value plummet. We always need to investigate at least the effects of slight changes in data in order to be sure that the data are not too critical (if they are very critical, our solution is valid only if we make very accurate measurements of the important data).

As an example, solve the above problem again, with the operating cost each year doubled (corresponding to driving 25,000 miles/year rather than the 10,000 on which the table above is based). Because of this additional mileage, the sale value is reduced to the sequence of numbers 3000, 2000, 1620, 1240, 930, 780, 540, 320, 200, 100, 20.

6. A MORE COMPLEX ROUTE-PLANNING PROBLEM

When there is a very large number of possible designs which must be analyzed to find the optimum, a direct approach is tedious, time-consuming, and frequently impractical. In such circumstances, we seek an algorithm which reduces radically the number of computations--an algorithm which permits step-by-step elimination of some of the alternatives until an optimum can be determined.

In this section, we consider one such example--a particular type of problem in which an idea called dynamic programming permits simple solution of a complex decision problem.

A power company, located at A in Fig. 14, plans the installation of a feeder line to serve a new factory, located at B. Streets have to be torn up and trenches dug for the new line. Some streets are paved with asphalt, others with concrete; the number of buried gas and telephone lines varies from street to street. On streets with heavy traffic, extra guards must be hired to direct traffic. The costs for the installation of the feeder line in any particular block are given by the numbers indicated in Fig. 14, from a minimum of $3,000 to a maximum of

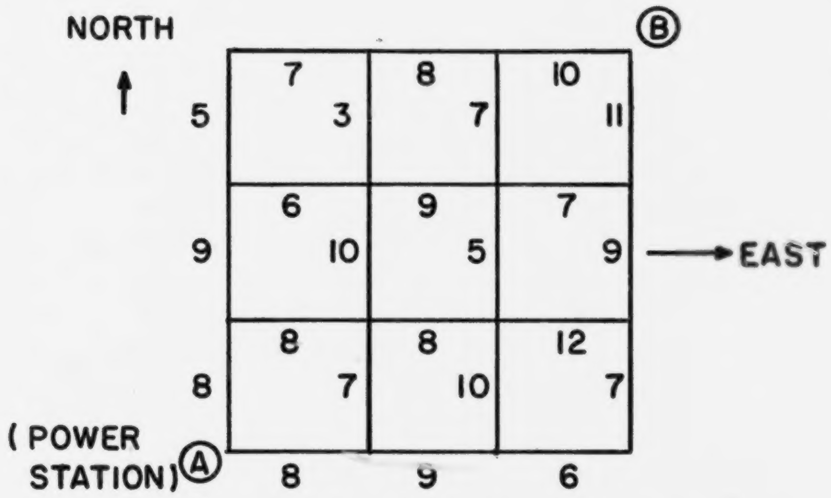

Fig. 14 Costs per block for new feeder line.

$12,000 (costs indicated in the figure are in thousands of dollars). It is required to determine the route which involves a minimum cost from the power station at A to the factory at B.

This problem can be solved by calculating the costs for all possible routes from A to B and then selecting the route with the minimum cost. In order to learn how difficult the problem is, we examine the number of possible routes from A to B, with the assumption that any feasible route will proceed either to the north or to the east (in other words, no route moves south or west, and we always move toward the ultimate destination at B). A simple procedure exists for computing the number of possible routes.

In Fig. 15 the point marked Ⓑ represents the goal to which a path must be selected from position X. Instead of considering X immediately, however, we first consider the two points E and N (east and north of X). If we want to move to Ⓑ from position E, there are 6 different paths by which the goal can be reached. These are shown in the diagram, and we can trace them through the streets, heading east and north at all times. If, on the other hand, we move to B from position N, only 4 different paths exist through the streets.

If our starting point is a position X, we can reach goal Ⓑ by moving to N with its four possible paths or by moving to E with its six available paths. From point X we may therefore select any one of 10 possible paths to reach goal Ⓑ.

Using this principle, we may proceed in a step-by-step fashion to determine the maximum number of paths. First, we note that from any of the intersections on the north edge of the grid, there is only one possible route to Ⓑ -- due east. From an intersection on the eastern edge, one can only proceed due north to get to Ⓑ. We place the number 1 at each of these boundary intersections on Fig. 16. We must now remember that only movements north or east are permitted. From Fig. 16, we next find that the possible routes from the intersection C to Ⓑ are therefore (1 + 1) = 2, which produces Fig. 17. From Fig. 17 we find, in turn, that the number of routes from intersection D to Ⓑ is 1 + 2 = 3 and from intersection E to Ⓑ is 2 + 1 = 3. The pattern is clear; starting from intersection C we move step-by-step to the left and down until each intersection is enumerated. The square (Fig. 18) now shows that there are 20 possible routes from Ⓐ to Ⓑ .*

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*If we removed our assumption that the routes proceed from A to B always to the north or to the east, there would be any number of possible paths (since we could loop around a square block any number of times).

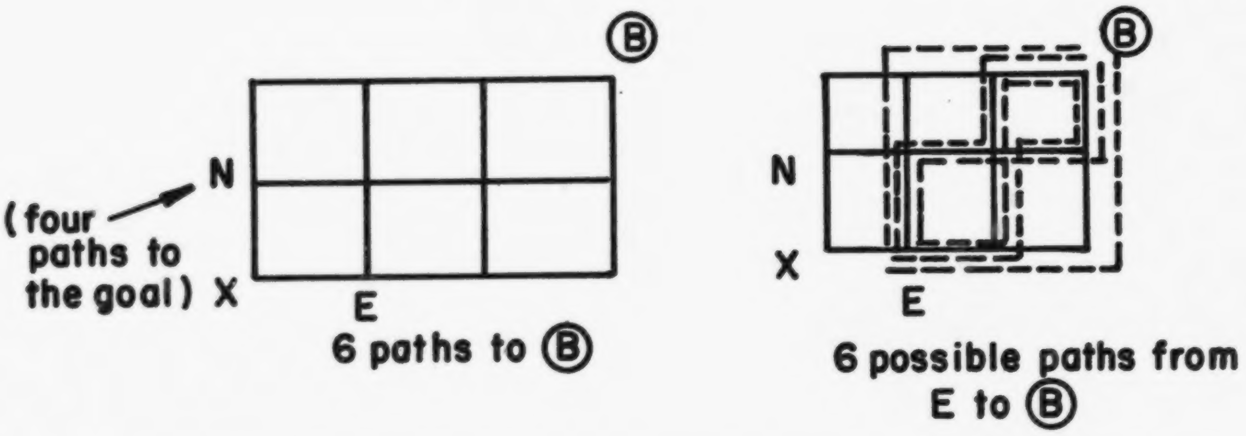

B
N
(four paths to the goal)
X
E
6 paths to B
B
N
X
E
6 possible paths from E to B

Fig. 15 Number of possible paths to the goal at B in terms of those at E and N.

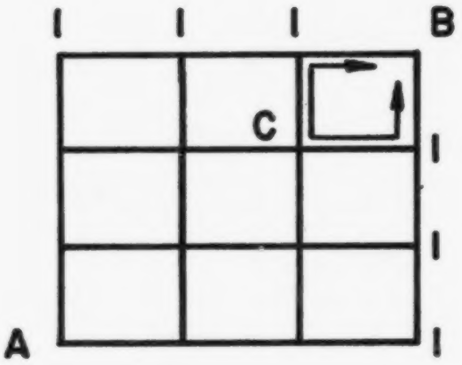


Fig. 16 From north and east boundary intersections there is only one route to the goal; from C there are 2.

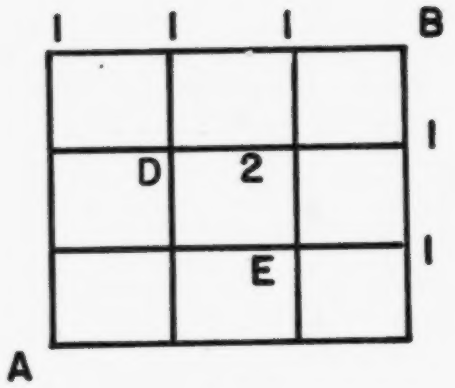

1
1
1
B
1
D
2
1
E
A

Fig. 17 A step in calculating number of routes.

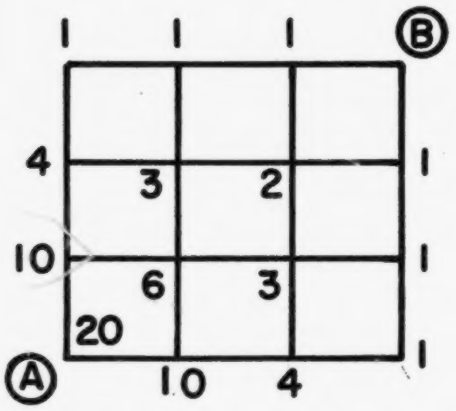


Fig. 18 Completed square giving number of routes from intersection to Ⓑ.

If this calculation is repeated for square grids with a larger number of blocks, the following table of values is determined:

No. blocks on a side	3	4	5	6	20
No. possible routes	20	70	252	924	137, 846, 528, 820

The power company would be faced with a gigantic task if it were to calculate the costs for all possible routes from Ⓐ to Ⓑ for a grid 20 blocks in each direction. Clearly, some method of reducing the enormous number of possibilities to a more manageable number is needed!

The Solution The solution is available if we work from the terminal Ⓑ backwards, rather than starting at Ⓐ and enumerating all paths. In other words, in Fig. 19, we first ask: if we are laying the pipeline and reach C, how do we proceed optimally from here? We then repeat the question for I, which is also one block from Ⓑ.

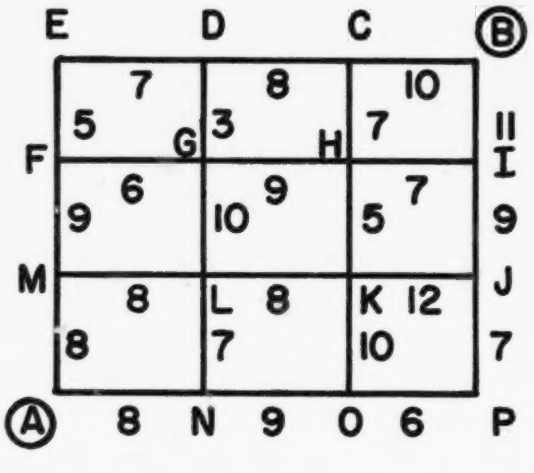


Fig. 19 The original problem with each intersection labelled.

The answers to both these questions are obvious. From C we must go east and the cost is 10; from I, north is the only allowable path and the cost is 11. We indicate these two optima on Fig. 20.

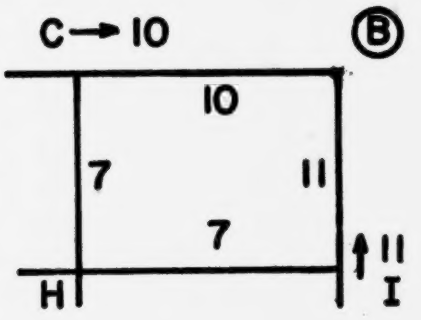


Fig. 20 Optimum paths from C and I.

Next we consider how to move if we are at intersection H. Here we can go north to C; from C we know east is the optimum. Hence north from H brings us to Ⓑ with a total cost of 7 + 10 = 17. East from H takes us to I at a cost of 7, then on to Ⓑ at a cost of 11--a total cost of 18. Hence, if we are at H we should move north and the cost to Ⓑ is 17 (Fig. 21).

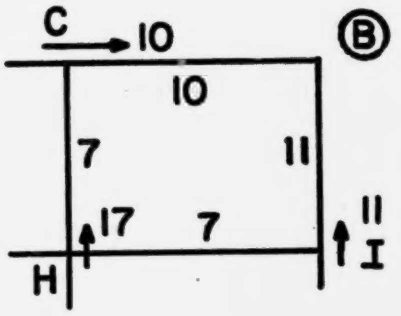


Fig. 21 Optimum path from H

Now we turn to the next set of intersections away from Ⓑ: the intersections at D, G, L, K, and J in Fig. 19. D and J are simple, since there is no choice (Fig. 22). G is calculated next: going north yields a total cost of 3 + 18 = 21;

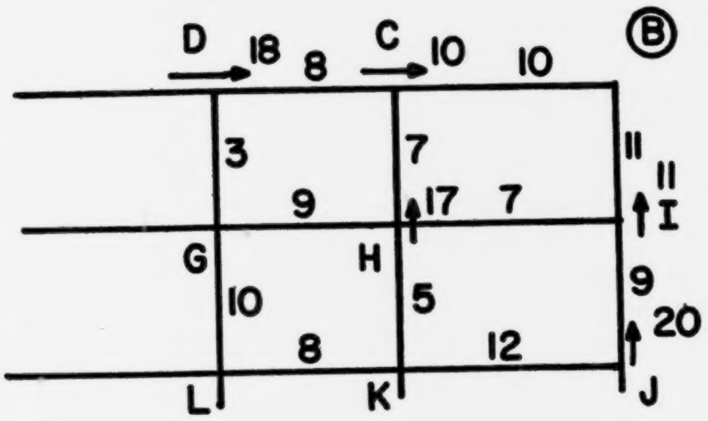


Fig. 22 Optimum paths from D and J

heading east yields 9 + 17 = 26 (from H we already know the minimum cost is 17). Hence the desired path from G is north at a cost of 21.

In the same way the path from K is calculated as north at a cost of 22. Once K and G are fixed, L is calculated and we obtain the data of Fig. 23.

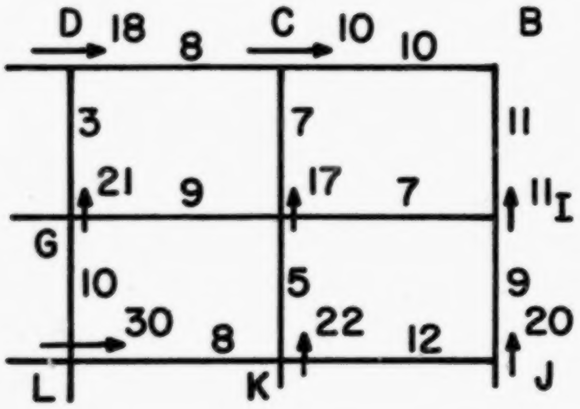

D
18
8
C
10
10
B
3
7
11
21
9
17
7
11
I
G
10
5
9
30
8
22
12
20
L
K
J

Fig. 23 Optimum paths for G, K, L added

We next consider the intersections E, P, F, O, M, N, and A. This process finally leads to the minimum cost values for every corner, shown in Fig. 24. The process also shows at each corner whether to travel north or east by arrowheads drawn upon the preferred path.

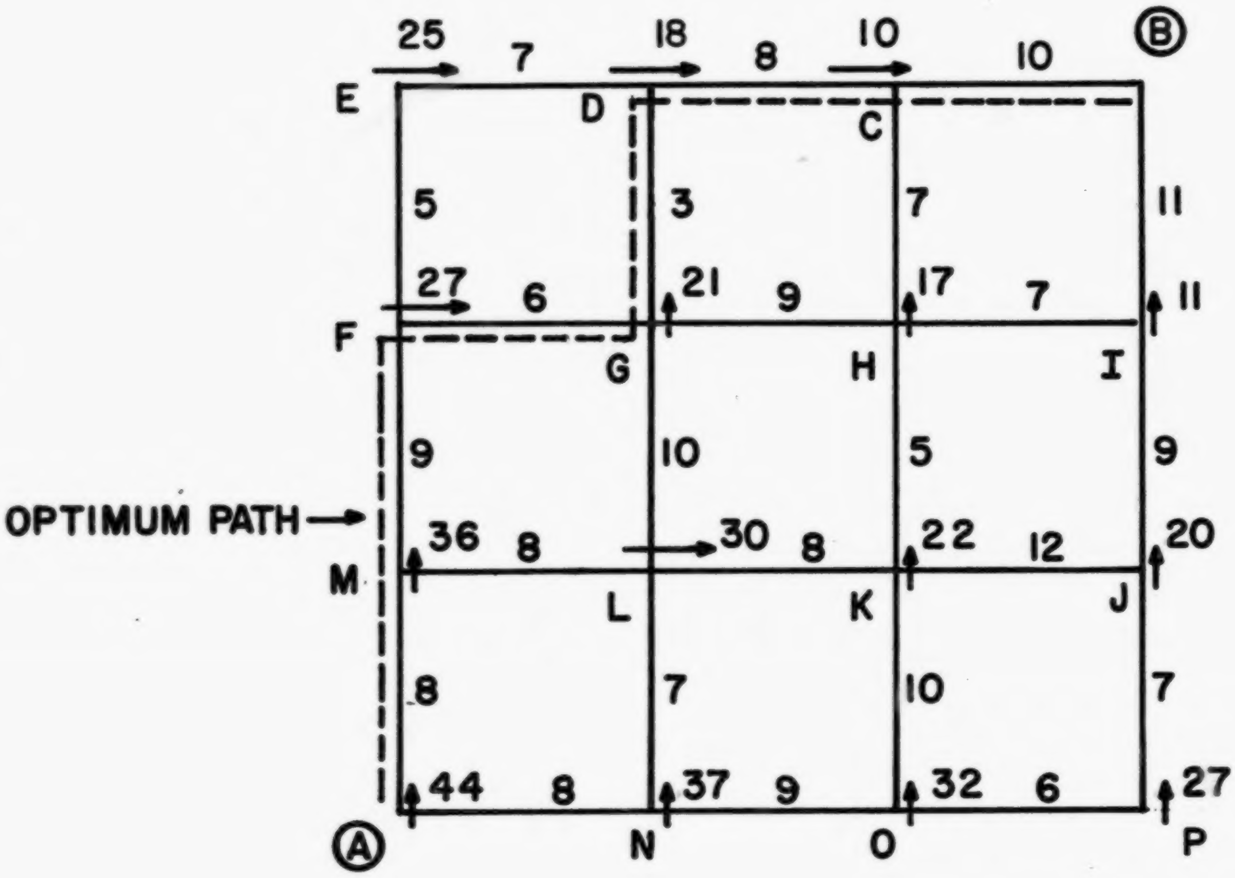

25
7
18
8
10
10
B
E
D
C
5
3
7
11
27
6
21
9
17
7
11
F
G
H
I
9
10
5
9
OPTIMUM PATH
36
8
30
8
22
12
20
M
L
K
J
8
7
10
7
44
8
37
9
32
6
27
A
N
O
P

Fig. 24 Minimum-cost values for the entire problem grid

Thus, Fig. 22 is the solution of the original problem: the determination of the minimum-cost path from A to B. The total cost of this optimum path is $44, 000.

Comments on the solution. The total grid possesses (as we saw at the beginning of this section) 20 possible paths from A to B. Instead of evaluating these 20 costs, one-by-one, we look at the problem as a set of binary (two-choice) decisions. There are nine of these in the total grid--at the intersections H, G, K, L, F, O, M, N, and Ⓐ in Fig. 24. Thus, solution of the problem requires only the comparison of nine pairs of numbers: these are, at each of these intersections, the costs if we go north or east.*

The saving in effort by the solution algorithm is dramatized more vividly in the case of a 20 x 20 grid. For such a problem, there are 137, 846, 528, 820 possible paths. With the algorithm, we need compare only 20 x 20 = 400 pairs of numbers. If we can carry out four comparisons per minute, the algorithmic solution takes less than two hours. If four path costs could be evaluated per minute (a rather high rate in a 20 x 20 problem), consideration of all possible paths would require over 65, 000 years of continuous work. The algorithm converts an unsolvable problem to a solvable one.

Because a variety of optimization problems can be solved in the same way as the above example, it is useful to reconsider briefly what steps are involved in the solution above. We are faced with a problem in which there is a large number of alternatives. We essentially try to find a way of looking at the problem as a sequence of simple decisions, rather than one complicated decision. We can realize this sequency by working from the termination Ⓑ back toward the origin Ⓐ .

As an example, we can consider the intersection L in Fig. 24. Before considering L, we have already determined the minimum-cost paths from G to B and from K to B (Fig. 25). If we are at L, we have only two choices: north to G or east to K. If we go north to G, we thereafter follow the already-determined, minimum-cost path to B (cost 21). Thus, from L the cost is 31 if we head north, 30 if we head east--we obviously should move eastward.

The key to success here is the fact that, once we determine the minimum-cost path from G to B, these are the path to follow and the cost regardless of how we reach G.** Recognition of this fact is what simplifies the decision at L.

---

*Actually, as in so many problems, more time is required to read a description of the solution than to carry it out. A problem such as Fig. 24, once the method of solution is understood, should require only a few minutes.

**In the literature on dynamic programming (name for the algorithm used in this example), this key idea is given the fancy name, the principle of optimality. The principle says that once an optimum path is determined from any interior point to the termination, this is the path to be followed regardless of how the interior point is reached.

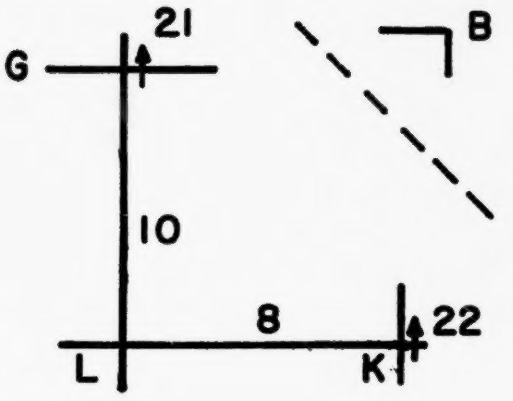


Fig. 25 Decision at L

The algorithm we have discussed, known as a "dynamic programming" algorithm, has taken on great importance with the advent of modern digital computers. In dynamic programming we replace the need for the simultaneous selection of a large number of variable factors by the process of selecting one variable at a time, starting with the end result, and working back to the starting condition. Dynamic programming is an important algorithm for the determination of the solution of a problem in optimization of a plan. Even with the fastest computer, one would not lightly consider the computation of the costs for the gigantic number of cases demanded by a "calculate all paths" process. With the dynamic programming algorithm, the calculation of a 20-block problem would also consume considerable amounts of time. But with a digital computer, this algorithm would require a few seconds of computer time and would be a practical method of solution.

Optimization problems that can be solved with dynamic programming algorithms occur in a wide variety of contexts. Airlines take advantage of the wind direction on the different legs of the transatlantic routes to determine minimum-time or minimum-fuel routes. Other examples are found in problems dealing with inventory control in warehouses, in the programming of expansions of telephone switching centers, and in the automatic control of industrial factories and chemical plants.

7. CONCLUSION

The basic objective of this chapter is the introduction of the elements of decision problems:

Model	Constraints
Criterion	Optimization

In order to emphasize these four aspects, several examples are described and solved--examples ranging from rather obvious and trivial exercises to the dynamic programming problem of the preceding section.

Optimization (or decision) problems have two fundamental characteristics. First, such problems constitute a focal element of modern technology and of business management, economics, social sciences, and other fields. Much of our personal and public lives is determined by significant decisions which confront us as individuals or our government or institutions (including business) in the establishment of public policies.

Second, because of the wide range of forms, optimization problems possess no straightforward method of solution. One must first understand the problem thoroughly, then seek an appropriate algorithm. In the last example (the dynamic programming problem), one can justifiably accuse us of resorting to a "trick" to find a solution. The idea of working from the termination backwards is certainly not obvious to most people.

If such problems can only be solved by tricks, how can the subject be taught or learned? As scientists, we should prefer to avoid such problems and concentrate our energies instead on problems in which systematic methods of solution can be written. As engineers or individuals interested in the impact of modern technology on the lives of everyone, we cannot disregard such problems; optimization decisions arise in the most interesting parts of modern technology: automation of medical diagnosis (e.g., automatic reading of electrocardiagrams), traffic control and transportation problems, efficient utilization of police and fire department personnel and equipment, business decisions, and so forth.

Because of the importance of this topic of optimization, the next chapter is devoted to a presentation of a very few algorithms which are useful in an important class of problems.

PROBLEMS

1.1 The following information represents the grades (to nearest 10%) of 300 students on an examination.

(a) Make a graphic model of the data on the right.

(b) If the performance criterion is that 80% of the students should pass, what would be the passing grade for this test?

(c) If we are using this test to help us select students for an honor class and school policy (a constraint) allows a maximum of only 10% of the students to be admitted to honor classes, what is the (optimum) passing grade for honor students?

Number of students	Test grade
10	20%
10	30
20	40
20	50
30	60
90	70
90	80
30	90

1.2 (Discussion problem). Write an algorithm for some school activity which you would like simplified. (E.g., how to keep records of lateness, cutting of classes, etc.) Keep in mind the four elements of decision-making.

1.3 General managers of baseball teams have the difficult job of making trades to improve their teams. In the spring of 1967 the Anaheim Angels traded a twenty-game-winning pitcher to the Minnesota Twins for a hard-hitting first baseman and an outfielder. There was much discussion about the deal. Put yourself in the position of the general manager of each team and cite the reasons (the criteria) for the decision to trade. (The players mentioned were Dean Chance, Don Mincher and Jimmy Hall).

1.4 (Discussion problem). Cite an example where the solution to a problem is counter to common sense.

1.5 A simple "model" of the car trade-in problem is to assume that the sale value of the car at the nth year is a fraction of the sale value at the (n-1) year; i.e.

$$v_n = av_{n-1} \text{ where } 0 < a < 1, \text{ (a = constant)},$$

and to assume that the operating cost for the nth year is increased by a fixed amount over the operating cost for the (n-1) year; i.e.

$$c_n = c_{n-1} + b \text{ (b = constant)}$$

Use this "model" for the case where

v = \$3000 (initial sale value)

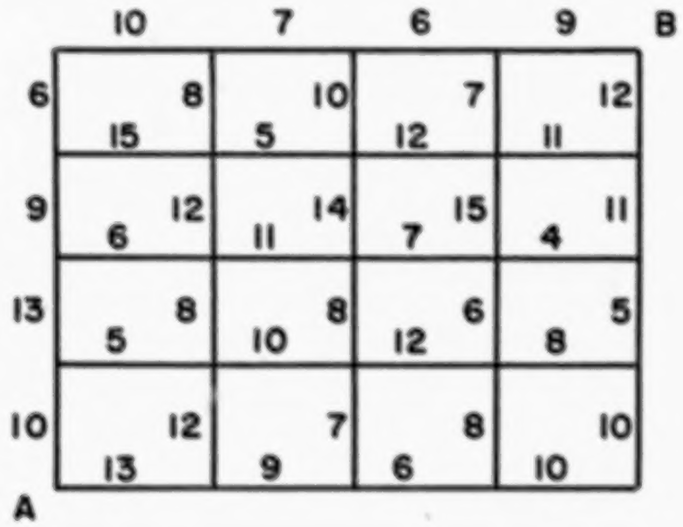
a = 0.80 (depreciation factor)

c = \$ 800 (operating cost for the first year)

b = \$ 50 (increase in operating cost per year)

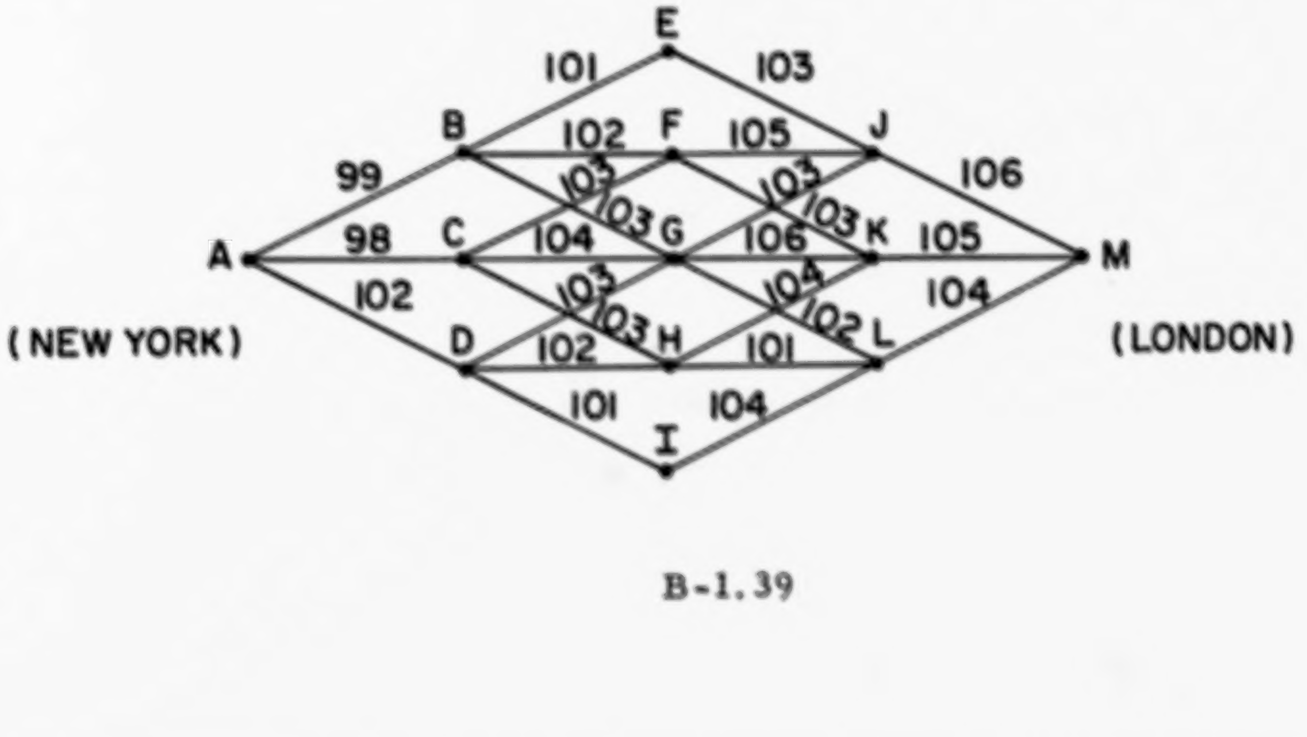
and determine the year to trade-in in order to minimize average operating cost per year.

1.6 In the four by four grid shown below, find the minimum-time path from A to B going only north and east. There are 70 possible routes, but you need to calculate only 24 numbers.


10
7
6
9
B
6
8
10
7
12
15
5
12
11
9
12
14
15
11
6
11
7
4
13
8
8
6
5
5
10
12
8
10
12
7
8
10
13
9
6
10
A

1.7 An unscrupulous cab-driver who is paid by time rather than distance wants to have the maximum time route from A to B in the previous problem, still traveling only north or east (to avoid suspicion). Can you find it for him?

1.8 Many of the airlines flying the North Atlantic now use computers to find minimum time jet flight paths taking into account the strong winds that usually occur at jet cruising altitudes, and the restrictions on possible routes imposed by air traffic control requirements. Savings on the order of 15 minutes on the nominal 7 hour flight are obtained this way. A grid of "check-points" is selected and all routes must consist of generally east-west straight-line segments between check-points. A simplified version of such a grid is shown below. Imagine that check-point A is New York and M is London (or Paris if you prefer). Check-points B through L are points in the ocean (located by giving their longitude and latitude). Using wind data from "weather ships" on the North Atlantic (they release balloons and track them as they rise) and characteristics of the particular jet airplane the airline uses, the flight planner computes the time to fly each segment (results


E
101
103
B
102
F
105
J
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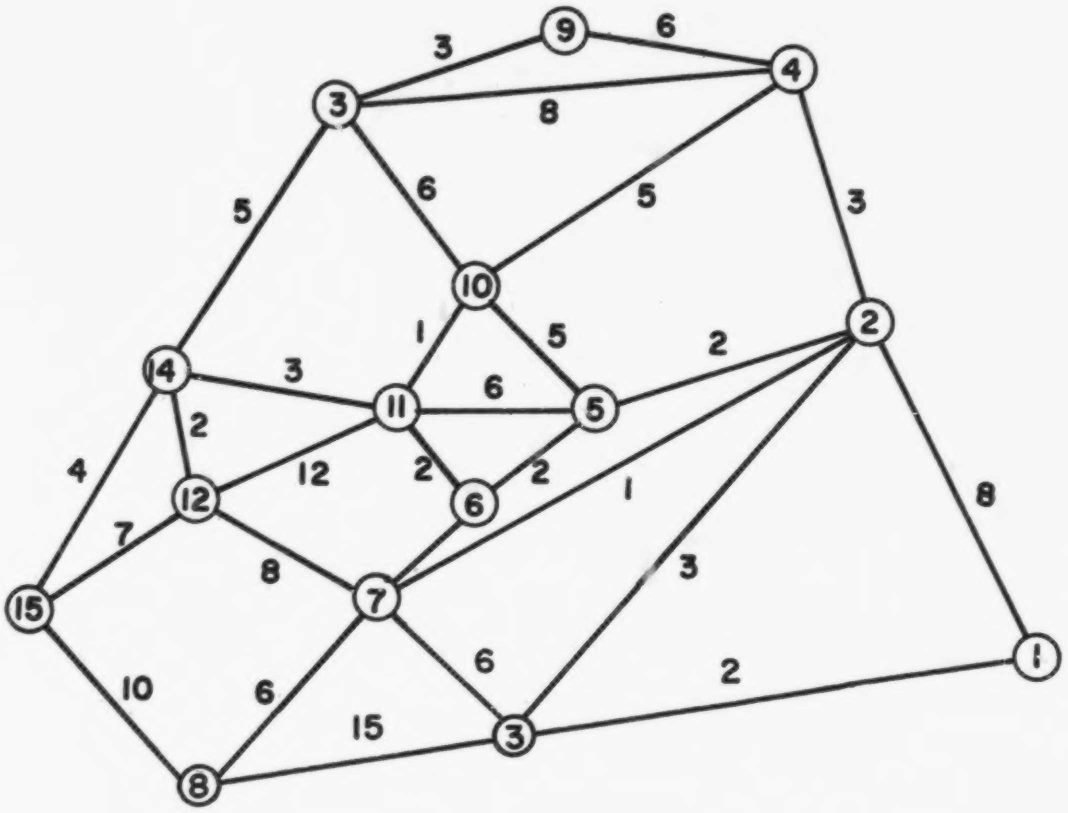
103
103
106
103
103
A
98
C
104
G
106
K
105
M
102
103
104
103
102
(NEW YORK)
D
102
H
101
L
(LONDON)
101
104
I

for east to west flights differ, of course, from west to east flights). He then has a "maze problem" very similar to the one treated in Section 6 to find the route that requires minimum time. In practice there are many more check-points that are shown in the figure below so that a computer solution is essential. For the data given in the figure (time in minutes), use the dynamic programming algorithm to find the minimum time route. (There are 19 possible routes.)

1.9 The text examples of a shortest route problem used a very special street pattern; namely, a square grid of streets. The technique developed for solving that problem can also be used on other street patterns. In the illustration shown below we have a pattern of streets and the time in minutes to travel each block is shown. We wish to find the shortest route from corner 15 to corner 1 where on any block we may only travel towards the right; e.g., we may travel from corner 7 to 6 but not from 6 to 7. Use exactly the same algorithm to solve this problem as was used on the square grid pattern.



1.10 Notice that in Problem 1.9 the route would have been 3 minutes shorter if the last step (2, 1) had been replaced by (2, 3, 1). However, this route traverses a block in a direction not permitted by the stated conditions. Suppose this restriction is relaxed: how can we find the best route when we may traverse a block in any direction? A method called relaxation can be used. We must have, to begin with, a time estimate at each junction that is certainly long enough (there must be room for optimization). At each intersection in turn, beginning with those nearest the destination, see if this time estimate can be improved by taking any other, previously forbidden, route, and if so enter in a chart the new estimate and the new direction. When the whole pattern has been surveyed, transfer the new times and directions to the map. For example, suppose we start with the final time estimates obtained in problem 1.9. The correction chart for the first few entries looks as follows:

junction	old estimate	arrow to	1st new estimate	arrow to
2	8	1	5	3
3	2	1	2	1
4	11	2	11	2
5	10	2	10	2

After making this improvement successively at all the junctions, we repeat the process again and again until no improvements can be made at any junction. We then have at each junction the minimum time to junction 1 and the arrows indicate the route. In this example, on the second time around we find that 4 and 5 have been revised to:

junction	1st new estimate	arrow to	2nd new estimate	arrow to
4	11	2	8	2
5	10	2	7	2

Complete this problem.

Chapter B-2
OPTIMIZATION

1. INTRODUCTION

In our daily lives, we are often required to make decisions on the basis of ill-defined or indefinable criteria. Some prefer rock-and-roll music, others prefer jazz, and still others prefer symphonies. When we decide which automobile, refrigerator, pair of shoes, can of peas, or candy bar to buy, we make the decision on the basis of criteria which we cannot define mathematically or express in words. Such criteria are called subjective criteria, since their form depends strongly upon the person (or subject) involved.

Engineers, too, are faced with the problem of arriving at decisions on the basis of subjective criteria. Such decisions usually are made by the exercise of "engineering judgment," which is a combination of intuition and past experience on the part of the engineer and of the company for which he works. The individuality of engineering judgment clearly is evident when we look at the relatively-wide differences among the several brands of automobiles available within a particular price range, or at the various designs of refrigerators or television sets on the market, or even at the significant design differences between American and Russian space vehicles. (Here, one might argue that these differences exist because of the lead which Russia gained some years ago. But within the U.S. space program, the U.S. Army and the U.S. Air Force, each, had satellite systems which differed markedly, even though both had access to the same technology).

Most engineering design activity currently relies on engineering judgment. There are, however, certain classes of problems for which engineers are able to define objective (i.e., mathematical) performance criteria. In such cases, it frequently is possible to determine the optimum design using well-known mathematical tools, by developing an appropriate algorithm (i.e., set of mathematical rules), determining the design parameters, and then proceeding with the algorithmic solution -- usually with the assistance of a computer, either analog or digital.

Some areas where criteria can be stated in mathematical form, and algorithms have been developed (and, hence, where the optimum choice may be made) are:

(1) Allocation of limited materials. In a petroleum refinery, the crude oil may be reduced to various combinations and proportions of a number of possible products (gasoline, kerosene, motor oil, etc.), and it is necessary to optimize (i.e., maximize) the profit.*

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* It is interesting to note here that such techniques are employed by a well-known sausage-maker. His recipe for frankfurters includes a number of ingredients for which the prices fluctuate widely from week to week. This manufacturer has developed a digital-computer program to determine the proportions of these ingredients to use for the week's production in order to maximize his profit. He, of course, imposes constraints on this solution to limit the amount of fat content to some maximum value and to keep variations of flavor within bounds.

(2) Route-planning problems. If a manufacturer of delicate instruments in Buffalo, New York, wishes to ship some galvanometers to Omaha, Nebraska, he may use trucks, railroads, airplanes, and Great-Lakes ships in various combinations and over a variety of routes. He may choose the route with lowest cost, or one that is most rapid, or one that will produce the least possible damage to his meters, or one according to an entirely different criterion.

(3) Queueing problems. Another group of problems which can be treated mathematically are queueing problems, which arise when it becomes necessary to design facilities which will be fewer than the demand for their use on some occasions, and greater than the demand for their use at other times -- as in the selection of the number of checkout counters in stores, telephone exchanges in a given area, toll booths at a bridge entrance, and air traffic control towers along an air route.

In the next three sections we discuss problems which involve simple inequalities (called "linear programming" problems). We start with a very simple problem and then gradually consider more significant examples of such problems. The second half of the chapter is devoted to a different algorithm and its application and then to a brief discussion of queueing problems.

The purpose of the chapter, then, is to develop a few types of algorithms useful in classes of important optimization problems. In the concluding section, we attempt to summarize the major points of this and the preceding chapter -- the material devoted to the general question of optimum decision-making.

2. A PRODUCTION PLANNING PROBLEM

An ice cream plant can make two flavors, vanilla and chocolate. The plant capacity is 1000 quarts per day, and the sales department says that it can sell any amount of vanilla up to 800 quarts and any amount of chocolate up to 600 quarts. If the profit per quart is 10¢ for vanilla and 13¢ for chocolate, what is the most profitable daily production?

Since the profit per quart is larger for the chocolate than for the vanilla, we should produce as much chocolate as we can sell. This would set the production of the plant at 600 quarts of chocolate. All excess production could then be applied to the vanilla ice cream, to add to the profit. The daily production of the plant is thus 600 quarts of chocolate and 400 quarts of vanilla -- not a very difficult problem.

We now solve this problem in a systematic manner, so that a technique can be developed which can be extended to more complicated problems.

We derive the general method of solution by making use of mathematical statements known as inequalities rather than equations. A statement of inequality uses the symbol $\geq$. $x \geq 20$ states that x is greater than or equal to 20; while $x > 20$ means simply that x is greater than 20.* It should be observed that an inequality does not have a single value for its solution, but a range of values: for example, $x > 20$ can be satisfied by a value of x of 21 or 22.5 or 2000, etc.

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*Similarly, $x \leq 20$ mean x is less than or equal to 20 (or, if we read from right to left, 20 is greater than or equal to x).

In order to establish to describe our problem in terms of mathematics, we let V = quarts of vanilla to be produced in one day and C = quarts of chocolate to be produced in one day. For our problem, 1000 = total quarts produced by the plant for one day.

The number of quarts of vanilla to be produced may not exceed 800, but may be less than that quantity. We express this by two statements of inequality:

$$800 \geq V \geq 0$$

In other words, the vanilla ice cream produced may be any amount between 0 and 800 quarts. Similarly the quantity of chocolate ice cream to be produced can be represented by:

$$600 \geq C \geq 0$$

The number of unknowns can be reduced from two to one by remembering that the total production must be 1000 quarts of ice cream. With this limitation, if V quarts of vanilla are produced, there can only be (1000-V) quarts of chocolate ice cream.

Our statements of limitations for production are thus:

$$0 \leq V \leq 800$$

$$0 \leq (1000 - V) \leq 600$$

At this point, we have a model for the decision problem. The model, as expressed by these two inequality relationships, states that there are several constraints on the quantity of vanilla, V, produced each day:

$V \geq 0$ (the amount of vanilla must be positive or zero, we cannot have a negative amount of vanilla)

$V \leq 800$ (we cannot sell more than 800 quarts of vanilla)

$1000 - V \geq 0$ (the amount of chocolate must be positive or zero)

$1000 - V \leq 600$ (we cannot sell more than 600 quarts of chocolate)

These four inequalities (the model) can be expressed somewhat more understandably if we manipulate the third and fourth so that V appears alone on one side of the inequality (as in the first two). For the third inequality above,

$$1000 - V \geq 0$$

means that V must be less than or equal to 1000. Hence this inequality is exactly equivalent to

$$V \leq 1000$$

(We can add V to both sides to obtain this relationship by direct manipulation).* In the fourth inequality,

$$1000 - V \leq 600$$

if we add V to both sides we obtain

$$1000 \leq 600 + V$$

We now subtract 600 from both sides and obtain

$$V \geq 400$$

Therefore, the model (the four inequalities above) can be written

$$\begin{array}{ll} V \geq 0 & \\ V \leq 800 & \\ & \underline{\text{Model}} \\ V \leq 1000 & \\ V \geq 400 & \end{array}$$

The second of these is evidently stronger than the third, the fourth stronger than the first: i.e., if the second is satisfied, the third is automatically O.K. Hence, the model can be simplified to:

$$\boxed{\begin{array}{l} V \geq 400 \\ V \leq 800 \end{array}} \qquad \underline{\text{Model}}$$

Now that the model is determined, we can turn our attention to the criterion: what is to be optimized? In this problem, we wish to maximize the profit. The problem statement specifies that the profit is 10¢ for each quart of vanilla and 13¢ per quart of chocolate. Thus, if P is used to represent profit (in cents), we can write

$$P = 10V + 13C$$

Once again, C is equal to 1000-V, and the criterion (profit) equation becomes

$$P = 10V + 13\,(1000 - V)$$

$$\boxed{P = 13000 - 3V} \qquad \underline{\text{Criterion}} \text{ (to be maximized)}$$

Finally, we must consider the constraints before we start the optimization. In this particular problem, there is no constraint specified (a constraint might be that the owner never permits production of less than 500 quarts of vanilla because of his personal desire to cater to his vanilla customers).*

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* We can add any number to both sides of an inequality; the number can be positive or negative, so we can also subtract any number from both sides. Although we do not need the fact here, we can also multiply or divide both sides by any positive (non-zero) number; if we multiply or divide by a negative number, the direction of the inequality is reversed (< becomes >, for example).

Solution. Now that the problem statement is complete we turn to the problem of optimization. We must select V to satisfy the two inequalities of the model, and simultaneously to maximize P. One approach (which is useful in more complex problems) is to consider a graphical representation of the equation $P = 13000 - 3V$: we plot P versus V as shown in Fig. 1 (the plot is made by substituting two values of V, say 0 and 800, calculating the corresponding values of P, 13000 and 10600, and then constructing the line through the two points).

On this graph, we introduce the model inequalities. The former states that V must be greater than or equal to 400; hence the line $V = 400$ separates the permissible region (to the right of that line) from the inadmissible. The second states we must select V to the left of the $V = 800$ line. Thus, the model requires we select V in the region called the "Feasible region" in Fig. 1.

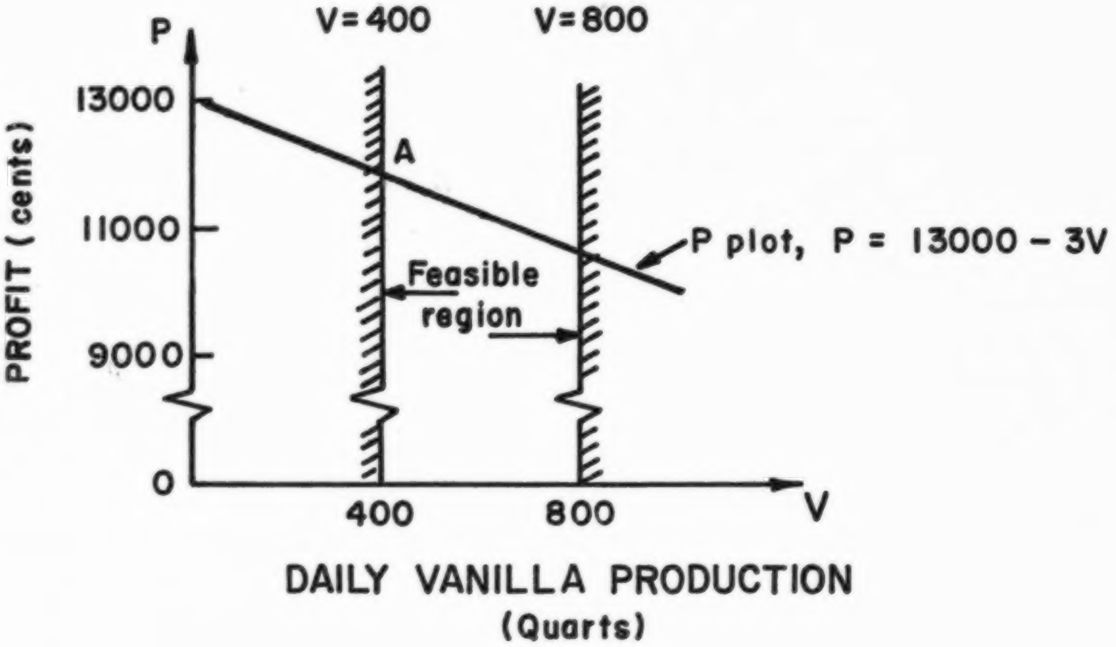

Fig. 1 Profit versus amount of vanilla ice cream.

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* One might argue, for example, that the factory limit of 1000 quarts/day of ice cream is a constraint rather than a part of the model. In problems of the nature we are discussing here, it makes no difference whether a given part of the problem statement is considered part of the model or a constraint (in this example, we arbitrarily put everything into the model). The only important factor is to include all significant aspects of the problem statement.

Figure 1 shows clearly the solution of our problem. We wish to maximize the profit; hence, operation should occur at point A on the edge of the feasible region. The figure shows that the smaller V, the larger the profit P; hence point B corresponds to minimum profit and point A to the desired maximum profit. From the figure, we can read at point A,

$$V = 400$$

The corresponding P can be read from the figure or, more accurately, can be calculated from the equation for P:

$$P = 13000 - 3V = 13000 - 3\ (400) = 11800 \text{ cents}$$

One interesting feature of this graphical solution is that the best point for operation occurs on the boundary of the feasible region. We see in the next sections that this feature is a characteristic of problems of the type we are considering in this first half of the chapter.

3. A TRANSPORATION PLANNING PROBLEM*

In retrospect, we can only marvel at the last section. There we take a ridiculously simple problem which can be solved by inspection and convert it to a complicated problem requiring the definition of a model and criterion, the manipulation of a set of inequalities, and a graphical solution. Clearly, it is essential to justify this ruining of a perfectly obvious problem, and in this section we turn to a more difficult problem in which we apply those techniques so laboriously developed in the preceding section.

The preceding problem involves a search for a "best" solution to a problem for which more than one solution is possible. The "best" solution is one which achieves either a maximum or a minimum under the conditions or restraints that are imposed. These restraints are expressed mathematically as either linear equations or linear inequalities. Such problems have been named "linear programming problems". **By a linear equation here, we mean each variable (x or y) appears only to the first power. For example, $3x + 4y = 2$ is a linear equation. The term linear also refers to the fact the plot is a straight line. In a similar fashion, a linear inequality is an inequality which becomes a linear equation when the inequality sign is replaced by an equals sign.

It is fascinating to discover the variety of design and planning problems that involve linear inequalities. In this section we treat a simple transportation planning problem and solve it by drawing a graph similar to the one used in solving the production planning problem of the previous section.

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*Example suggested by "Operations Research for Students of Business", notes by James R. Jackson, published by University of California of Los Angeles.

**Our examples of linear programming problems in this chapter are all sufficiently simple to permit manual solution. In practice, problems frequently involve a large number of variables and must be solved with high-speed computers.

A grain dealer owns 50,000 bushels of wheat in Grand Forks, North Dakota, and 40,000 bushels in Chicao. He has sold 20,000 bushels to a customer in Denver, 36,000 bushels to a customer in Miami, and the remaining 34,000 bushels to a customer in New York. He wishes to determine the minimum-cost shipping schedule, on the basis of the freight rates in cents per bushel shown in Table 1. For example, it cost 42¢ per bushel for shipments from Grand Forks to Denver. Different modes of shipment involve rates which are not proportional to the distance between the cities.

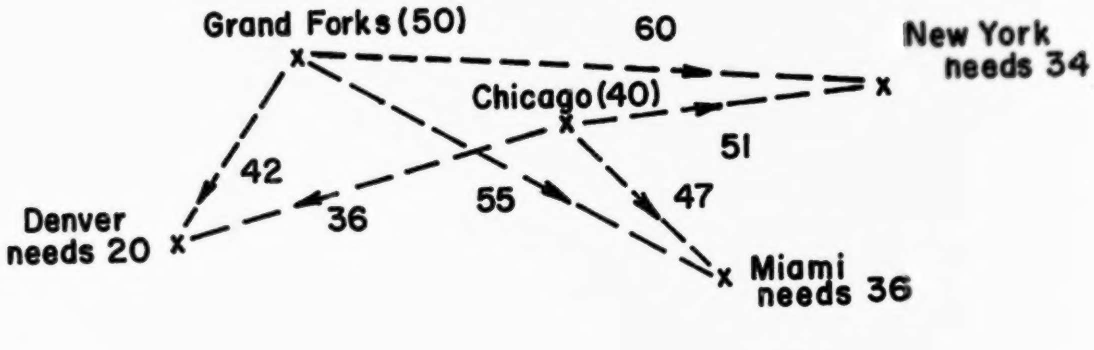

Grand Forks (50)
60
New York needs 34
Chicago (40)
51
42
36
55
47
Denver needs 20
Miami needs 36

Fig. 2 Sources and needs of wheat and shipping costs (numbers in thousands of bushels and in cents per bushel)

From \ To	Denver	Miami	New York
Grand Forks	42	55	60
Chicago	36	47	51

Table 1. Wheat freight rates in cents per bushel.

In this problem, the quantity of wheat which has been sold is just equal to the quantity of wheat in storage. This need not necessarily be the case in practice, but we have designed this problem to illustrate the general method with the use of the simplest mathematics.

For convenience we can combine our data into a table, which indicates quantities and costs as shown in Table 2. The figure in the upper right-hand corner of each square is the freight rate between the two cities.

	Quantity needed	20,000	36,000	34,000
Quantity in Storage	To From	Denver	Miami	New York
50,000	Grand Forks	42	55	60
40,000	Chicago	36	47	51

Table 2. Data arrangement for solving grain problem.

Our problem is to find the quantity of wheat to fit into each of the 6 squares so that (a) the amounts in the first row add up to 50,000 and the amounts in the second row add up to 40,000 (i. e., the total amounts to be shipped from Grand Forks and Chicago, respectively); (b) the amounts in the first, second, and third columns add up to 20,000, 36,000, and 34,000, respectively (i. e., the amounts to be delivered to Denver, Miami and New York, respectively); (c) the total freight cost is a minimum (this cost is obtained by multiplying the amount in each square by the rate in the upper right-hand corner and adding these six numbers together).

It is not sufficient to select the numbers of each box so that the sums of the terms in the rows and columns are correct (a feasible solution). We must also minimize the total cost. With "cut and try" methods, we might find such a solution. However, a systematic approach takes less time and, for problems with more shipping points and with more destinations, a systematic approach (an algorithm) and a computer are essential.

We can simplify the numerical values by expressing the quantities in units of 1000 bushels. Thus 20 units of wheat must be delivered to Denver, 36 units to Miami and 34 units to New York. In Table 3 these values, in terms of units of 1000, have been shown in the circles. If we designate the quantity of wheat shipped from Grand Forks to Denver as x units, then $(20-x)$ units must be shipped from Chicago to Denver, to complete the order. Similarly, if we designate the amount shipped from Grand Forks to Miami as y, then the amount from Chicago to Miami must be $36-y$. Now the amount from Grand Forks to New York must be $50-x-y$ if the total out of Grand Forks is to be 50. The amount from Chicago to New York must then be $34-(50-x-y) = x+y-16$. (We automatically satisfy the requirement that the total shipped from Chicago is 40, since the total amount sold equals the amount owned.)

	Quantity to be Delivered →	⑳	㊱	㉞
Quantity in storage	To From	Denver	Miami	New York
㊿	Grand Forks	42 x	55 y	60 $50-x-y$
㊵	Chicago	36 $20-x$	47 $36-y$	51 $x+y-16$

Table 3. Table 2 with data entered and unknown quantities x and y defined.

Table 3 now represents the quantities of wheat (in thousands of bushels) to be shipped from each storage house (Grand Forks and Chicago) to each destination (Denver, Miami, and New York). Because of the way the six entries are selected, each row and column adds up properly. Thus, our task in optimization is now to select x and y and, hence, each of these six entries:

$$x,\ y,\ 50-x-y,\ 20-x,\ 36-y,\ x+y-16$$

There is one additional part of the model (or the constraints -- again it is immaterial whether we assign a relationship to the model or the constraints section of the problem). Each of the entries in Table 3 (the six items listed just above) must be positive or zero; we cannot ship negative quantities of wheat. Therefore, in addition to Table 3, we must add the six inequalities:

$$x \geq 0$$

$$y \geq 0$$

$$50 - x - y \geq 0$$

$$20 - x \geq 0$$

$$36 - y \geq 0$$

$$x + y - 16 \geq 0$$

These inequalities can be rewritten in the following form, if we rearrange terms:

$$x \geq 0$$

$$y \geq 0$$

$$\text{By adding terms to both sides of the inequalities} \left\{ \begin{array}{l} y \leq -x + 50 \\ x \leq 20 \\ y \leq 36 \\ y \geq -x + 16 \end{array} \right.$$

We now have the model and constraints, represented by Table 3 and the six inequalities above. Before considering the criterion function (the cost to be minimized), we can represent our six inequalities on a graph of y versus x. Each inequality states that we must work on one side of a straight line in this plane (these are linear inequalities; they divide the plane into two regions by a straight line represented by the linear equation formed when the inequality sign is replaced by an equals sign).

For example, the first inequality

$$x \geq 0$$

requires we operate to the right of the y axis; the fourth

$$x \leq 20$$

places us to the left of the vertical line $x = 20$. The effect of these two inequalities is to restrict our choice of x and y to the region shown in Fig. 3.

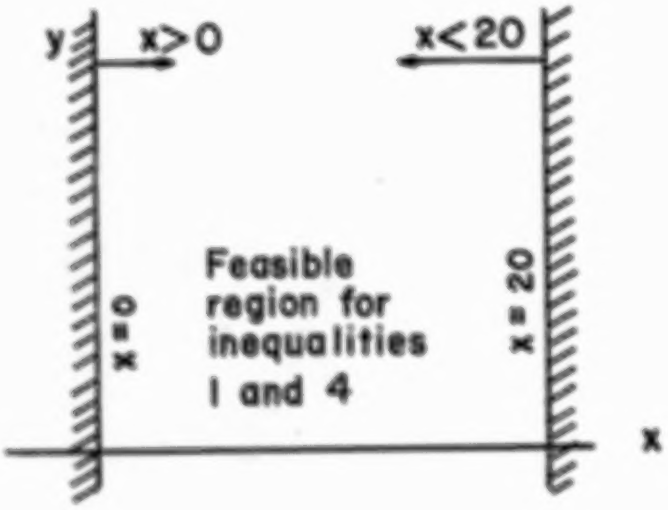


Fig. 3. Plot of the vertical lines $x = 0$ and $x = 20$

Each inequality thus divides the x-y plane in half by a straight line; the equation of this line is obtained by replacing the inequality sign ($\leq$ or $\geq$) by an equals sign. Once the line is drawn, we can determine which side is permissible by inspection of the original inequality.

Figure 4 shows the six straight lines and the corresponding permissible regions. Satisfaction simultaneously of the six inequalities requires that we operate (select x and y) either within the "Feasible region" indicated in Fig. 4 or on the boundary of this region.

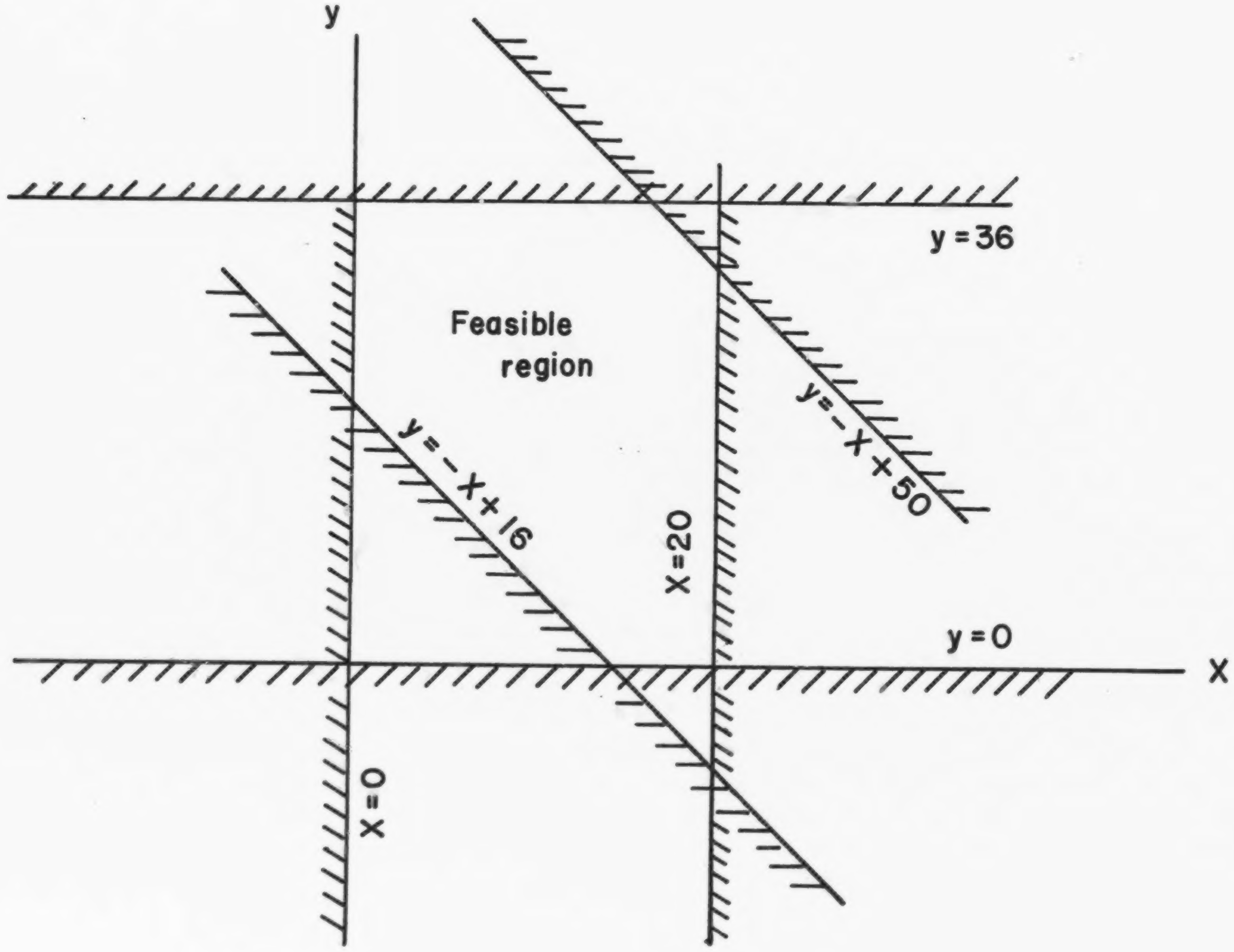


Fig. 4 Graphical representation of the six inequalities of the model

Criterion function. Figure 4 and Table 3, together, now represent the model and the constraints of the total problem. Before we can start the optimization we need to determine an equation for the criterion -- the cost. Optimization then consists of looking for a point (x and y) within the feasible region at which the cost is minimized.

The cost of each shipment in each of the squares of Table 3 can be expressed by multiplying the number of bushels shipped times the cost of shipping for each bushel. The sum of all these individual shipping charges then gives an expression which represents the total cost of the entire shipment. Thus, the shipment from Grand Forks to Denver, which involves x units of wheat (1000 bushels per unit) costs $(1000x)\cdot 42$ cents or $\frac{1000}{100}(42x)$ dollars $= 10\cdot(42x)$ dollars. In a similar fashion, we can compute the cost of shipment between any two cities for each of the squares. The total cost of shipping is given by the expression

$$C = 10\cdot 42x + 55y + 60(50-x-7) + 36(20-x) + 47(36-y) + 51(x + y - 16)$$

If this expression is simplified by basic algebra, we obtain the equation:

$$C = 45960 - 30x - 10y \text{ (in dollars)}$$

Rearrangement of terms produces an equation:

$$y = -3x + \left(4596 - \frac{C}{10}\right)$$

This is a linear equation with a slope of -3 and an intercept on the y axis of $\left(4596 - \frac{C}{10}\right)$. Any acceptable solution must have values for x and for y which satisfy this equation or lie on the line which represents this equation. For any particular value of x, y will depend on the value we assign to C -- the smaller we make C the larger the value for y will be.

We can reverse this statement and say that large values for y represent small values for C. In other words to determine the smallest value for C (the shipping cost) we must graph the equation so that (1) it has a slope of -3, (2) it lies in the feasible region, and (3) it cuts the y axis at the largest value. Such a line can then be used to determine the value of x, y and C to meet the constraints.

In Fig. 5, two dotted lines of the correct slope and in the feasible region are shown. The dotted line to the right, when extended, produces the largest y intercept and yet falls in the feasible region. This line therefore represents the "line of minimum cost". It passes through the intersection of the lines

$$y = -x + 50 \text{ and } x = 20.$$

When these equations are solved simultaneously, we find that $(x = 20, y = 30)$ is the solution for minimum cost, with

$$C = 45960 - (30)(20) - (10)(30) = 45060.$$

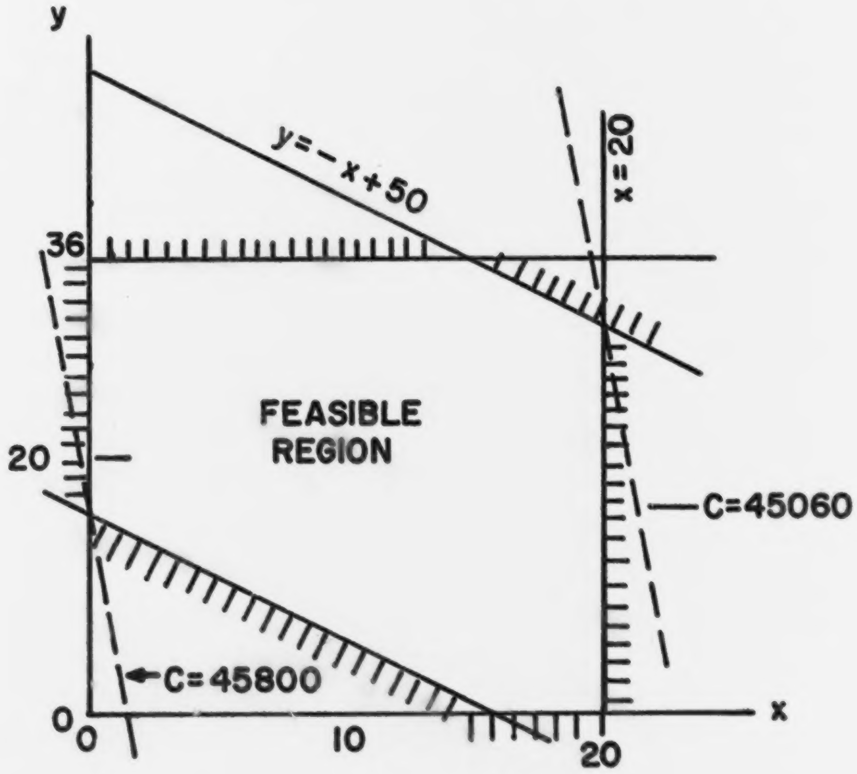


Fig. 5 Feasible values of x and y.

The minimum-cost solution is shown in Table 4. It is interesting that no wheat should be shipped from Chicago to Denver even though such shipment involves the lowest rate per bushel. (As discussed in the preceding chapter, intuition is often not very valuable in optimization problems).

To From	Denver	Miami	New York	
Grand Forks	42 20	55 30	60 0	50
Chicago	36 0	47 6	51 34	40
	20	36	34	

Table 4 Minimum-cost solution for grain problems.

Using the same reasoning as above, we conclude that the largest value of C is associated with the least value of the y intercept. The cost line to the left of Fig. 5 does have the smallest value of the y intercept which is within the region of feasible solutions. The point through which this cost line passes is $(x = 0, y = 16)$; substituting these values into the cost equation reveals that the greatest possible cost of transportation is $45,800.

One may be tempted to scoff at the small savings obtained by our analysis. The difference between the best and the worst feasible solutions is only $740 out of about $45,000. However, this 1.6% difference can be a substantial percentage of the profit involved in the sales (many industrial operations operate with relatively small profit percentages). If the maximum profit is 5% ($2250), we realize one third of that merely by optimum planning of the transportation.

The example of this section represents the essential elements of the class of optimization problems of interest in the subject of linear programming. In the following section, we conclude this portion of the chapter with a brief discussion of the scope and nature of linear programming.

4. LINEAR PROGRAMMING PROBLEMS

Linear programming problems are optimization problems characterized by a model and constraints consisting of a set of linear equations and inequalities, and by a criterion function (to be maximized or minimized) which is a linear combination of the variables. In this discussion, the term linear refers to equations or inequalities in which the variables $(x, y, \ldots)$ appear alone and to the first power (the equations are straight lines, as in the examples of the preceding section).

More realistic blending and transportation problems may involve up to 100 variables. In such cases it is impossible to draw a graph of the feasible region. One of the features we observe in the simple problems treated above carries over to the more complicated problems: that is, the point which represents an optimum solution occurs at the boundary, and not in the interior, of the feasible region. The solution is usually represented by a point at a vertex of the polygon which encloses the area of feasible solutions: i.e., at a point where two of the inequalities are equalities. In rare cases the solution may be along an entire "edge" instead of at a "vertex".

Linear programming problems which involve more than two variables are quite common in engineering as well as in industry and would require long and tedious computations for their solution. The computer has made possible the treatment of such problems, so that decision-making in engineering and in industry becomes more effective because more of the limitations which exist in the real world can be included in the mathematical model.

The computer must be programmed for such problems, and an algorithm for linear programming problems, named the "simplex" algorithm, has been developed which depends on the property of "vertex solutions" described above: namely that the minimum or maximum solution is represented by a point at a vertex or point of intersection. If n inequalities define the region of feasible solutions, these inequalities are first converted into equations by defining additional variables (e.g., $x + y \leq 50$ becomes $x + y + z = 50$ where z has a positive or zero value). The computer is then programmed to solve these linear equations to determine the vertices and the value of the criterion function at each vertex in sequence. In addition, the computer is programmed to move automatically from one vertex to the next in such a way that the criterion function continues to decrease (or increase, if we are trying to maximize).

Since the location of a vertex may require the solution of several dozen simultaneous, linear equations, the necessity of a computer to handle the routine computations is obvious. The importance of the computer exists in even relatively simple problems (such as the transportation problem of the last section) if we want to find the optimum transportation plan with several different sets of transportation costs (or changes in the numbers involved in the model or the criterion function). Such an investigation is particularly important if we are not sure of these numbers (e.g., shipping costs might vary from day to day or might depend on the availability of trucks at the moment we place our orders).

5. MINIMUM WIRE LENGTH

In this section, we continue the discussion of optimization by considering an algorithm for an entirely different class of problems. We consider a community which, for all practical purposes, is stretched out along one road. Each building requires a certain number of telephones. Each telephone is connected by a wire to a switching center. When a telephone call is made, the ends of the appropriate wires are connected by switches in the switching center. Where should we construct the switching center in order to minimize the amount of wire?

This idealized situation is generalized in the problems (at the end of the chapter) to a slightly more realistic situation. The real situation is extremely complicated since we must consider factors like: price of land, expected growth of the community, cost reduction due to the fact that a cable with n wires does not cost n times as much as a cable with one wire, the fact that in some areas cables can be hung on poles while in others the cable must be buried, etc.

Figure 6 shows the simplest possible problem of any interest: namely two buildings, one building with two telephones (small squares) and the other

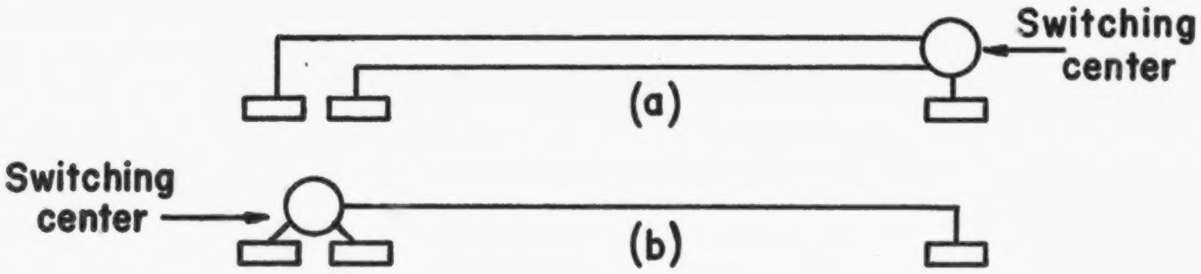
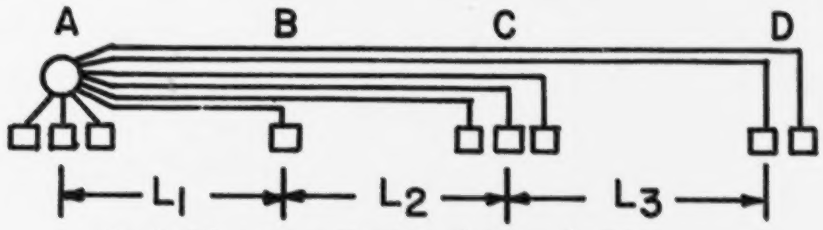


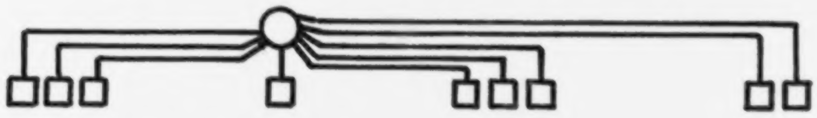
Fig. 6. Possible locations of switching center for two buildings.

building with one telephone. If the switching center (the large circle) is located in the building with one telephone, we need two wires between buildings. If the switching center is located in the building with two telephones, we need only one wire between buildings. Obviously the latter location uses less wire.

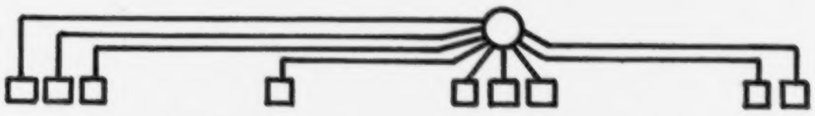
Figure 7 shows a slightly more complicated case involving four buildings with 3, 1, 3, and 2 telephones, respectively. The four possible locations of the switching center are shown. We let the lengths of the segments 1, 2, and 3 be L_1, L_2, and L_3, respectively.



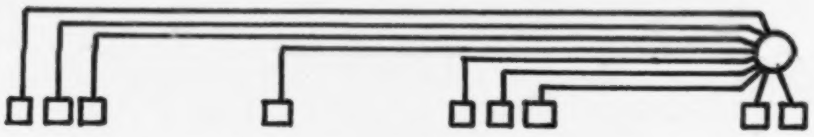
(a) Wire length $= 6L_1 + 5L_2 + 2L_3$



(b) Wire length $= 3L_1 + 5L_2 + 2L_3$



(c) Wire length $= 3L_1 + 4L_2 + 2L_3$



(d) Wire length $= 3L_1 + 4L_2 + 7L_3$

Fig. 7 Switching-center location for four buildings.

In Fig. 7(a), the total length of wire which is required is shown as $6L_1 + 5L_2 + 2L_3$. If we now shift the switching center to building B, as shown in Fig. 7(b), the total length of wire is $3L_1 + 5L_2 + 2L_3$. If we compare this value to the original value above, we note that this new position for the switching center involves less wire: a reduction of $3L_1$ units of length, since L_2 and L_3 are not affected by the change.

If we now shift the switching center to building C, the length of wire required is shown in Fig. 7(c) as $3L_1 + 4L_2 + 2L_3$. This new length is less than that with the switching center at building B by an amount L_2. The L_1 and L_3 lengths are not affected by the shift from B to C.

If we now try to shorten the wire length further by placing the switching center at building D, the wire length is found to be greater than the length required from building C by $5L_3$ units, since now the L_1 and L_2 values are not changed. Obviously the location of the switching center which requires the least wire is that at building C. This location is determined without regard to the actual values of L_1, L_2 or L_3 (these lengths can have any non-negative values).

Can we observe anything about these examples that would help us in more complicated problems with many buildings and many telephones? In Fig. 7 there are three "segments" (intervals) between buildings. We note the following facts:

The discovery that the actual lengths of the distances between buildings is not involved in the placement of the switching center for minimum wire length is very interesting. It means that the only factor (which determines the minimum that we seek) must be the number of telephones on either side of the switching center. If we begin our search with the assumption that the switching center is at the building at the extreme left, the first shift of position changes the first length from $6L_1$ to $3L_1$, all other lengths remain unaffected. In this case, we reduce the telephone line length.

A shift of the switching center to building C does not affect the length $3L_1$, but requires four lengths of L_2. It is also apparent that the two wires ($2L_3$) are not affected. Thus the second shift produces a change only in the second term which is equal to the number of phones to the left of C multiplied by L_2. Since the term $4L_2$ replaces the term $5L_2$ of the B location (all other terms remain unaffected), location C is preferable to location B.

How does a shift to D affect the length of line? The $3L_1$ and $4L_2$ terms are not disturbed, but we now have 7 telephones to the left of the switching center, each of which requires an additional length L_3 to reach the center. This replacement of $2L_3$ by a new value $7L_3$ obviously represents an increase in length compared to location C, and this is true, regardless of the actual value of L_1, L_2, or L_3.

This leads to a simple algorithm to find the optimum location of a switching center:

1. In each segment (between buildings), we determine the number of telephones to the left and to the right of the segment, and write these numbers below the segment as (N_L, N_R).

2. We place an arrow below the number pair pointing from the smaller number toward the larger number.

3. We find the building where the arrows point to it from both sides; this is the location of the switching center that gives the shortest length of wire.

Let us try the algorithm on the examples of Figs. 6 and 7. In Fig. 6, there are 2 telephones to the left and 1 to the right of the central segment so we write (2, 1) and put an arrow underneath pointing to the left (from the smaller number 1 toward the larger number 2) as shown in Fig. 8. In the "segment" to the left of the central segment there are no telephones to the left and 3 to the right so we write (0, 3) there and put an arrow to the right. Thus the building on the left has arrows pointing at it from both sides and is the best location..

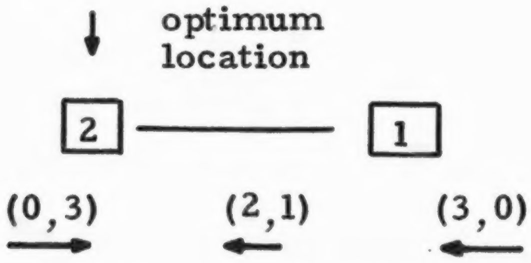


Fig. 8. Optimum switching-center location for two buildings.

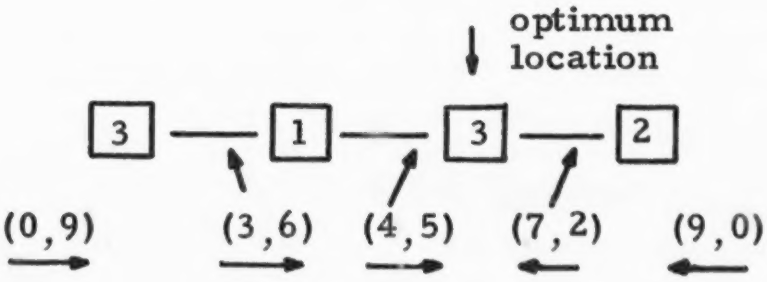


Fig. 9. Optimum switching-center location for four buildings.

In Fig. 9 for the four-building example, arrows from both sides point to the second building from the right as the best location.

In Fig. 10, we show an example with more buildings and, in the problems at the end of the chapter, we consider the extension to cases where the buildings are not all on a single line.

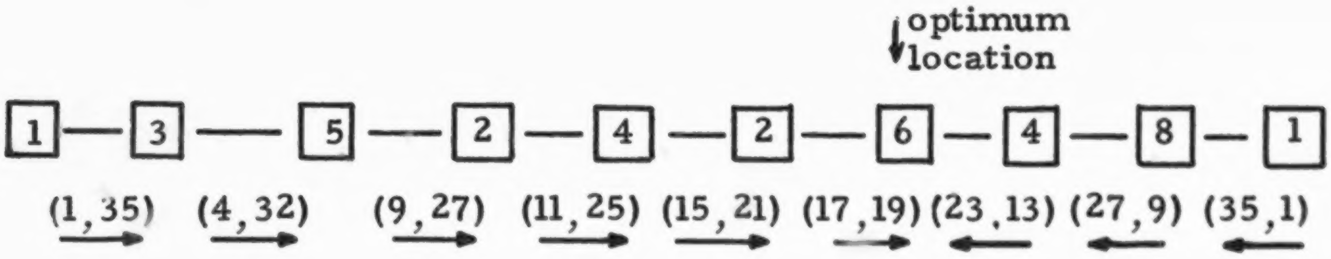


Fig. 10 A 10-building example (it is helpful to note that the sum of N_L and N_R always equals the total number of telephones).

You may ask, "What about a location between buildings?" We can easily show this is not helpful by putting in a "fictitious" building (to house the switching center) with zero telephones in it. Using our algorithm above, we find the same number pair (N_L, N_R) on both sides of this building and hence the arrows point in the same direction on both sides; therefore, this vacant "building" can not be the optimum location. We conclude that the best location is always at a building. Whenever an algorithm has been developed for a type of problem, the computer may be programmed for this algorithm and problems may then be solved quickly and accurately.

6. A PRODUCTION PLANNING PROBLEM

There are problems which require optimization, but in which the information available for making decisions is not precisely known in advance. In this section and the following, we conclude our discussion of optimization with two examples of problems in which the model is probabilistic (i. e., the model can be described only in terms of the probability that certain events will happen).

For example, the sale of bread at a bakery may vary from day to day in a very irregular manner. On some days as few as 10 batches of fresh loaves are sold. On other days a maximum of 14 batches of fresh loaves can be sold. A single batch of bread contains 10 loaves.

If a loaf of bread costs 12¢ to bake and is sold when fresh for 22¢ there is a profit of 10¢ per loaf, or 1 dollar per batch. On the other hand, unsold fresh bread must be sold as "day old" bread at 8¢ per loaf, which involves a loss of 4¢ per loaf or 40¢ per batch. If the owners of the bakery wish to maximize their average daily profit over a period of many days, how many batches should be baked? Baking only 10 batches will leave nothing for "day old" sales since they never sell less than 10 batches. But the sales figures show that such small sales were few. On only 5 days in 100 days did this occur. On other days they could sell larger quantities and thus increase the overall profit. As a matter of record their sales in the 100 days show:

10 batches of fresh loaves were sold on 5 of the days
11 batches of " " " " " 20 of the days
12 batches of " " " " " 50 of the days
13 batches of " " " " " 20 of the days
14 batches of " " " " " 5 of the days
100

The owners expect the sales to be about the same for the next few months.

This is a problem involving probabilities. The sales records indicate that the bakery can always sell at least 10 batches of fresh loaves. They can sell 11 batches 95% of the time (20 + 50 + 20 + 5 = 95), 12 batches 75% of the time (50 + 20 + 5), 13 batches 25% of the time (20 + 5), 14 batches 5% of the time, and they never sell 15 batches or more. Now if a batch is sold fresh it gives a profit of 100 cents = 10 x (22 - 12), whereas if it is sold to the "day-old-baked-goods" store it results in a loss of 40 cents = 10 x (12-8).

What is the average profit that could be made from the eleventh batch? Clearly since it is sold fresh 95% of the time and sold to the "day-old-baked-goods" store 5% of the time, the average profit will be

$$0.95 \times 100 - 0.05 \times 40 = 93.$$

Similarly, the twelfth batch will bring an average profit of

$$0.75 \times 100 - 0.25 \times 40 = 65$$

The thirteenth will have an average profit of

$$0.25 \times 100 - 0.75 \times 40 = -5$$

This is an average loss of 5 cents. Similarly, we can show that the fourteenth batch results in an average loss of 33 cents. Since, on the average, the profit on the 13th batch is negative, it is obviously not worth baking, on the average. Thus the optimum number of batches to make is 12; i.e., 120 loaves. The average expected profit is

$$\begin{aligned} 10 \times 100 + 93 + 65 &= 1158 \text{ cents} \\ &= \$11.58 \end{aligned}$$

The example completed above is relatively straightforward. The interesting feature of this problem is the fact that the system model is not known precisely: we cannot say with certainty that on any given day in the future, it will be possible to sell 13 batches. All we do know is that, in the next 100 days, there should be about 20 in which we can sell 13 batches.

In many of the situations requiring decisions in the real world, such probabilistic models are unavoidable. When a man and his wife decide to travel to a distant city without their small children, they often debate whether to travel together or separately. The decision depends on the evaluation of the probability of a serious accident and consideration of the value they place on being sure at least one parent makes the trip safely in order to care for the children in the future.

7. QUEUEING PROBLEMS

This idea of models involving probability can be illustrated by a group of studies called queueing problems. These relate to the formation of queues or waiting lines when customers arrive at a service facility and there is only a limited number of service personnel. Queueing problems arise in such diverse situations as:

(1) Toll booths on a parkway, where long line of cars may build up during periods of heavy traffic flow.

(2) A barbershop or hairdresser operating without appointments. Even though the barber may be idle a significant part of the time, there are other times when a queue of three or four customers forms (the queue seldom becomes longer because prospective customers turn away rather than wait).

(3) An airport in which only one runway can be used for take-off and landings. Because of traffic demands generated by men anxious to return home for the weekend, many major airports in the United States have very large queues of planes circling and of aircraft waiting to take-off every Friday afternoon about 5 p.m.

(4) A telephone exchange. On Christmas and Mother's Day, long-distance toll lines are frequently saturated.*

(5) A production facility dependent, for its successful operation, on a number of machines (e.g., 10) which must be maintained by two mechanics. Each machine breaks down occasionally and can be repaired by a mechanic (with an amount of time required for servicing which varies according to the nature of the difficulty). In this case, the queue consists of inoperative machines being repaired or waiting to be repaired (the customer for the service is the machine).

All of these areas are described by:

(a) Customers arriving to be serviced

(b) Service stations or personnel available in a limited number to give the service, and

(c) A possible queue, if customers arrive faster than they can be serviced during a given period.

In the study of such systems, we might ask a variety of questions, such as:

(1) With given level of business (average number of customers) and specified service facilities (e.g., number of toll booths), what is the average size of the queue?

(2) How often will the queue be longer than a given length (how often does the barber lose possible customers who look in the window and then disappear if there are more than two men waiting)?

(3) What is gained by adding another service station (an additional toll booth or an extra barber and chair)?

(4) What can be gained by reducing the time required to service customers?

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* The telephone-system example is particularly appropriate, since the major portion of the basic work on queueing problems was developed for guiding the design of telephone systems.

We should like to be able to answer such questions mathematically (or by computer studies), since in many cases it is just not feasible to experiment on the actual system. We can hardly expect political approval for constructing another runway at an airport if we merely express the desire to experiment in order to see how such an addition would decrease the number of planes stacked over the airport on Friday afternoon. Thus, queueing theory is concerned with seeking solutions to questions (such as cited above) on the basis of studies of the model.

In this section, we can only introduce the subject and give a general indication of the type of results which can be realized. In particular, we restrict our consideration here to a very simple sub-group of queueing problems in which we can obtain simple answers; in more complex problems, we have to use the digital computer to obtain simulations (computer models) of the actual system. In particular, the following paragraphs consider only the single question: what is the average queue length in a very simple class of problems?

Arrival of customers

In our attempts to simplify the problem, we first consider a very special way in which customers arrive at the service facility. Specifically, we assume that they arrive completely at random and independent of one another. In other words, if we represent the average number of customers arriving in a unit time, we assume that in any short interval dt the probability of a customer arriving is αdt. For example, if $\alpha = 0.25$/minute (on the average 1/4 customer arrives per minute or one customer every four minutes), the probability of a customer arriving in any given 1/10 minute is $(1/4) \cdot (1/10)$ or 1/40; this number is correct regardless of what 0.1-minute interval we choose -- even if we choose an interval just after the arrival of another customer.

All we are saying in the preceding paragraph is that the customers arrive at a certain average rate (α), but exactly when they arrive is pure chance.* Occasionally several customers arrive in close succession; on other occasions, there are long intervals between arrivals.

This assumption of random arrivals is used because the solution of the queueing problem is thereby greatly simplified mathematically. Actually, however, there are many cases in which the assumption is very close to the actual situation. Telephone calls during a busy period may be placed at random; customers do often arrive at a barbershop nearly at random. When highway traffic density is not too great (so cars do not clog up behind an occasional, slow vehicle), cars may arrive at a toll station according to our assumption.

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* If we study probability, we find that this assumption is described by saying there is an "exponential distribution of times between arrivals."

Servicing time

Our second major assumption is that there is only one service facility or person: only one barber, one runway, or one toll collector to service the arriving customers. Furthermore, we assume that the times required to service the various customers are either:

(a) All equal, or

(b) Vary in exactly the same way as the inter-arrival times of the customers (with an average value, then, some less and others greater than this average at random).

In other words, we assume that every customer requires the same length of time (a case which would be approximately true for landing airplanes or haircuts) or that the required service time varies in a random fashion (the probability of the end of the service period lying in a particular time interval does not depend on the location of that interval).

Actually, we could complete this section by considering only the characteristic of equal service times. The only reason we add the random variation is to show in the examples below the effect of different customer service times (an effect with which we are all familiar as a result of waiting in line behind the little old lady who has a thousand questions she must ask).

Utilization

Before considering the problem of the queue length (the ultimate goal of this discussion), we need to ask one preliminary question: can the service station do the required job of handling the arriving customers? In other words, is the service facility (with its specified servicing time) capable of processing the customers (with their average arrival rate)? Or, does the system break down completely, with the number of waiting customers tending to grow without limit?

The answer to this question depends only on a ratio which we call β:

$$\beta = \frac{\text{average servicing time}}{\text{average inter-arrival time } (1/\alpha)}$$

If $\beta<1$, the service facility can process the customers; if $\beta>1$, the system cannot work and the concept of a queue is meaningless. In other words, on the average the service facility must be able to process customers faster than they are arriving on the average.

This result is hardly startling. If airplanes arrive at an airport at an average rate of two per minute and they can land on the single, available runway at a maximum rate of one per minute (corresponding to a servicing time of one minute per customer), it is clear that the system cannot operate successfully: the sky will gradually fill with aircraft.

The quantity β, which is called the utilization factor in books on this subject of queueing, is a measure of the fraction of time the service facility is used. Thus, if $\beta = 0.7$, the service personnel and equipment are being utilized 70% of the time -- in an eight-hour day, the barber (for example) would be giving haircuts 5.6 of the hours.

Length of the queue

We are now ready to discuss the queueing problem, and in particular the average length of the queue. In the preceding paragraphs, we have considered the arrival of the customers (at random with an average rate of α), the servicing time, and the utilization factor β. In order to describe the problem in specific terms, we consider the barbershop situation.

We make the following assumptions:

(1) There is one barber.

(2) Customers arrive at random, at an average rate of one every 20 minutes (hence, $\alpha = 1/20$).

(3) The service time is 14 minutes for each customer (i.e., a haircut requires 14 minutes, including the time to seat the customer, to accept his money and receive the tip, and so forth). The service time is identical for all customers.

We are interested in the question of how long the queue will be on the average in this barbershop. In particular, we are interested in what happens after a steady, normal operation has developed. When the barbershop first opens for business at 8:30 a.m., the first customer clearly experiences no wait at all. The second customer certainly does not have to wait more than 14 minutes, and very likely he has a much shorter wait. After the shop has been open for several hours, however, the effects of the initial opening die out, and the system operates in what is called a steady-state fashion.

In this steady-state operation, the actual situation is, of course, governed by probability. Since the customers arrive absolutely at random, we cannot predict for any particular customer whether he will have to wait or not. Instead, we can say that there is a certain probability he will be served immediately, or a different probability that he will be served in less than ten minutes, etc. Alternatively, we can determine the average length of the queue.

This average queue length, q, is given by the formula*

$$q = \frac{\beta(1 - \frac{1}{2}\beta)}{1 - \beta}$$

This is the formula valid for our particular problem: one service facility, random arrivals, and constant servicing time.

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*We do not derive the formula here, since that would involve an extensive detour into mathematics which is not of primary interest.

For our particular barbershop problem, β has the value 0.7 [14 minutes/ 20 minutes = (servicing time) / (average inter-arrival time)]. Incidentally, if β were greater than unity, the problem would be done: the single barber cannot handle the customers and the queue length is irrelevant. Since β is less than unity (indeed, the barber is busy only 70% of the time), we can substitute in the above equation to find q:

$$q = \frac{0.7\,(1 - \frac{1}{2}\,0.7)}{1 - 0.7}$$

On the average, the queue length is 1.51 customers.

Our barber may well be very satisfied with this state of affairs. He is working 70% of the time and most customers do not have an unreasonable wait for service. He is, however, troubled by the situation which develops on Saturdays, when for some reason the shop seems to be filled with impatient customers. With his elementary knowledge of queueing problems, he watches again the situation on Saturday and finds that he has 40 customers in an average ten-hour day. In other words, customers on Saturday arrive at the rate of one every 15 minutes (rather than every 20, as during the week).

This decrease in average inter-arrival time means that

$$\beta = \frac{14}{15} = 0.933$$

He realizes now why he is exhausted every Saturday night: he is working 93.3% of the time on Saturday, or with only 4 free minutes each hour on the average. Furthermore, substitution of this value of β in the equation for average queue length reveals

$$q = \frac{0.933\,(1 - 0.467)}{0.067} = 7.4$$

(This average queue length is now so long the validity of the model is open to suspicion: can the barbershop hold all these men waiting for haircuts? Furthermore, since some people experience no wait, there must occasionally be individuals who arrive with a queue of perhaps 12 customers).

In order to improve the Saturday situation, our barber has two obvious alternatives. He can try to hire an additional barber to work only on Saturday (this alternative is not particularly attractive since every other barber shop is looking for Saturday-only employees, wage rates are high, and extra equipment is needed). The second possibility is to reduce the servicing time, by speeding up his own work (while customers might occasionally be unhappy, the customer dissatisfaction stemming from the waiting time is already great). If he can bring his servicing time down to 10.5 minutes, he can restore the 0.7 value for β to reduce the average queue length again to the weekday 1.51.

Comments on the example

Two general comments should be made about this example, and indeed refer to optimization problems in general. First, there is always a question about the validity of the model. In this problem, we have made various assumptions -- e.g., the randomness of customer arrivals and the equal servicing times. What happens when these assumptions are not true in the actual system we are studying? Are our numerical results of any value?

Certainly no barbershop operates exactly as we have assumed here. For example, if there are four men waiting, very few customers would be likely to enter to increase the queue length. In the real world, servicing times are not equal, even in a barbershop or for a telephone exchange.

In spite of such aspects, which mean that our model is only a rough representation of the actual world, the numerical answers obtained by queueing theory are often extremely valuable as guides to system design and optimization. In practice, we analyse a situation with known characteristics, when we can compare the theoretical results with the actual situation. If this agreement is good, we assume the model is adequate and then use that model to study the effects of system changes (an additional barber, a reduction in servicing time, etc.).

The second general comment relates to the rather simple, specific problem discussed above. Can these same methods be applied to more complex and interesting problems or to answer other questions about the system (e.g., we might ask what fraction of the customers will arrive at the barbershop when the queue length is two or greater -- these might well be the prospective customers turned away by discouragement).

As mentioned earlier in this section, one way to broaden the problem is to assume a distribution of servicing times, rather than a constant time. If, for example, we find that the servicing times are distributed at random (just as the inter-arrival times), with an average servicing time known. we find that the average queue length is given by the equation

$$q = \frac{\beta}{1-\beta}$$

rather than the formula given earlier for constant servicing time. For the two values of β given in the barbershop example, the resulting values of q are:

β	q (constant servicing time)	q (random servicing time)
0.7	1.51	2.33
0.933	7.4	14

The queue length is increased markedly when occasional customers require a long time for servicing.

Finally, this entire subject of queueing problems is related to optimization in a way which may seem somewhat indirect when we only have the above, simple examples as background. Queueing theory is a basis for optimum design: a study of these problems indicates, for example, how many toll gates are required to yield the simplest (or most economical) system which services the customers without objectionably long queues. More advanced consideration of queueing problems also indicates the quantitative value of adopting different priority policies: should we take the customers on a first-come, first-served basis or on the basis of the fast customers processed first; what is gained by using an express counter in a bank or supermarket for customers who can be rapidly serviced?

8. CONCLUDING COMMENTS

We are now at the end of two chapters devoted to the fundamental ideas underlying a variety of problems calling for optimum decisions. In general, the formulation of these problems requires a model of the system to be designed and a listing of the constraints which are imposed. Once this model is chosen, the criterion function is selected -- i.e., we decide how to measure the quality of the system, or what specifically it is that we wish to optimize. At this point, we are ready for the actual act of optimization: the determination of the design such that the criterion function is maximized (or minimized).

In this and the preceding chapter, the models are usually conceptually simple, although they may involve many variables and it may be very difficult to determine suitable values for the parameters (the numbers of the model). Because the models are relatively easy to understand, we can focus our attention on methods of solution -- and particularly the optimization portion of the problem.

The problems considered in these two chapters all lie within the field of engineering called operations research. This name is somewhat unfortunate,* since the field is part of applied science and engineering and does not necessarily involve research at all.

In the chapters which follow, we extend these ideas of optimization and modelling to a much broader class of problems than we have considered in these two chapters. In particular, we are interested there in dynamic systems -- systems which move or change with time. In such cases, the modelling part is a difficult and focal portion of the problem, and most of our attention is focussed on the understanding of the idea of modelling. It is difficult to discuss optimization in detail until we have acquired a rather deep background in basic engineering concepts.

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* It arose because the field evolved from research into military operations during World War II.

PROBLEMS

2.1 Suppose that a radio manufacturer turns out only two types of radio: a standard model, selling at a profit of $20 each, and a luxury model, selling at a profit of $30 each. The factory has two assembly lines, but their capacity is limited. It is possible to produce at most either 8 standard radios or 5 luxury radios per day on one assembly line. The manufacturer is faced by another constraint: owing to limited skilled labor supply he has only 12 employees, so the available labor amounts to 12 man-days per day. To assemble a standard radio requires one man-day, but it takes two man-days to make a luxury radio.

(a) How many radios of each type should he produce in order to maximize his profit?

(b) What will this maximum profit be?

2.2 In the transportation problem of Section 3, suppose that the amount of wheat at Grand Forks is 30,000 bushels and at Chicago is 60,000 bushels. Find the minimum cost shipping plan.

2.3 An oil company has 200 thousand barrels of oil stored in Kuwait (on the Persian Gulf), 150 thousand barrels stored in Galveston, Texas and 100 thousand barrels stored in Caracas, Venezuela. A customer in New York would like 250 thousand barrels and a customer in London would like the remaining 200 thousand barrels. The shipping costs in cents per barrel are shown below. Find the minimum cost shipment schedule.

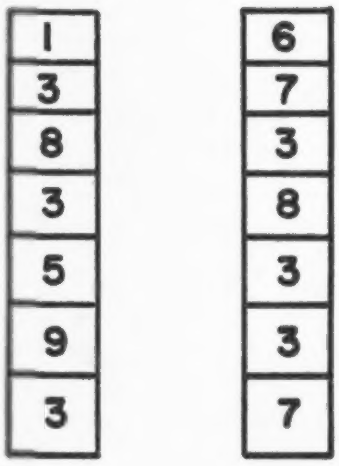
	Kuwait	Galveston	Caracas
New York	38	10	18
London	34	22	25

2.4 a. Find the optimal location for the switching center when the number of telephones in each building (reading from left to right) is

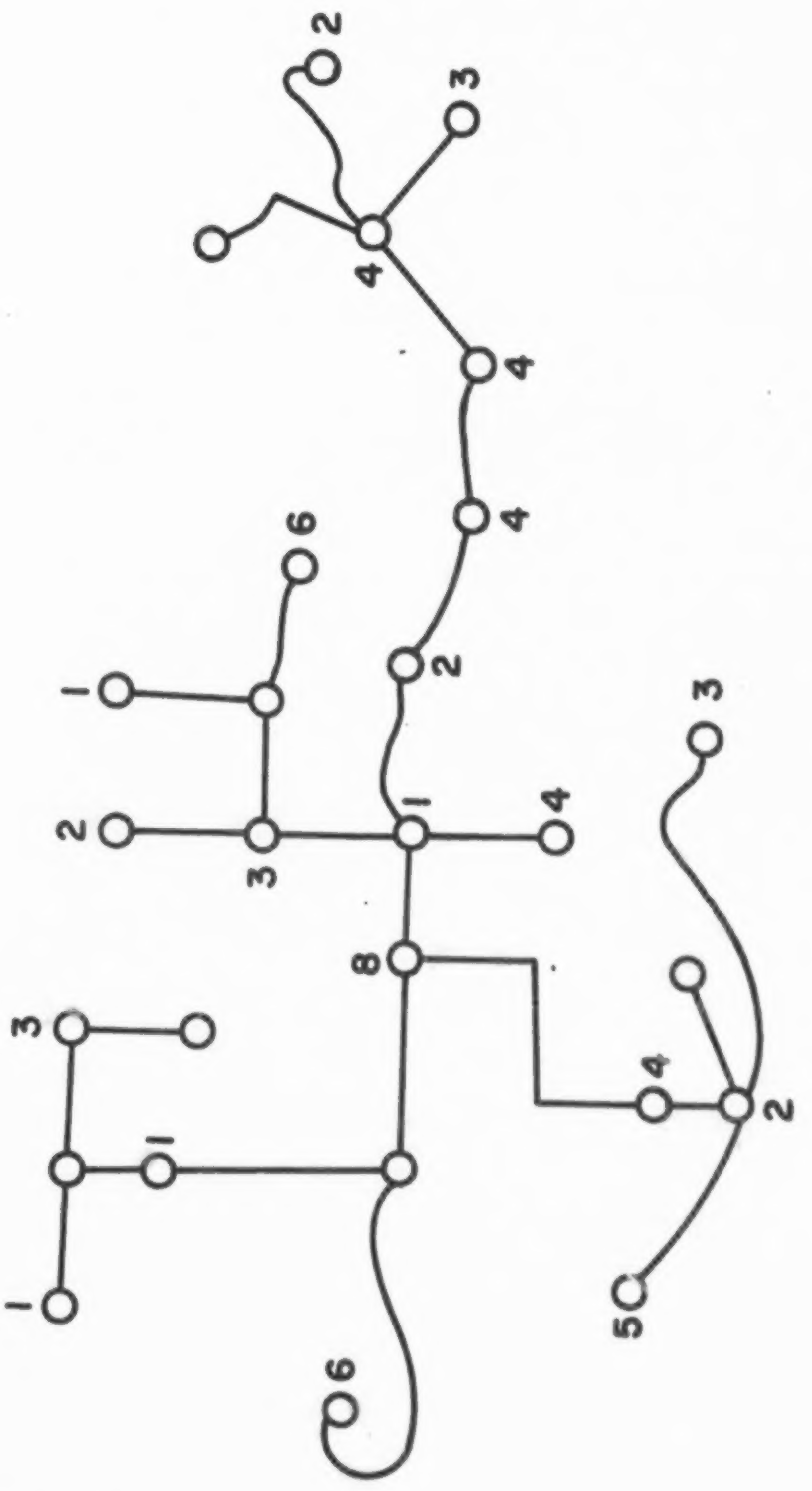
8, 7, 8, 2, 2, 5, 4, 6, 7, 1.

b. Did you obtain all the optimal locations?

2.5 Consider the diagram on the right. Two rows of buildings are separated by a street. Find the optimal location for the switching center if there is the following constraint: a wire that goes across the street must be underground. (The expense would limit the numbers of tunnels across the street to one.)



2.6 (For Special Credit) The restriction in Section 5 that the buildings must lie on a single road (note that the road need not be straight) can be relaxed. Suppose the buildings lie on a road network as illustrated below.


2
3
4
4
4
6
1
2
2
3
1
4
3
8
3
1
4
2
1
6
5

Each small circle is a building. Notice that there is exactly one path between any two buildings. A configuration with this property is called a tree. The community has many other roads but telephone lines can be laid only on the indicated roads because (a) of town laws, (b) of geographic obstructions, (c) of economic factors and (d) for a tree we know how to solve the problem of optimal location of a switching center.

a. For the single road town problem, the optimal location was characterized by two inequalities. What characterizes the optimal location for the tree town problem? You can obtain the correct inequalities by the same analysis used in the text.

b. Find the optimal switching center location for the figure above.

c. Give an algorithm for making the necessary calculations in a systematic and efficient way.

2.7 A newstand buys a certain weekly magazine for 20¢ and sells it for 35¢. Left-over magazines can be returned at the end of the week for a refund of 5¢. Their records show that they sell:

46-50 magazines	5% of the time
51-55 "	10% of the time
56-60 "	20% of the time
61-65 "	30% of the time
66-70 "	15% of the time
71-75 "	12% of the time
76-80 "	8% of the time
	100

How many magazines (to the nearest five) should the newstand order to maximize the average weekly profit? What is the maximum average weekly profit?

2.8 In a barber shop the service time is 15 minutes per customer. The cost for a haircut is $2.00.

a) If the average inter-arrival time is 20 minutes, find the percent of time the barber will be working.

b) Assuming that we are talking about constant service time, what is the average queue length?

c) What is the gross income per day (neglecting tips), assuming an 8-hour day?

d) In order to stay in business the barber must gross at least $40/day. If he wishes to keep his percent of working time constant and does not wish to speed up his service time, how much must business increase before he can afford to hire another man, costing $20/day?

e) If he had not hired a new man when business increased by the amount in part (d), what would be the average queue length (still assuming constant service time)?

2.9 In the previous problem we assumed constant service time.

a) Use the same figures and calculate the queue lengths of (b) and (e) assuming random service time.

b) Which is a more realistic model for serving time, constant or random?

c) How could the barber decrease his queue length without getting more help?

Chapter B-3

MODELING

In Chapter B-1 we learn that the making of decisions, an everyday need that too often is satisfied by chance or whim, can be done in an orderly and reasonable way. The process is to derive a model, to select criteria, to examine the constraints that exist, and after making a trial solution to attempt to optimize this solution. In this chapter and the next one we study models in more detail.

1. THE NATURE OF MODELS

According to an old story, six blind men who had never seen an elephant tried to decide what it was.* The first man, feeling the elephant's flat, vertical side, concluded that the beast was similar to a wall. The second man touched a round, smooth, sharp tusk, and decided that an elephant is similar to a spear. Grasping the squirming trunk, the third blind man said that the animal resembled a snake. The fourth man, who touched a knee, observed that elephants resembled trees. From an exploration of the ear of the elephant, the fifth man was convinced that the animal had the shape of a fan, while an examination of the tail convinced the sixth blind man that an elephant was similar to a rope.

Each, of course, was partially correct, but insofar as a complete representation of the elephant was concerned, all were wrong. Each man, after observing the "real world", formulated a description, i.e., a model of the real world. But since the observations were incomplete, the models were incorrect.

Every time we describe an object, we really construct a model. We perceive and think in terms of images, but these images are models of the world about us. Information about the real world can only come through our senses. From this information the mind infers interrelationships which produce the effects we observe. These inferences constitute the models.

The model provides an efficient way of viewing, for it is not required to tell us everything about what we observe, but only what we believe to be useful. No model is ever a complete representation; it must be only a simplified version of the real world. Therefore its usefulness will be limited because it includes only those properties of the real world which are of importance at the moment. We comprehend our environment through the models that our minds construct.

An important factor in human progress is man's capacity to construct models which are more versatile than the models our minds build for us automatically. More and more we build purposeful models, based on scientific data and measurements. These are essential if man wishes to obtain a more complete understanding and greater control of the environment in which he lives.

Models begin as conceptions; that is, they are ideas about the structure and nature of something. Once an idea of the structure and nature of a thing is conceived, it may be expressed in many different ways -- we may have different models. Some, as we have seen, are verbal models. A map is a model: a graphical model. Other models are mathematical, in which quantitative expressions are used to describe and to show relationships in a highly precise way. We shall also discover that models are frequently developed which employ computers, others make use of electric circuits, or of hydraulic, mechanical or chemical systems.

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*The Blind Men and the Elephant (J. G. Saxe 1816-87).

An aircraft represents so complicated an aerodynamic problem that a complete mathematical description may be impossible. Therefore, it is usually modeled by constructing a small-scale version of metal or wood for testing in a wind tunnel. In each case the attempt is made to provide a simplified but accurate representation of the really essential features of the thing being described. The secret of effective modeling lies in knowing what to include and what to omit.

Models are used, not only to describe a set of ideas, but also to evaluate and to predict the behavior of systems before they are built. This procedure can save enormous amounts of time and money. It can avoid expensive failures and permit the best design to be found without the need for construction of many versions of the real thing. Models evolve, and it is customary to go through a process of making successive refinements to find a more suitable model. A familiar example is the testing of a number of small-scale model airplanes in a wind tunnel before final design is accepted.

In other cases too, such as in the development of a model for a nerve cell, there is need for successive refinement. A preliminary model is designed, it is tested against the real nerve cell, then the model is modified so that it becomes more realistic in its behavior. There is thus a continued process of successive approximations until an accurate and revealing fit between the model and the nerve cell is obtained. In the process of model construction it is essential to alternate back and forth between the real world and the model. Without this continued testing and refining, models can give misleading results, and if models are inaccurately conceived or too simply structured the results will be unrealistic and useless. Properly developed, models are necessary tools without which we cannot react to the world around us.

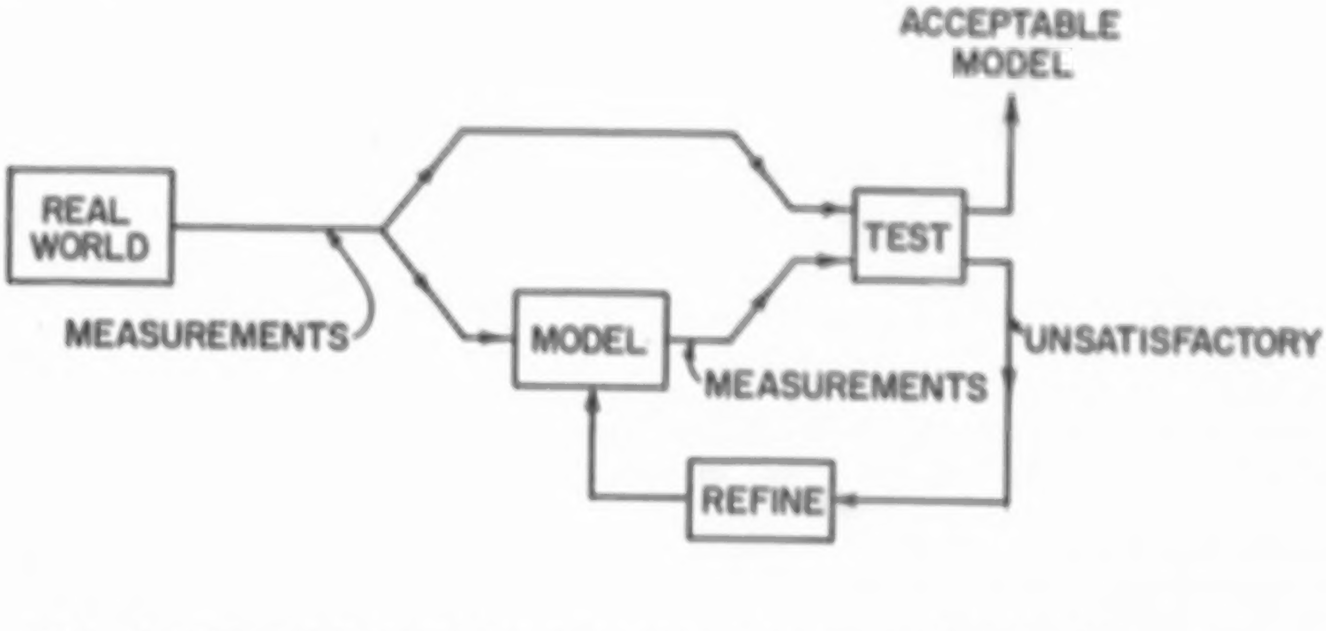

Fig. 1 The model-making process, shown as a block-diagram.

The essential parts of the model-making process are illustrated in Fig. 1. Measurements or observations of the real world are used to develop a model. After a preliminary model is made, measurements and predictions made with it are compared to the behaviour of the real world. In most cases these tests show that the model is not completely satisfactory, so that it must be refined. This process is repeated until comparison indicates that the model is acceptable. The process is then considered as complete.

In the modeling of a nerve cell, or of the growth of a population of people, the real-world measurements are made on a system which already exists. In this case our model-making process is intended to produce a model which accurately matches the real world. In the case where scale-model airplanes or spacecraft are modeled, the real world object may not yet exist, and the box marked "Real World" in Fig. 1 theoretically contains the real object which we imagine and wish to achieve, as well as all pertinent facts about the real world (such as the properties of air, characteristics of flight systems which have already been built, and the characteristics of various materials and fuels). The model building procedures will be no different from those already discussed.

Models can be descriptive, as in verbal, graphical, or mathematical representations. They can also be functional (they "really work"), as in scaled down airplanes for use in wind tunnels or working replicas of nerve cells. In the succeeding sections, we see how both descriptive and functional models are developed and used. We investigate some simple models to observe how they are developed and used to improve our understanding of the real world and how they help in the design, manufacture, and operation of devices and systems.

2. THE GRAPH AS A DESCRIPTIVE MODEL

When data are collected about some aspect of the real world, we observe properties that can lead to the formulation of a model. Suppose we wish to determine if a simple relationship exists between the heights and the weights of 17-year-old men. After making several experimental measurements, we may secure a set of related numbers such as 5'6", 130 lb; 6'1", 180 lb; 5'7", 155 lb; etc. But it is difficult to discover any systematic relationship in this way. Even though we may have a reasonable expectation that as height increases, weight will increase, this verbal model is vague and imprecise. Suppose now that we make a graphical plot of the data, as in Fig. 2. Each point represents the height-weight data for one man. We notice now that the points are not scattered but seem to be closely grouped. Of course there are several points which fall some distance from most of the others, but in general there appears to be a particular organization of points. What we can say about the relationship between height and weight from this presentation of our data?

Notice that a straight line can be drawn through the points to represent averages for these data. This is shown in Fig. 3, where the line is dashed outside the region where the points lie. The vertical scale has been changed, which causes the cluster of points to be slightly rotated. We can observe a relationship between height and average weight. This graphical picture of the data with the derived straight-line average gives us a graphical model of the relationship.

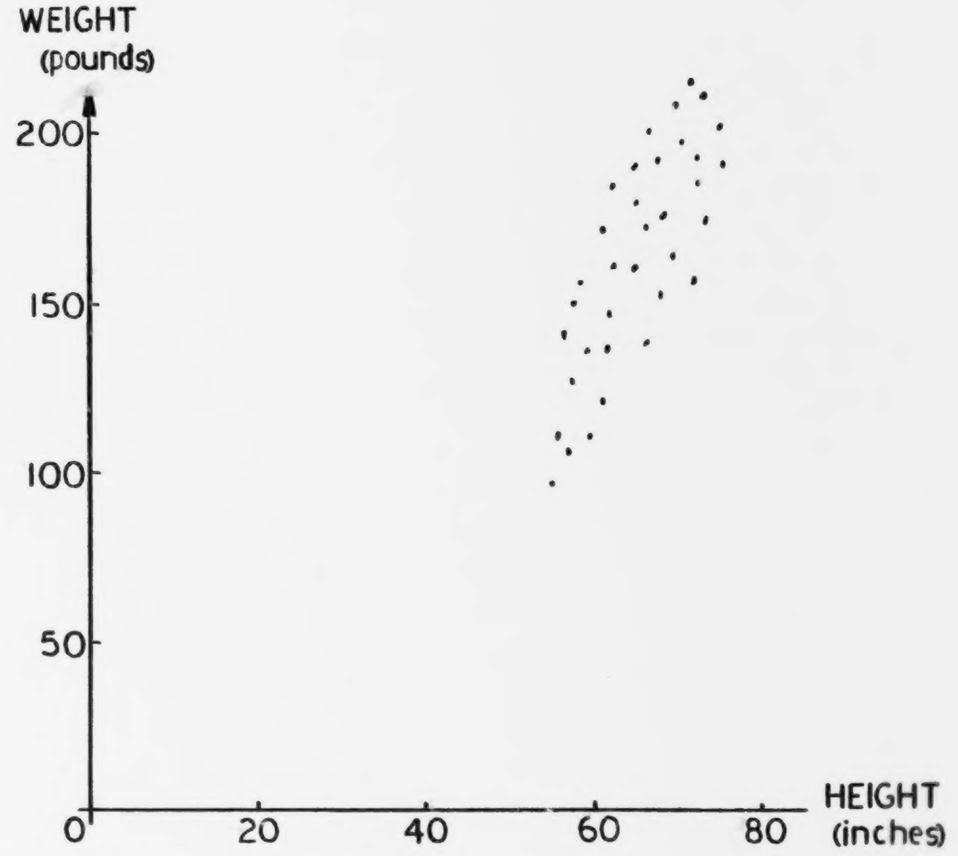

Fig. 2 Height-weight data for 17-year-old men.

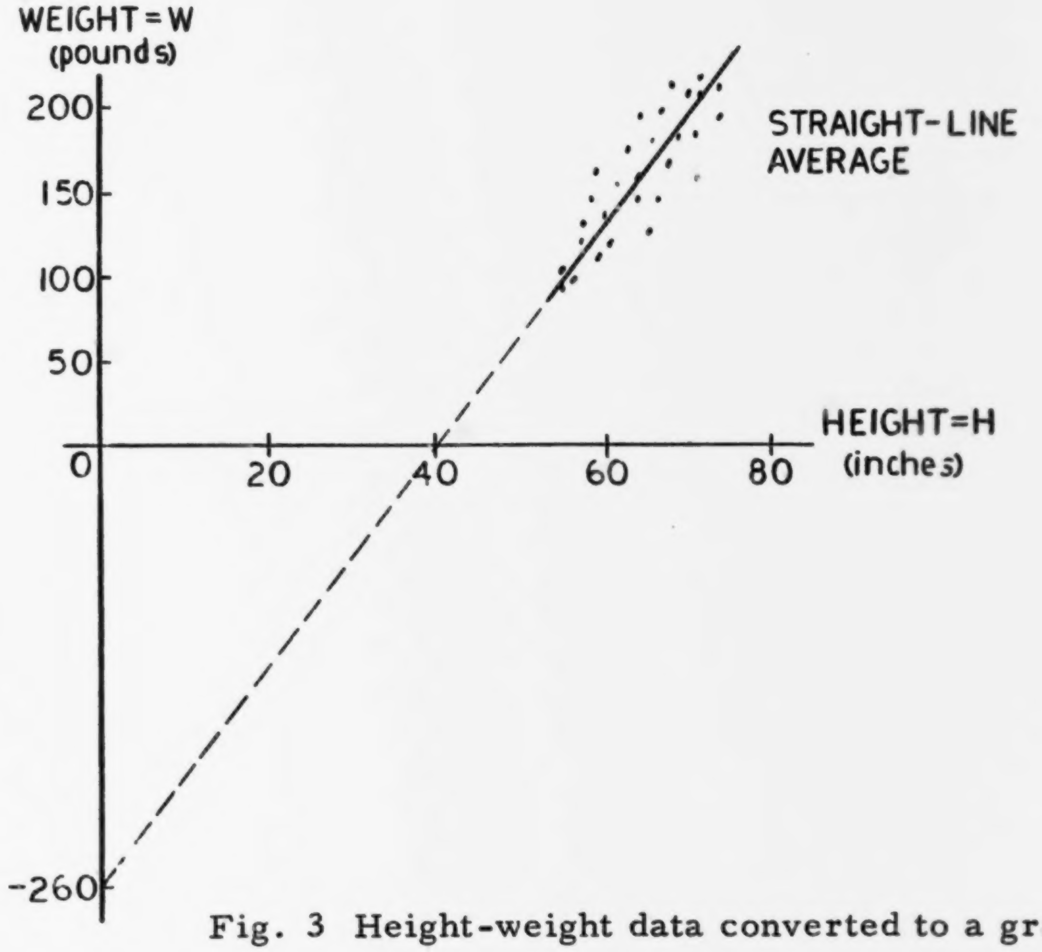

Fig. 3 Height-weight data converted to a graph.

This graphical model presents a clear but simplified description of the real world. It can be used as the basis for some reasonable predictions. From the straight-line average we can estimate the probable weight of a 17-year-old man even though we know only his height. Let us suppose that we wish to estimate the weight of someone who is 5'1-1/2" tall. The corresponding weight for this height can be immediately obtained from our graph. Notice that the graphical model permits us to estimate the weight of such an individual, even if our original data did not include any individual of this height. Thus we may predict from our graphical model that an individual who is 5'1-1/2" tall will probably weigh about 140 lb.

Once we have obtained the graph which represents the average, we can often find a mathematical equation which describes it. Since our graph is a straight line, we know from algebra that the equation has the general form $y = mx + b$, or in this particular case,

$$w = ah + b$$

where w is the weight, h is the height, a is the slope, and b is the vertical axis intercept. To complete the equation, we must therefore determine the value of the constants a and b. Since b represents the y intercept of the line, examination of the graph reveals that the line cuts through the Y axis at a value of -260. This reduces the equation to

$$w = ah - 260$$

To determine the numerical value of a which represents the slope of the line, we select any point on the graph, read the related values of w and h for that point. If we substitute the values of w, h and b in the general equation for the graph, we can solve for the value of a and thus establish the equation for any and all other values of w and h.

Suppose we select on the graph in Fig. 3 the point $h = 70''$ and $w = 195$ lb. We can substitute these values into the general equation for the straight line and solve for the value of a.

Thus:

$$w = ah - 260$$

or

$$195 = a(70) - 260$$

$$a = 6.5$$

The equation for our graph is then

$$w = 6.5h - 260$$

For any given height h, we can determine the related weight w.

We can thus use this model to predict points which were not originally in in our data sample. But these predictions must be carefully examined. For instance, the line we have drawn tells us that an individual who is 48" tall will weigh about 52 lb; worse yet, a 24" person can be expected to tip the scale at an impressive minus 104 lb! Surely these are curious figures for 17-year-old men. What is wrong with our model?

The straight line average that we drew to fit the data was extended so that the Y intercept (-260) could be found. But we are not entitled to say that all points on the entire line must represent real situations. Actually, the only valid use we can make of this model is to predict within our data field. We know that anywhere inside of the cluster of measured points we are in the neighborhood of a real-world possibility. However, we run into the danger of unrealistic prediction if we apply the model beyond the region that has been measured. This is why part of the graph was drawn with a dashed line.

Our model, therefore, has its limitations. It must not be used to predict beyond the region of results experimentally obtained unless there are very good reasons to believe that real-world laws are not being violated. To test this model (as indicated in Fig. 1) requires that we obtain a fresh sample of 17-year-old men, either the data from the same school for other years or data from other schools. Then we either enter them on the plot or compare them with predictions made from the algebraic model.

This is a mathematical model and it makes available in more compact form exactly the same relationship that was displayed by the graphical model. Both of these models are more useful than the verbal model with which we started.

One of the most interesting aspects of this model is the fact that it turns out to be so simple. This is really quite unexpected. If we think of the people we meet while walking down a busy city street, we know that all sizes and weights are intermingled in a quite random way. It is true that the sample studied was passed through two strainers, so to speak (age and sex), to make it more manageable. Even with these restrictions, however, the raw data coming from the physical education office showed little to suggest any other treatment than finding simple averages.

In the example just treated, we have illustrated the nature and the utility of graphical and mathematical models as means for establishing a basis for prediction. To examine the process of model construction or modeling in greater depth, we now consider a quite different example, but one that can, nevertheless, be modeled in a somewhat similar way. We develop a graphical model of the traffic flow in a school.

3. A DESCRIPTIVE MODEL OF TRAFFIC FLOW

We turn to a problem of quite another sort, that of traffic flow. One of the many stubborn parts of the urban problem we have already looked at briefly is the question of how to handle motor traffic in the streets. Some cities have gone so far as to ban all automobiles from certain streets (except for delivery vehicles in the small hours of the morning, when no shoppers will be inconvenienced or endangered). This doesn't so much solve the problem as eliminate it -- at least from those streets. Furthermore, it substitutes other problems, for example for the elderly and infirm. The urban traffic problem also includes those of air traffic into and out of airports (many planes using one runway or at best rather few, the difficulty complicated by weather), and of railway traffic.

If a traffic engineer is to improve present conditions he must be able to predict the results of the changes he suggests. To do this he must construct a model of the traffic flow in and around the airport, railroad yard, or city. Since such a model is too complicated for our purpose, we use a study of the simpler circumstances within a school; even this we limit to what goes on at a single corridor intersection, such as that shown in Fig. 4. However, the limited model we derive for a single intersection could be extended (often by means of new measurements where circumstances are different) to a whole building. The resulting larger model would be of practical use to school administrators and schedule-makers.

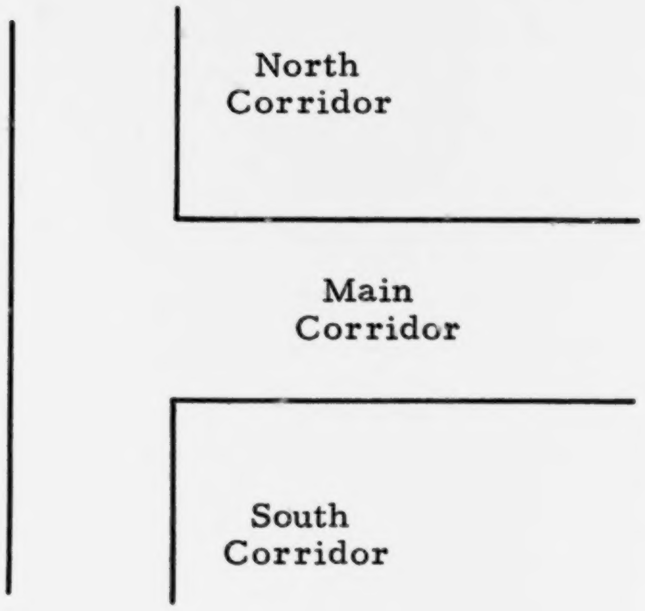

Fig. 4 Schematic illustration of a school corridor intersection

In order to construct our model, we must determine the important factors which affect the behavior of the system. We are primarily interested in the rate at which people (teachers and custodians are people too) pass from one corridor to another; in other words, in the density of traffic as measured in people per minute. The measurement will be made by sensors, devices which respond in some way whenever a person passes. An example would be the device often called an electric eye, but for short-term service a much more practical sensor is a person stationed at the proper spot.

Fig. 5 shows the intersection with measurement points identified. For reasons which are obvious to high school students, traffic problems usually arise at intersections rather in the main corridors*. What we need for the present is

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*As Joe passes position T_2 going east he sees that Susan is going south past T_1; he is immediately sure that he cannot survive math class next period if he doesn't have the answer to his invitation to the party on Friday night, and so he reverses his field and meets her in the middle of the intersection. Therefore they both hold up traffic (and he is counted three times at T_2; he might even stroll a few yards down the south corridor with Sue and add two more tallies to his total).

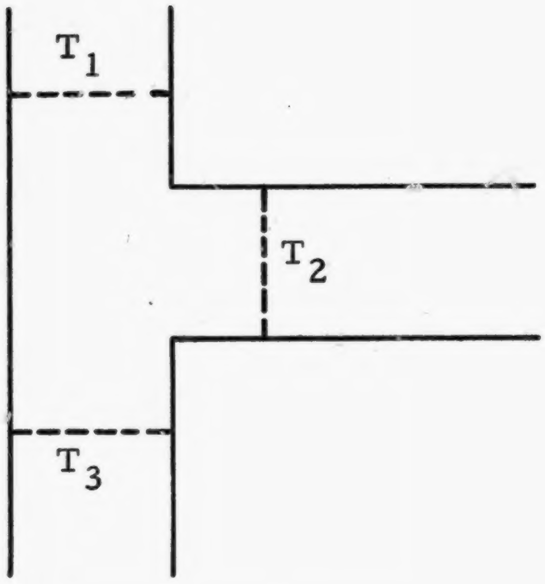

Fig. 5 Where traffic density is measured

a count, minute by minute during the time when classes are being changed, and for a minute (or more) before and after.

Table 1 shows a possible result of such a set of measurements. The actual number of people who go through the intersection during the counting period

Minute number	Counting station			Total counts
	T_1	T_2	T_3	
1	26 counts	50 counts	30 counts	106
2	42	63	47	152
3	61	102	55	218
4	112	184	73	369
5	38	42	28	108
6	22	17	9	48
			Total for period:	1001

Table 1 Detailed traffic count at one intersection, end of first period

is about half that shown, because each person was counted twice, going in and going out (if we forget people like Joe in the footnote). It is immediately evident that most of the traffic, but not all, occurred during the interval between classes. The data can be made easier to see at a glance, however, by constructing a bar graph (sometimes called a histogram), Fig. 6. The height of each vertical column is proportional to half the total count for the minute indicated below it, half the total for the reason just explained. The school administrator confronted with such a bar graph might well be suspicious of the large traffic during the minute before the bell rang for class changing: are some teachers dismissing their classes early? He might also be disturbed by evidence of a good deal of tardiness, and would no doubt send for the records of tardinesses reported in order to be sure that all teachers are making it a practice to turn in such reports.

Further information, useful to the scheduling officer, can be obtained if another bar graph is made, this time of total counts for each period during the

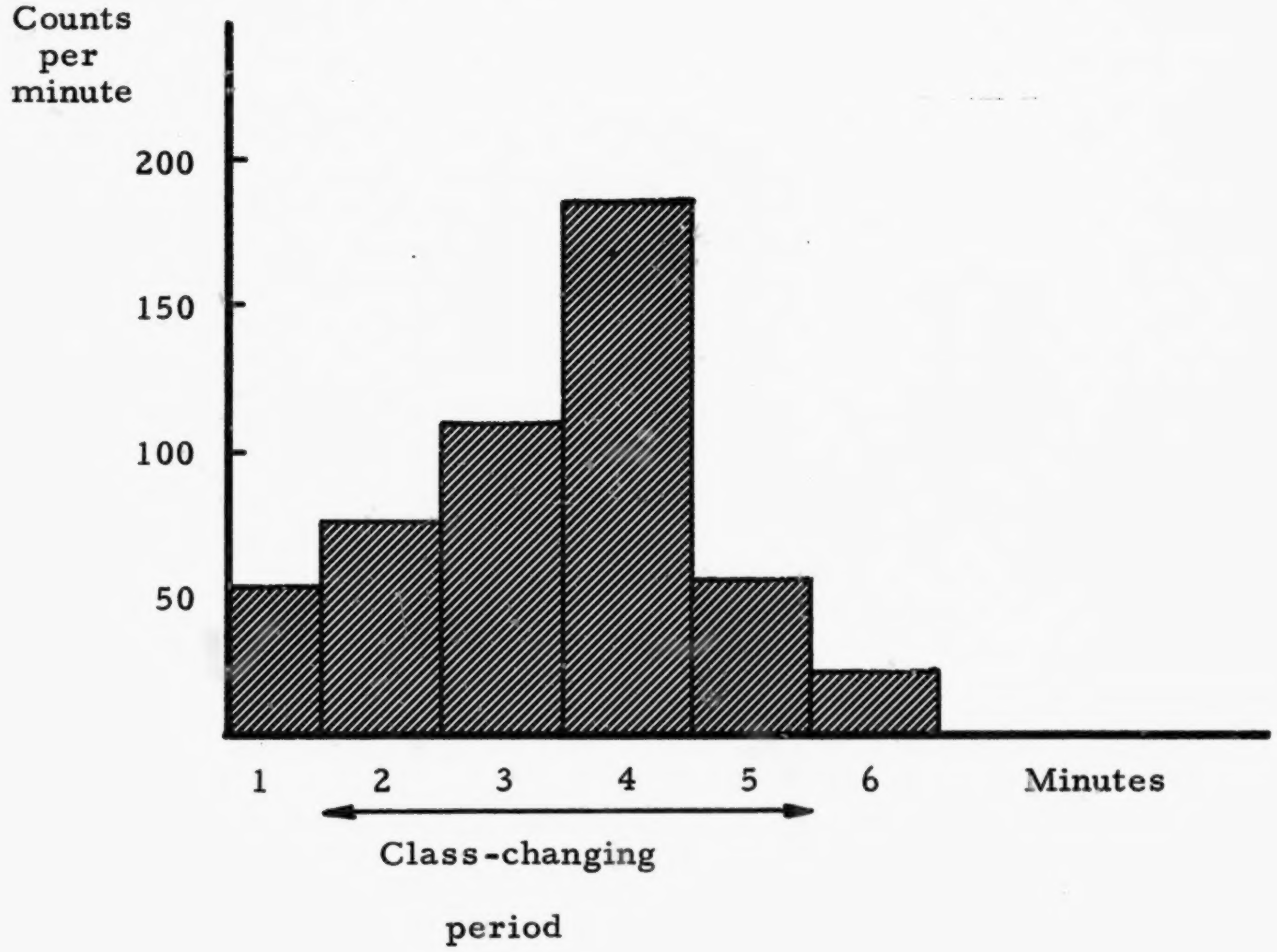

Fig. Bar graph of traffic at one intersection, end of first period

day (Fig. 7). Here the peaks shown for traffic at the ends of the first and fourth periods might suggest altering the room assignments in such a way as to lessen the traffic through this intersection at these times. For example, he might plan to have more of the students move from one classroom to another in the same corridor, without having to pass the intersection at all.

It is probably obvious that a full study by this method of a school's traffic pattern requires that data be obtained at every intersection for every class-changing period throughout the day or even the entire week. The school principal, confronted with such a staggering sheaf of graphs, would probably quietly file them away in the basement or else ask his School Board for an administrative assistant to "make a study in depth" of a problem which he perhaps never before knew existed.

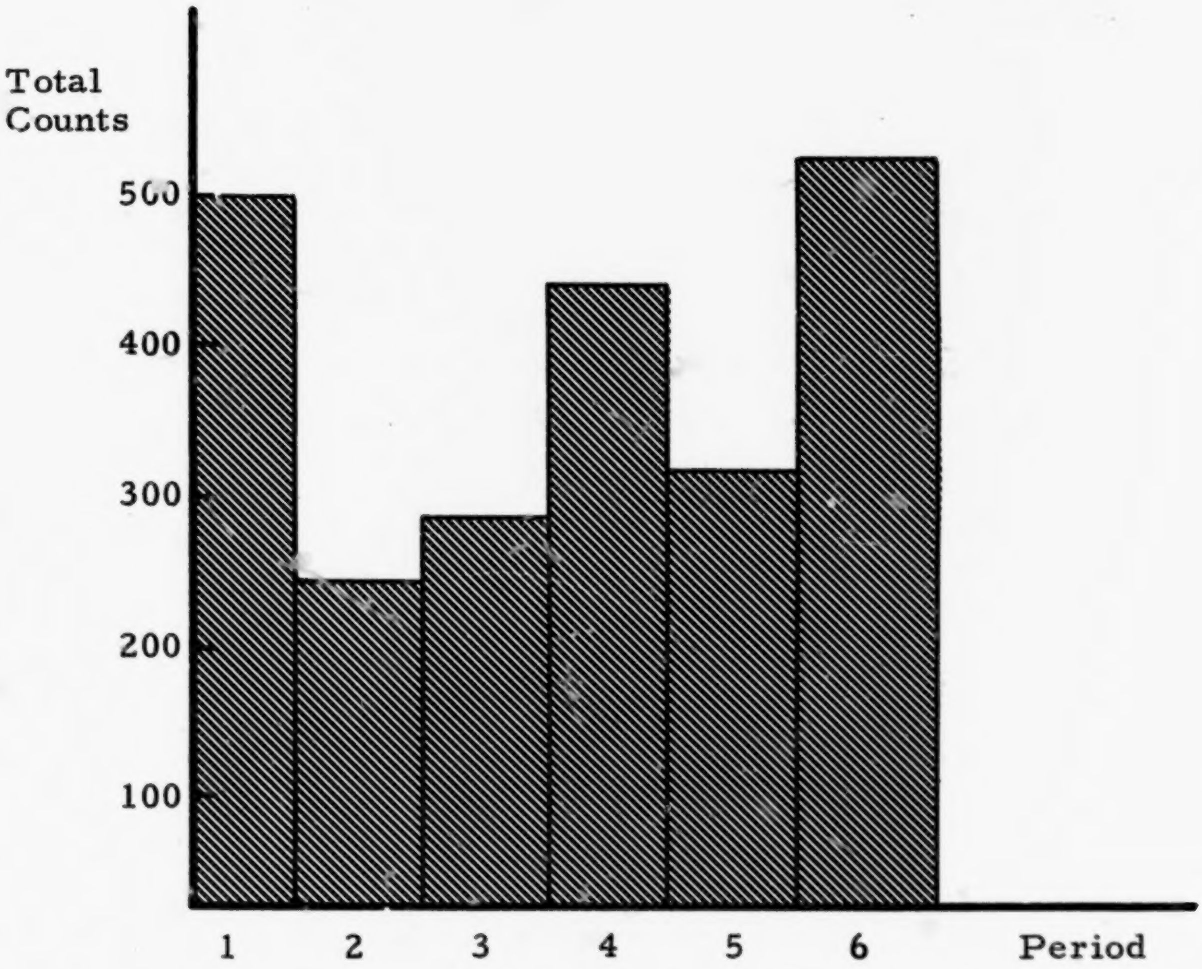

Fig. 7 Bar graph of total daily traffic at one intersection

4. A DESCRIPTIVE MODEL FOR AIRFLOW

Breathing is a complex process. Can we develop a simple model to represent the behaviour of the respiratory system? In the following example we consider the flow of the air through the trachea and bronchial tubes of an animal. From the throat to the lungs, air is carried first through a tube called the trachea (windpipe). The air then divides through two bronchial tubes and moves to the two lungs. The arrangement is shown in Fig. 8.

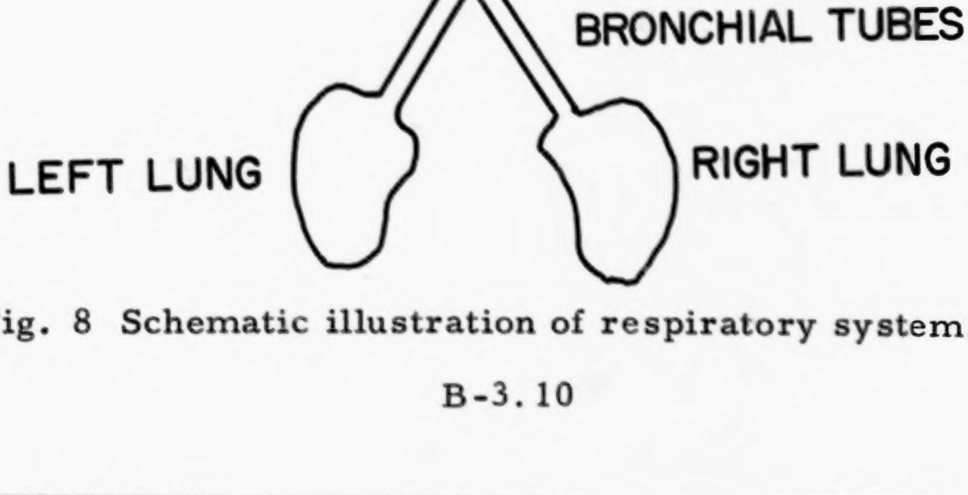
AIR SUPPLY
THROAT
TRACHEA
BRONCHIAL TUBES
LEFT LUNG
RIGHT LUNG

Fig. 8 Schematic illustration of respiratory system.

The model that we develop for this system will be useful for understanding the breathing activity of animals; for studying the effects of damaged or diseased lungs; for predicting the effects of clogged bronchial tubes; and for designing artificial devices to replace any of these elements.

In order to construct our model, we must determine the important factors which affect the behavior of the system. Clearly we are interested in the rate at which air flows in each part of the system.

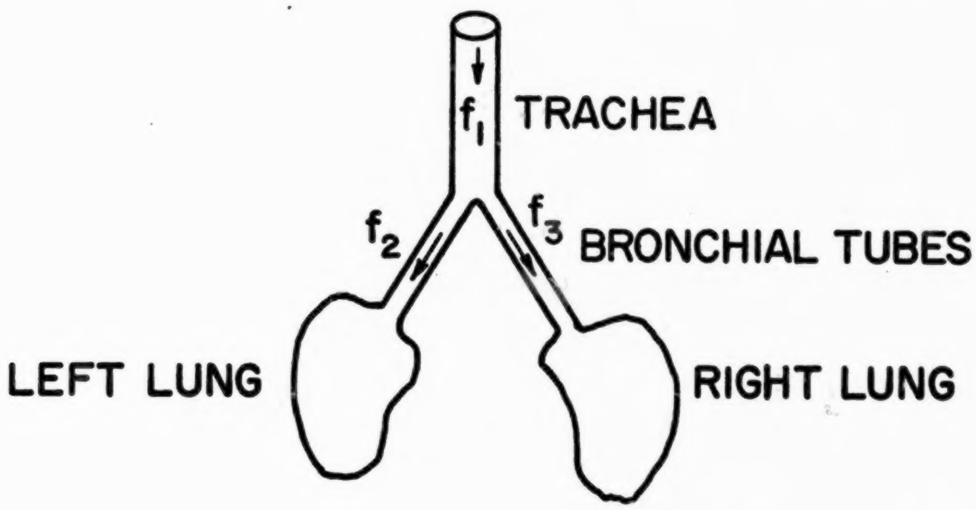

Fig 9 Definition of the flow-rate variables. (f_1, f_2, f_3, are flow rates in cubic inches per second or similar units.)

Inspection of Fig. 9 reveals that we may distinguish three different air flows, which we can label f_1, f_2, f_3. The figure shows the direction in which the f's are to be measured; for example if

$f_1 = 30$, f_2, $= 32$, and $f_3 = -2$ (all in cubic centimeters per second)

we mean that air enters the upper portion of the trachea at the rate of 30 cubic centimeters per second, air is entering the left lung at the rate of 32 cubic centimeters/second, and air is leaving the right lung at the rate of 2 cubic centimeters/second (the minus sign on f_3 means air is actually flowing opposite to the direction of the arrow in Fig. 9, since the arrow indicates our convention for positive direction).

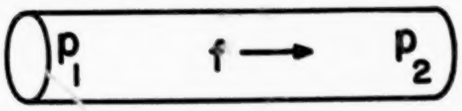

Fig. 10 Flow and pressure variables for flow through an inelastic tube (we make the simplifing assumption that the respiratory tubes are inelastic).

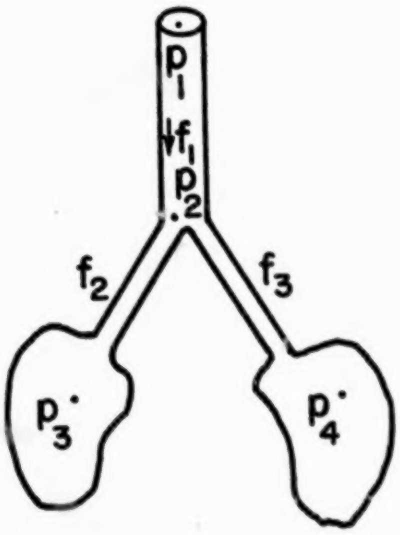

Fig. 11 Definition of all variables.

To push air through a tube, we must exert a pressure on the air. Air pressure is often measured in inches of mercury, because the height of the mercury column in a standard barometer is normally about 30 inches (at sea level). Other units are centimeters or millimeters of mercury (about 76 or 760), or millibars, often seen on weather maps. Normal sea-level pressure is about 1013 millibars. Another unit increasingly used is the torr*: 1 torr = 1 millimeter of mercury. In this example we use millimeters of mercury, because thus we have fewer decimal places to contend with. The quantity of air flowing through a tube (Fig. 10) depends on the pressure difference between p_1 and p_2. The phenomenon is similar to our home water supply system. When the faucet is open, water flows through the system from the pumps or water-tower storage to our outlet because the pressure at the pumps is high (typically 75 pounds/square inch or more) compared to the atmospheric pressure in the basin (about 15 pounds/square inch). The greater the pressure difference between pump and outlet the faster the water will flow.

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*Named for Evangelista Torricelli (1608-41), the first to construct a mercury barometer.

This relationship between rate of flow and pressure difference in a fluid makes it possible to describe our trachea-bronchial system quantitatively. We must express the numerical value of the pressure at various points in the system. In Fig. 11 these pressures are shown as p_1, p_2, p_3, p_4. In the same figure we can then observe that the flow f_1 through the trachea depends on the difference in pressure between p_1 and p_2. In a similar fashion f_2 and f_3 will depend on (p_2-p_3) and (p_2-p_4) respectively. Our mathematical model of air flow in the respiratory system should however indicate the exact relationship between flow and pressure differences in the trachea as well as in the bronchial tubes. The relationship may not necessarily be the same in these parts of the system but they must be expressed in the mathematical model.

Development of the Model

We are now faced with the problem of determining the numerical constants required to complete the mathematical model. Let us consider the trachea. We know that f_1 depends on $(p_1 - p_2)$, but we need to determine the relationship in terms of numerical values.

To measure, we anesthetize the animal under test and temporarily paralyze the system so that the flow of air can be controlled. We then operate on the animal and insert instruments at the points shown in Fig. 12. The air flow is measured at the mouth, and the pressure is measured at both ends of the trachea. Actually we ought to measure air flow in the trachea directly but in practice it is considerably simpler to make measurements in the mouth; we assume that the air flowing through the mouth is the same as that flowing through the trachea.

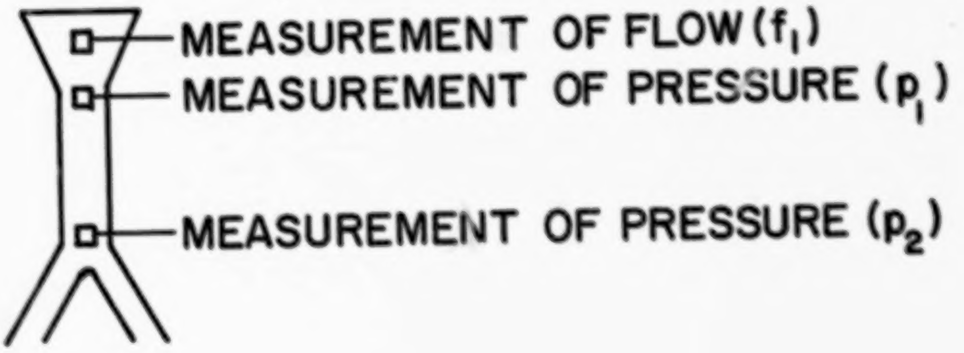
MEASUREMENT OF FLOW (f_1)
MEASUREMENT OF PRESSURE (p_1)
MEASUREMENT OF PRESSURE (p_2)

Fig. 12 Pressure and flow measuring devices inserted in trachea.

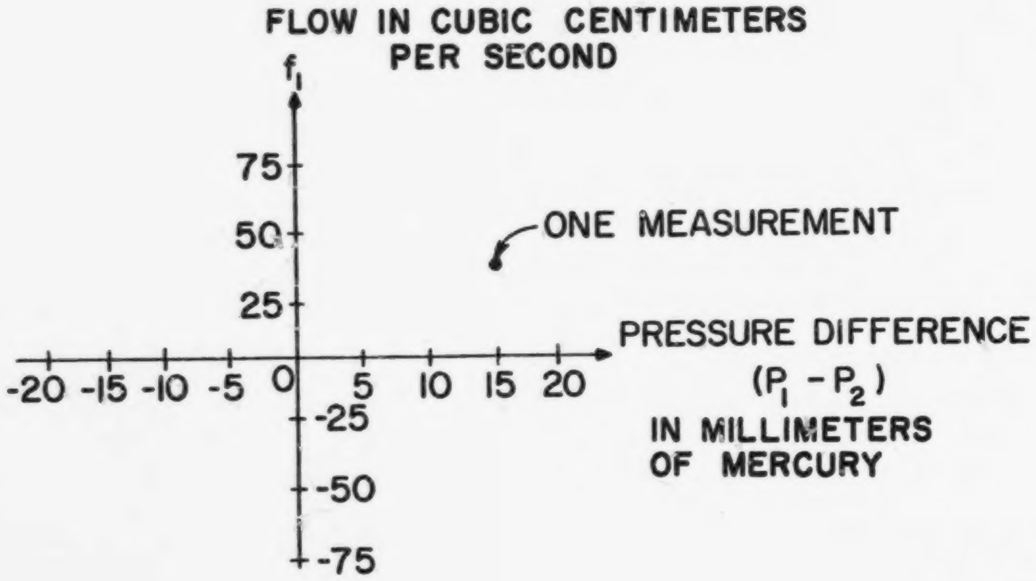

Fig. 13 One datum point of measured pressure difference for a particular flow rate.

Air is made to flow at a measured rate, let us say at a value of 37 cubic centimeters per second, and we measure the pressure at each end of the trachea at the same time. If the difference between these two pressures is 15 millimeters of mercury, we can represent this information as a single point on the graph. (In Fig. 13 we show the experimental quantities graphically, as was done in the height-weight example.)

If we make a number of such measurements of flow versus pressure difference, we observe that a straight-line approximation can be drawn which fits all of the points rather well. The result is shown in Fig. 14.

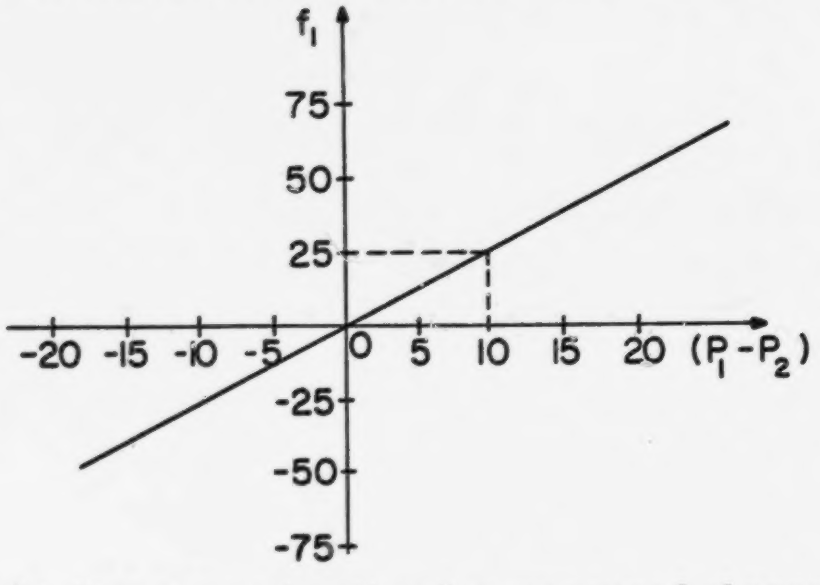

Fig. 14 Straight-line approximation for measured characteristics.

Having decided that a straight line fits the experimental points reasonably well we can once again find the algebraic equation which is approximately equivalent to the graphical model. This equation must be of the form

$$f_1 = a(p_1 - p_2) + b.$$

The line passes through zero on the vertical axis, so that $\underline{b} = 0$. To find $\underline{a}$, we pick a convenient point (Fig. 14), say where $(p_1 - p_2) = 10$ and find that $f_1 = 25$. From this we calculate that $\underline{a} = 2.5$, and so the mathematical model is

$$f_1 = 2.5\,(p_1 - p_2).$$

Similar measurements yield mathematical models for the bronchial tubes in the same way. The corresponding equations are:

$$f_2 = 1.7\,(p_2 - p_3)$$

$$f_3 = 1.4\,(p_2 - p_4)$$

If both bronchial tubes were alike, we should expect the numbers in the two equations to be equal. In this model, however, one bronchial tube is smaller than the other. The coefficients 1.4 and 1.7 indicate that the air flow is less in one tube than in the other for equal pressure differences. These are measurements for a single animal; in a valid study we should measure a number of healthy animals to determine whether there was a consistent difference between the left and right bronchial tubes. Disease or damage to the tubes would change these coefficients.

The complete model for the trachea-bronchi system thus consists of three measured relationships between air flow and air pressure. In addition, we recognize that the air flow in the trachea divides in two at the bronchial tubes. The complete mathematical model for describing the air flow as related to the pressures at various points in the respiratory system is:

$$\begin{aligned} f_1 &= 2.5\,(p_1 - p_2) \\ f_2 &= 1.7\,(p_2 - p_3) \\ f_3 &= 1.4\,(p_2 - p_4) \\ f_1 &= f_2 + f_3 \end{aligned}$$

Use of the Model

The model can be used directly to determine the flows when the external pressures (p_1 at the throat, p_3 and p_4 at the two lungs) are specified. For example, we measure p_1 as 760 millimeters of mercury. At the same time p_3 is 730 mm of mercury and p_4 is 750 mm of mercury (the pressure in the left lung is somewhat lower than in the right lung). The pressure conditions are then as shown in Fig. 15; we want to use the model to find all three flows and the unmeasured pressure p_2.

As matters now stand, the model consists of these four equations in four unknowns (f_1, f_2, f_3, p_2).

Let us first calculate p_2. When the known numbers are substituted for p_1, p_3, and p_4 we have:

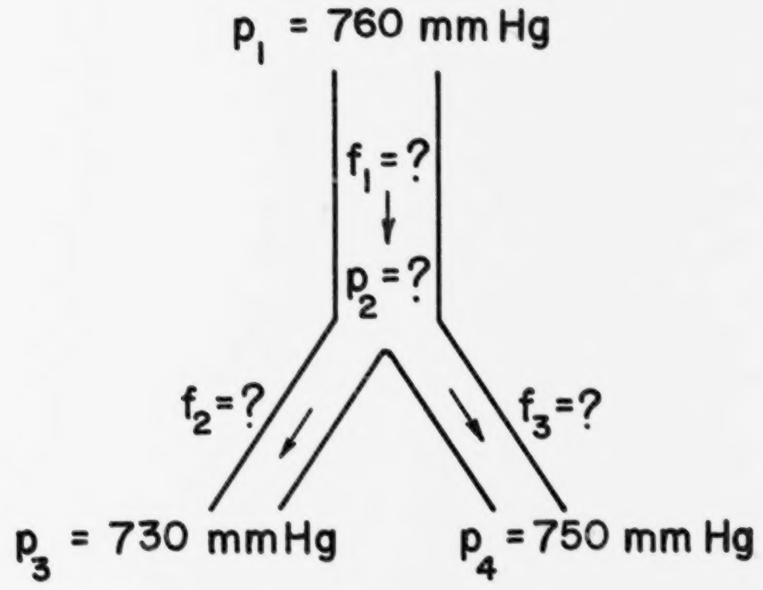

Fig. 15 Pressure conditions measured at throat and lungs. The model must be used to find the pressure at the bottom of the trachea as well as all three rates of air flow.

$$f_1 = 2.5\,(760 - p_2)$$
$$f_2 = 1.7\,(p_2 - 730)$$
$$f_3 = 1.4\,(p_2 - 750)$$
$$f_1 = f_2 + f_3$$

If we substitute each of the first three equations in the fourth, we obtain

$$2.5\,(760 - p_2) = 1.7\,(p_2 - 730) + 1.4\,(p_2 - 750)$$

or

$$1900 - 2.5\,p_2 = 1.7\,p_2 - 1241 + 1.4\,p_2 - 1050$$
$$4193 = 5.6\,p_2$$
$$p_2 = 748.4 \text{ mm of mercury}$$

Now that p_2 is known, we can find the three rates of flow

f_1 = 29.0 cu.cm./sec. f_2 = 31.3 cu.cm./sec. f_3= -2.2 cu.cm./sec.

The model has permitted us to make relatively simple and straight-forward determinations of flows for the given pressure conditions, revealing some important quantities in the system which were not directly measured.

The model is, of course, only an approximate representation of the real system. After deriving the model, the scientist or engineer would certainly compare its predicted performance with that of the real system. (If major discrepancies were to appear, then either the model must be improved or a more complex model must be developed.) Furthermore, he must evaluate how well the model represents the system in different animals, since the measurements taken only on one sample of a population may not be truly representative of respiratory systems of this structure.

The important point to be observed in this example is that a few basic measurements on a system which has several interacting relationships can often give us the means by which we can devise a mathematical model. Once a model is developed, we can predict the performance of the real system under a variety of external conditions.

5. DYNAMIC MODELS

The models of the last three sections are simple statements of algebraic or graphical relationships. They represent static systems in which the relationships between the factors do not change during the time interval involved in our observation of the behavior of the systems. Very often however we wish to use the idea of modeling for a system in which the relationships among the various factors do change during our period of observation. In this section, we consider examples of such models.

Engineers and scientists use the term dynamic to describe any system in which change or motion is important. The growth of a person or of a population of people, the cooling of a cup of coffee, and the functioning of a nuclear reactor are all examples of dynamic systems. Most of the interesting systems are dynamic; we live in a world characterized by increasingly rapid change.

6. A POPULATION MODEL

It has been estimated that since the appearance of man on the earth, a total of 15 billion human beings have existed. With a world population of nearly 3 billion today, 20% of all the people who have ever lived are alive today. Our population is obviously growing at an explosive rate.

Demography, or the study of population, is of increasing concern to economists, sociologists, political scientists, engineers, and many others who must understand the present and plan for the future. Models of population change are exceedingly important to such study, for they make possible analysis and prediction which can lead to more effective planning for the many goods and services that people need.

Suppose that we wish to obtain a simple model which would let us estimate the world population at some future date. The present average rate of population increase for the entire world is estimated to be close to 2% per year, and we assume in this section that this rate of increase does not change. In 1960 the world population was approximately 3 billion (that is, 3 followed by 9 zeros, or 3×10^9).

If the rate of increase is 2% per year, then in 1961 the increase is 0.02(3,000,000,000), or 60,000,000 more people, making a total of 3,060,000,000. We can then calculate the increase for the next year and for all succeeding years. A table of the results is shown in Table 2.

We see clearly that since a constant percentage of the year's starting population is taken and that the population is larger each year, the numerical increase becomes continually larger. In fact, if we were to compute values for

YEAR	POPULATION AT START OF YEAR	INCREASE	POPULATION AT END OF YEAR
1960	3,000,000,000	60,000,000	3,060,000,000
1961	3,060,000,000	61,200,000	3,121,200,000
1962	3,121,200,000	62,424,000	3,183,624,000
1963	3,183,624,000	63,672,480	3,247,296,480
1964	3,247,296,480	64,945,930	3,312,242,410
1965	3,312,242,410	66,244,848	3,378,487,258
1966	3,378,487,258	67,569,745	3,446,057,003
1967	3,446,057,003	68,921,140	3,514,978,143
1968	3,514,978,143	70,299,562	3,585,277,705
1969	3,585,277,705	71,705,554	3,656,983,259

Table 2 Estimated world population, 1960 to 1969.*

the table far enough, we should find that by the year 1995 (less than three decades away), the predicted population would be six billion; that is, the population would double in about 35 years. Carrying the calculation further, we should notice that this doubling occurs every 35 years. This would be true for any population number we start with so long as the rate of increase is exactly 2% per year. If the rate of increase were 3%, the doubling would occur in 23.5 years. A 3% rate of increase would cause the population to increase by a factor of 2.81 in thirty-five years instead of by a factor of 2. We have merely restated the same fact in a different form.

There is a simpler way of expressing these results using a special symbol $\sum$, (called "sigma," a letter of the Greek alphabet). This symbol stands for summation, (i.e., addition); since we have been adding or accumulating the population increase each year, the last column in our table can be called a column of summations. A shorthand way of indicating the procedure for computing the population for the third year is to write:

$$\left\{\begin{array}{l}\text{Total population}\\ \text{at end of 1962}\end{array}\right\} = \left\{\begin{array}{c}\text{Population at}\\ \text{beginning of}\\ \text{1960}\end{array}\right\} \left\{\begin{array}{l}+\text{ increase in 1960}\\ +\text{ increase in 1961}\\ +\text{ increase in 1962}\end{array}\right\}$$

$$\left\{\begin{array}{l}\text{Total population}\\ \text{at end of 1962}\end{array}\right\} = \left\{\begin{array}{c}\text{Population at}\\ \text{beginning of}\\ \text{1960}\end{array}\right\} \left\{+\begin{array}{l}\text{The sum of the increase}\\ \text{in 1960, 1961, and}\\ \text{1962}\end{array}\right\}$$

Using the summation sign $\sum$ we can write even more compactly: Total population = initial population + $\sum$ (increases in 1960, 1961, and 1962).

If we let P be the symbol for the total population; P_o be the initial value of the population at the beginning of 1960; and p_i be the population increase for the year represented by the subscript i, we can write the entire expression as:

$$P = P_o + \sum_{1960}^{1962} p_i$$

In other words, the total population at the end of 1962 is the initial value plus the sum of the increases for each year: 1960, 1961, 1962.

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*Compare with these estimates: in July, 1967, the Population Reference Bureau used UN and other statistics to estimate that in the summer of 1966 world population was 3.34 billion, an increase in one year of 65 million.

From our table we can substitute the appropriate values into the above equations:

$$P = 3,000,000,000 + (60,000,000 + 61,200,000 + 62,424,000)$$
$$= 3,183,624,000.$$

Are these numbers that out equations produce really accurate? Consider for instance the entry in Table 2 for the population at the beginning of 1966. The model predicts exactly 3, 378, 487, 258 people which is a precise value. We have, however, ignored tha fact that the numerical values with which we started were only approximations; the 3 billion initial population was a rounded number, and the 2% was an estimated average growth rate. If P_o was exactly 3, 000, 000, 000 and the rate of increase was exactly 2. 000, 000, 000,% then we should obtain 10 meaningful digits in our answer. But since precision was lacking in our measurement of both the starting population and the rate of increase, the results can have only a limited number of significant figures. We must, therefore, be content to use rounded numbers. The extent of precision when two numbers are multiplied is restricted by the number with the smaller precision. If in this case it is the 2% figure, and we assume that we are certain of its value to three significant figures, (i.e., 2.00%), then the rounded number having acceptable accuracy is not 3.378487258 billion but only 3.38 billion.

If we continue our example with a 2% rate of increase, we find that in the year 2060, just one hundred years from our starting date, the calculated population will be nearly 22 billion, and by the year 2160, it will reach the enormous sum of 157 billion! With a doubling of population in 35 years, the growth after two centuries results in a population which is more than 50 times the original population.

We have already observed that a graphical plot of data can quickly and easily reveal relationships that may be difficult to observe in a table of numbers. We now construct such a plot and introduce some variety with a new table; one which displays the average population predicted for the beginning of each decade from 1960 to 2060 in rounded numbers as in Table 3.

As a matter of convenience, we have used the population for the first year of the decade, although the actual number continually grows. In the United States, where a census is made every tenth year, the count obtained is often considered to be the legal population until the next census is completed, even though the Census Bureau issues an annual estimate of the current number of our people. These

DECADE	POPULATION (IN BILLIONS)
1960-1970	3.00
1970-1980	3.65
1980-1990	4.46
1990-2000	5.44
2000-2010	6.64
2010-2020	8.04
2020-2030	9.86
2030-2040	12.0
2040-2050	14.7
2050-2060	17.9

Table 3 Estimated world population, average for each decade, 1960-2060.

values are plotted as a bar graph in Fig. 16. The height of each vertical column is proportional to the population total for that decade as given by the table. It is interesting to notice that not only do the heights of the bars go up in each ten year period, but that the steps become increasingly larger.

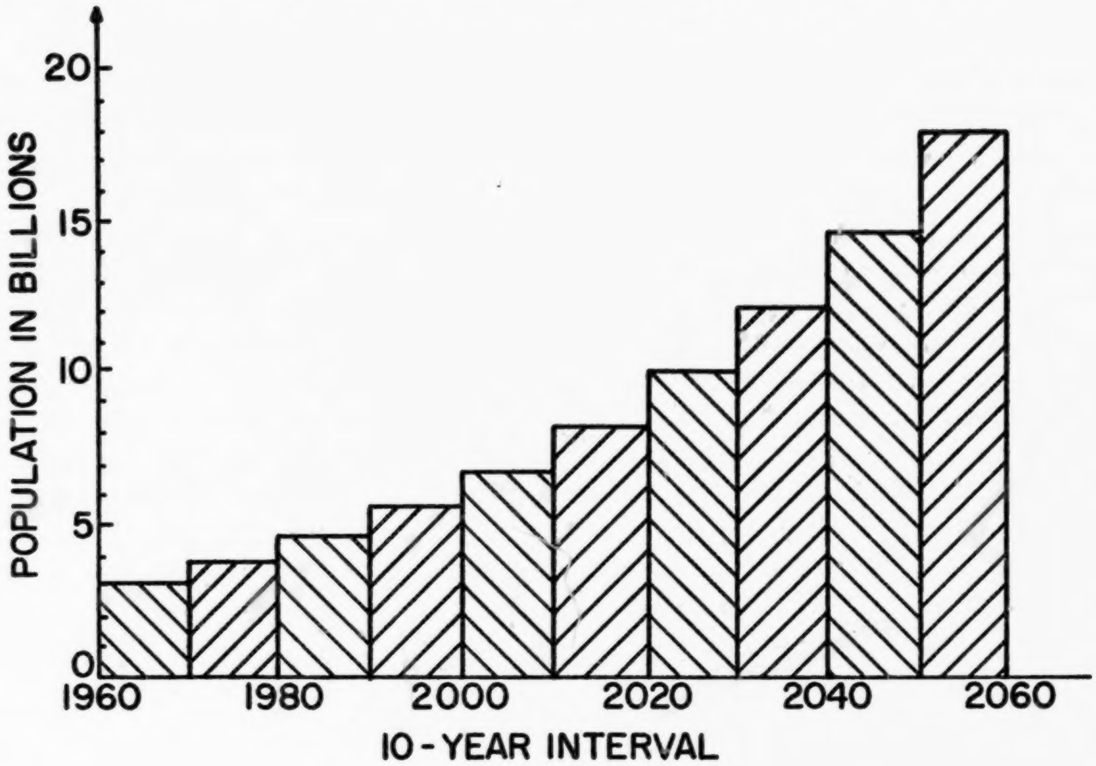

Fig. 16 Estimated growth of world population

But there is a more significant aspect to this graph. It looks as though a reasonably smooth average line could be drawn to connect the tops of the bars, just as in the earlier height-weight model it was possible to draw in a straight-line average which provided a fairly good fit to all of the datum points. In Fig. 17 the top left corners of the bars in Fig. 16 (i.e., the population estimates for the first year of each decade) are plotted, and then joined by a smooth curve. This gives, of course, only an approximate model of the real situation, because the growth fluctuates irregularly from year to year*.

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*It might be argued that it is, strictly speaking, improper to draw such a curve at all. In mathematics, a curve on a graph represents a continuous function, meaning that between any two points on the curve, no matter how close together they may look, there is actually an infinite number of other points. But a population is a step-function: it can change only by whole numbers, and from this point of view should properly be shown only by a series of bar graphs side by side, one for every day or every hour or every minute or.....If such a graph could really be constructed (and obviously it could not, because information is lacking), the upper ends of the bars would be found to end in a sort of jiggly manner, because the change in population, even from one millisecond to the next, is not quite regular. But when inspected from a distance of a few yards, the irregularities would tend to fade from view. We may justify drawing a smooth curve for what is in truth a step-function, then, by maintaining that we have shown how the latter would look from a little way off. From a slightly different point of view, we may look at the curve as a kind of best guess, justifiable as a way to get ahead with the discussion in such a case as this one where the uncertainties of measurement, the gaps in our knowledge, are without doubt at least as great as the fluctuations which the true value of world population undergoes.

Now the extremely fast growth of population is quite clear. It is important to notice that even though the percentage increase remains constant at 2% per year, the continually larger increases each year produce a curve which becomes steeper and steeper. This curve is different from those we found in our previous models. The previous plots were linear. The population curve of Fig. 17, however, is not a straight line; it is non-linear.

Furthermore, this is a particular kind of non-linear curve. It was produced by a set of numbers in which each new value of the variable is obtained by adding a constant percentage of the previous value to that particular value. We have a growth that is proportional to the accumulated size; "the bigger it gets the faster it grows". This snowballing relationship is call exponential, and the curve of Fig. 17 is therefore known as an exponential curve. This is a very important non-linear curve; it represents a model which is encountered very frequently in nature and in engineering; we find other examples of it later in this chapter.

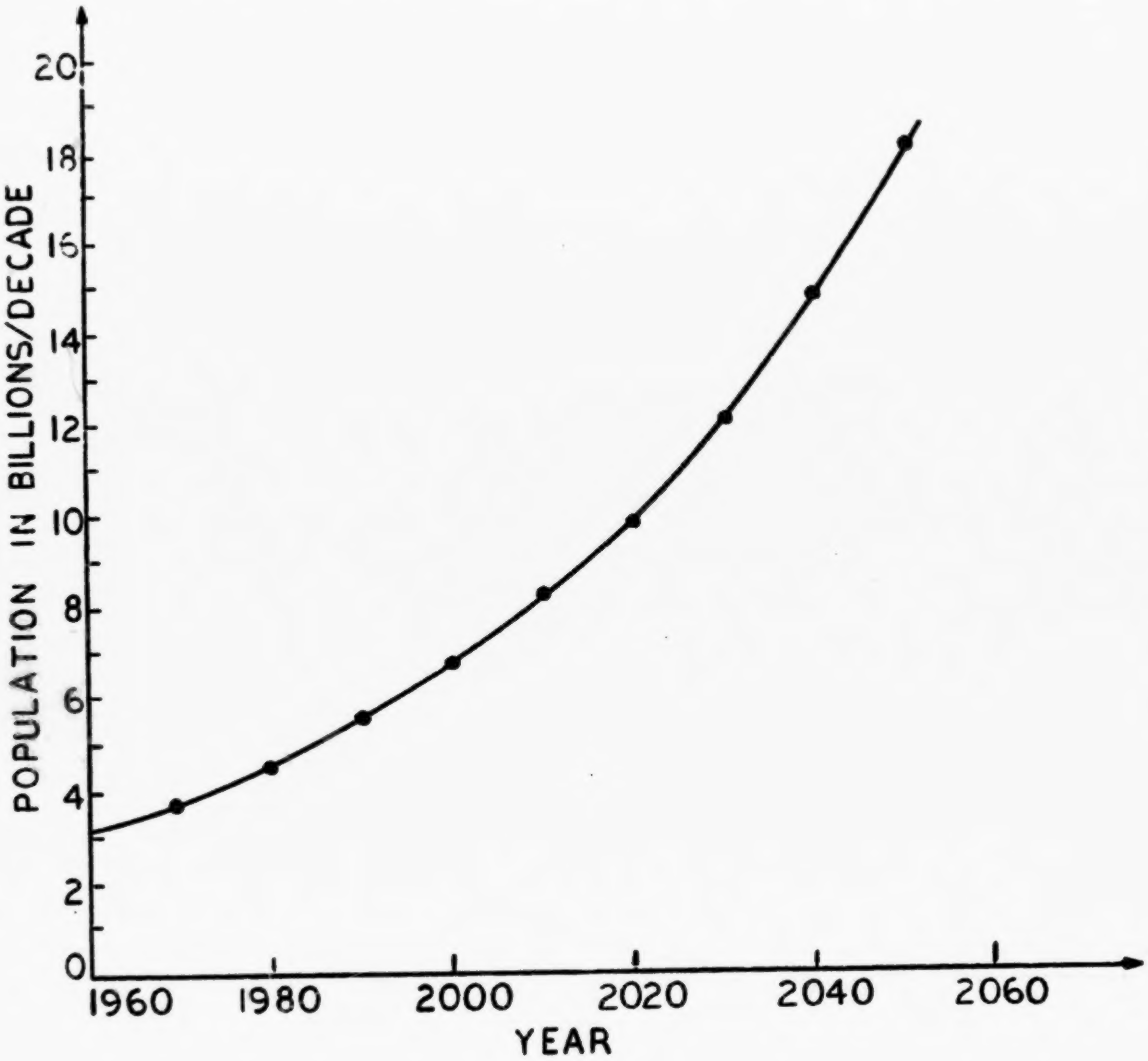

Fig. 17 Fitting a smooth average line to the population growth graph.

We now consider another plot of population increase, from 1700 to 2165. This is shown in Fig. 18. The eye tends to follow the curve upward to the right; but it is also important to be aware of the fact that the graph drops as we look along it to the left, or as we go backwards in time. In fact, prior to 1800, the height of curve on the scale of this graph is so small that it is difficult to measure its value. This reflects the fact that a "population explosion" has occurred: the population of the earth in the past was extremely small relative to the present population. The curve makes more reasonable the earlier statement that approximately 20% of all the people who have ever lived are alive today. From a larger scale copy of the curve we could find the even more striking fact that the population increase from 1940 to 1963 (just 23 years) is greater than the total estimated population of the world in 1800!

Suppose now that we continue to calculate the population to the year 2700, a period only slightly more than 700 years from now. This represents about the same time difference as that between the present and the time of Marco Polo. The graphical results of the computations are shown in Fig. 19.

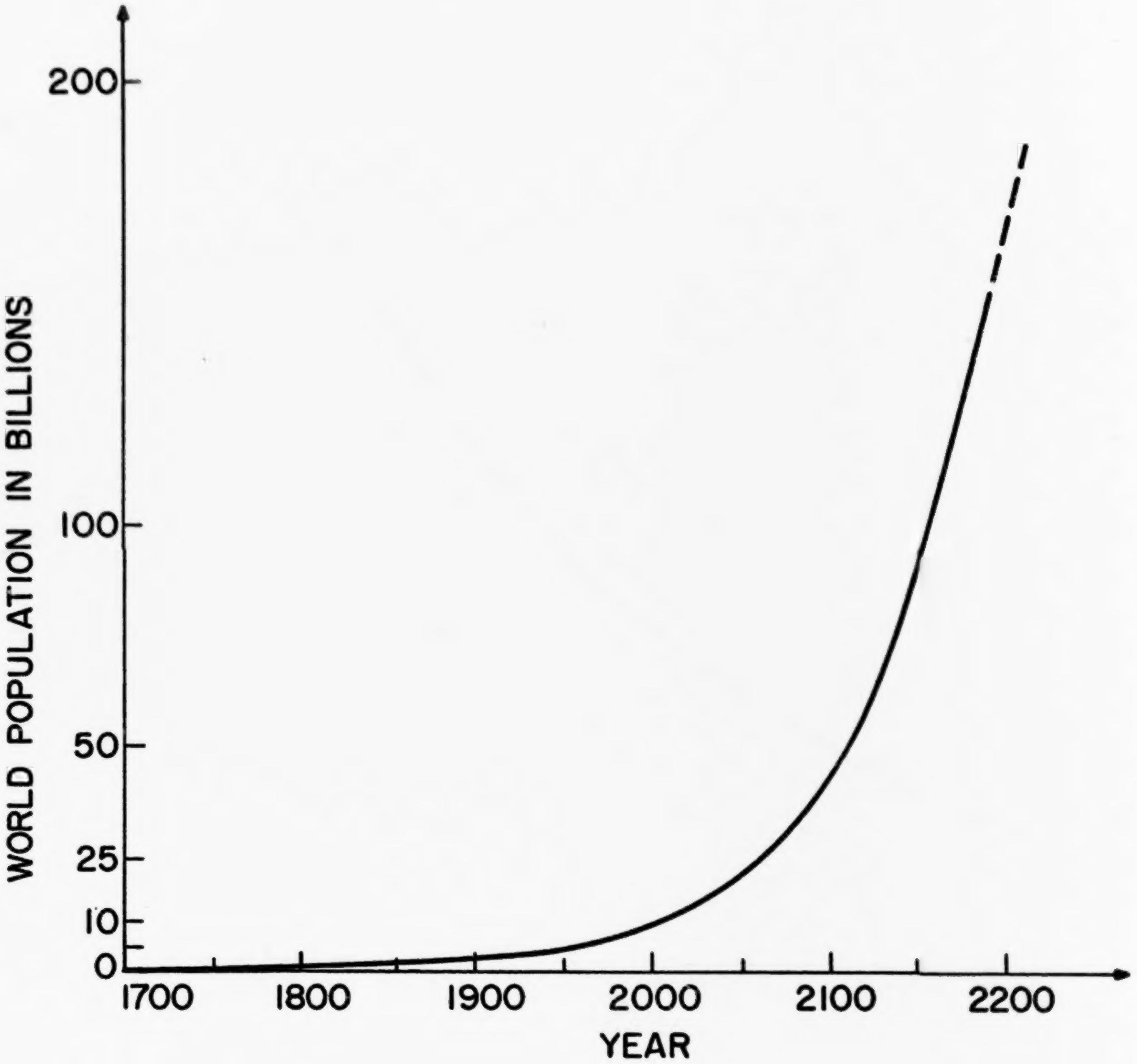

Fig. 18 The growth of world population from 1700 to 2165. Prior to the present time the curve is a reasonable representation of historical fact. Later values are predicted from a model.

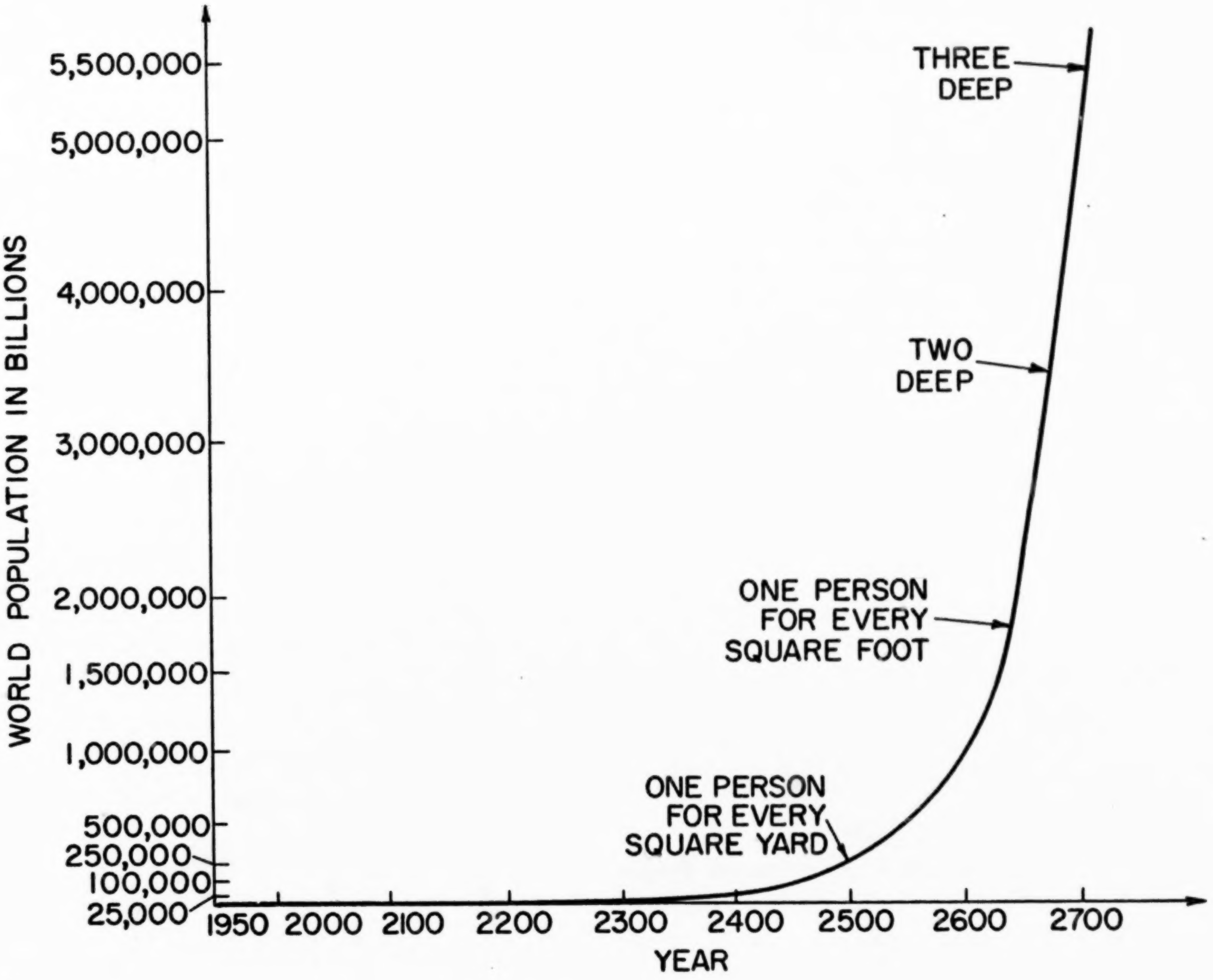

Fig. 19 Modeling prediction of world population to the year 2700.

Can this really be expected? The curve shoots up at a fantastic rate. Notice that the vertical scale on the left is much larger than that in the preceding figure--so much so that the steeply rising curve to the year 2165 (Fig. 18) is now compressed to a degree that permits no measurable value. Our new exponential curve has reached such proportions by the year 2700 that if we tried to plot it on the scale of Fig. 18 it would require a sheet of paper twenty-seven thousand times as high, or 11 thousand feet (more than 2 miles) high instead of 5 inches.

What does this curve of Fig. 19 tell us? By the year 2510 we should expect to have a world population of nearly 200, 000 billion people, by 2635 about 1, 800, 000 billion people; thirty-five years after that it will have doubled to approximately 3, 600, 000 billion; and in the year 2692 the model predicts a 5, 450, 000 billion population.

How large is 5,450,000 billion? We can express it in many ways. The number when written completely would appear as:

$$5,450,000,000,000,000$$

It may be written as 5.45×10^{15}, but one doesn't get a good "feeling" for the enormous size.

Here is one picture that helps to visualize the magnitude of the number. There are roughly 31 million seconds in each year. If we counted one thousand persons per second it would take us

$$\frac{5.45 \times 10^{15} \text{ (people to be counted)}}{10^3 \times 31 \times 10^6 \text{ (counts per year)}} = 1.7 \times 10^5 \text{ years.}$$

That is, it would require about 176,000 years to complete the census!

There is yet another way to grasp the significance of this estimate of 5,450,000 billion population. Let us ask where these people will be; how much room will they have? The surface of the earth contains approximately 1,860,000 billion square feet. About 80% of this area is covered by water, but let us suppose that all of the surface were land. We can calculate that in the year 2510 when the population is 200,000 billion there will be

$$\frac{1,860,000 \text{ (billion square feet)}}{200,000 \text{ (billion persons)}} = 9.3 \text{ square feet per person}$$

or about one person per square yard all over the earth. Worse yet, in 2635 each person will only have one square foot in which to stand, and in 2670 if they insist on retaining that much real estate they will be standing on each others' shoulders two deep. And only 22 years later they will be three deep. Now, if we do not assume that these people can tread water but instead must occupy the land area only (1/5 of the total area,) then in 2692 we should expect to see totem poles 15 persons high on every square foot!

7. AN IMPROVED POPULATION MODEL

Obviously the model we developed in the last section is incorrect. Beyond a certain point at any rate it leads to impossible conclusions. Quite clearly there must be some limiting factors which will prevent a population increase to a value that is ridiculous. Actually the model is too simple, because we did not take into account several important factors that tend to limit our predictions.

To learn more about these factors, it is helpful to examine functional models of the world population. Such models are easy to find in a biology laboratory. Any small organism that reproduces rapidly will do. Fruit flies, yeast, bacteria are commonly used examples. Here we describe a population model using yeast. First the experimenter must prepare a food supply, a "nutrient medium". For many yeast species this may be simply a weak sugar syrup slightly modified by addition of other substances. Then there is need, obviously, for a jar in which to keep the yeast as they gorge themselves, and for Adam and Eve, so to speak; the syrup must be inoculated with a few yeast cells to start with. It is perhaps not so obvious that the temperature should be kept constant, nor that the medium should be gently but constantly stirred.

It is hardly possible to take a census of yeast cells as one does of people. Instead, a sampling technique is used. Knowing the starting volume of his experiment, the investigator can withdraw a definite, very small, percentage of it and count the yeast cells in that. Since the solution of food has been stirred, he can safely assume that his sample is typical, and that he can simply multiply by the proper factor to learn the total population. Such models as this one are particularly convenient because they take up very little space; moreover, it is easy to try different circumstances ("to vary the parameters", as the professional puts it). It becomes possible to answer such questions as these: What is the rate of increase of population when the experiment begins? Does this rate remain constant as the population becomes larger? Does it matter whether the available space for the organisms remains constant or is made to increase as the population grows? Yeast cells produce an alcohol (there are many kinds of alcohol) from the sugar they consume; what is the effect of leaving the alcohol to accumulate in the nutrient medium? of removing all but a constant fraction? of removing all of it as it forms? (Removal can be rather easily accomplished by continually pumping fresh nutrient medium in and at the same time allowing the used medium to trickle out through a filter).

Which of these possible experiments cast light on our graphical model of world population? First, we know that the entire land surface of the earth is not inhabited but that it is not unlimited (there is room for population to increase but the space will be used up some day). This is modeled in the yeast case by using bigger jars (and more medium) up to a certain point but then no more. Second, we know that food production can be increased for human beings but not without limit. We can supply more sugar to the yeast on a schedule that we think is comparable to the future history of the world; even better, we can try many schedules. In short, we can test our model and thus refine it, by comparing it with what we already know about the course of development of the human population.

Now it turns out that such experiments as those described are practically always alike in one feature. Growth is roughly exponential at first; the rate of increase is not necessarily constant, but the population curve is closely similar to that of Fig. 18. However, if the experiment lasts long enough the rate of growth sooner or later begins to lessen and in time reaches zero. The curve stops its exponential growth, trends to the right more and more, and tends to level out, as suggested in Fig. 20. Because this curve has a kind of S-shape it is known as a sigmoid (from the Greek letter S, which is called sigma).

The basic reason for the change of shape shown is overcrowding. Without unlimited space in which to grow, and unlimited food to support life, the individual yeast cell has neither room nor food to allow it to reach normal size. No doubt there are other reasons, but they are less important.

It is interesting to know that the sigmoid shape is found in many other kinds of growth cases. For one example, if a coil of wire is wound around an iron bar and a slowly increasing electric current is sent through the coil, a graph of the magnetism induced in the bar plotted against the current shows the same general

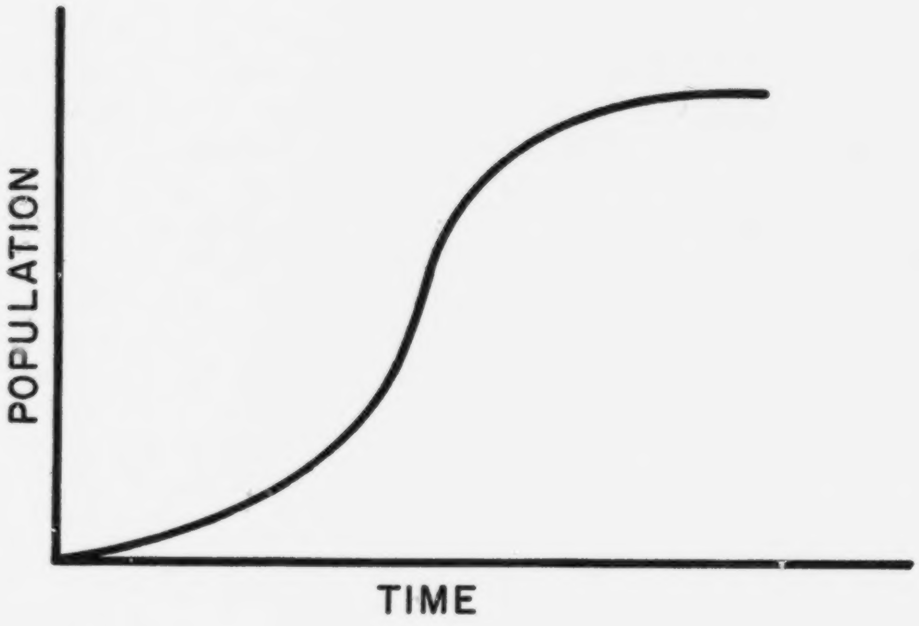

Fig. 20 Population growth in laboratory studies.

behavior: the bar's magnetism grows for a time and then the curve flattens out; the bar is said to become saturated, after which its magnetism can become no stronger (though the total magnetism does, because the current in the coil has its own magnetism independent of that in the bar).

Perhaps the first man to recognize the sigmoid character of population growth (though he did not express it in this way) was Thomas Robert Malthus. He was an Englishman who lived from 1766 to 1834, and who wrote a gloomy essay pointing out that a time must come when population will outrun food. Then the growth of population would be stopped, he believed, by wide-spread epidemics, or starvation, or war, or some combination of these. Instead of a truly sigmoid curve, however, his curve would probably actually turn downward. He predicted that the end of the growth period would come during the nineteenth century. That it did not was owing to the discovery of chemical fertilizers; with these an acre of ground will bring forth several times as much food as was possible in Malthus' day. We can see, however, that some limit, at some time, must be reached in the number of pounds of food that can be won from an acre of ground; that the supplies of potash and phosphate easily recovered must someday disappear, so the cost of fertilizers must rise (the third major fertilizer element, nitrogen, is available without foreseeable limit from the air); and hence that Malthus may yet prove to be right. The end may be less gloomy, however, if the practice of family limitation, birth control, is ever adopted in the parts of the world where the growth rate is even now excessive - up to 4% a year.

It is not very difficult to take the algebraic expression for population growth that was written in the last section and modify it to a more general form; this general equation can then be modified again into another one which, when plotted, turns our population-growth curve into a sigmoid. However, it may be argued that to show all this here would be to plant so many trees, so to speak, that the forest would be hidden. Since we care chiefly about the forest, we leave the trees in the nursery.

8. USES OF POPULATION MODELS

Predictive population models are often used with great success in governmental planning at all levels -- town, state, and federal. For example, the design of a suitable transportation system for a region of a country requires that the engineer have available reasonably reliable predictions of population distribution in order to assess future transportation needs (for transporting people and the materials which people require).

In such a problem, the complete system model includes population models for hundreds or thousands of separate towns. The complete model is often a mathematical model composed of many equations. Some of these are similar to the equations we have used and some are more complicated. Not only must birth and death rates (net growth) be considered, but the relationships among other factors must be included. There are also influences which make the populations of towns interdependent. If one town becomes unduly crowded, there is a strong tendency for neighboring towns to grow more rapidly. Immigration and emigration rates thus are important considerations for the development of an accurate dynamic model, i.e., a model in which the interrelationship of factors changes with time.

Then too, in a population problem of this sort, the engineer has a multiple responsibility: he must formulate the model, decide which factors are to be included, guide the collection of essential data, interpret the results of the modeling studies in terms of recommended developments, and finally evolve a plan which takes into account the technical solution suggested by the model studies and at the same time the economic, political, and social constraints -- all of which may place limitations on the adoption of the complete technical solution.

9. MODEL APPLICABILITY

Many Models for One System

It is possible for one system to be represented by a number of different models. As in the case of the blind men and the elephant, no one model describes the real thing completely, but separate models of sub-systems are often necessary and useful.

Consider an air-conditioner. One model can be developed which is based on heat flow; how heat is extracted from a room; how the fluid in the unit changes its temperature as it absorbs heat; and how this heat is then transferred outside of the room. This model must include such factors as expected temperature ranges, the characteristics of the refrigeration unit, blower, and intake and outlet duct air flows.

Another model to describe the same system might be a control model which includes the thermostat, the various relays and contacts, and the electrical network which links the electrical parts of the system.

Yet another model of an air conditioner could be developed for a study of its mechanical behavior. For example, we may wish to know how much noise and vibration the equipment will produce and how to design the air conditioner to minimize the noise and vibration. For this objective the model would include a number of factors such as the characteristics of the moving parts, their mountings, the location of shock absorbers, and the geometrical arrangements of the openings, the absorbent surfaces, and the baffles.

One Model for Many Systems

What is there in common between the way in which a cup of coffee cools, the way in which the numbers of chain letters increase, and the way in which a human head grows? Just as it is possible for one system to be described by several different models, so one model frequently is applicable to many kinds of systems. In Fig. 21 there are models of three different processes which one would not ordinarily think of as being similar. In (a) is shown how the temperature of a cup of coffee drops as the coffee cools to room temperature. The initial temperature is just below the boiling point. It drops rapidly at the start, then more and more slowly. After a half hour, the temperature of the coffee has dropped to within a few degrees of room temperature.

The illustration in (b) describes a system in which the production of an item doubles at each step. During the early part of the curve the values are not readily observed because of the scale of the graph axis, but as the number of steps goes from 1 to 2 to 3 to 4, the number of items increases from 2 to 4 to 8 to 16. The increases become larger with each step; for instance in going from 9 to 10 steps the number of items doubles from 512 to 1024. This is a model for the chain-letter process, where an individual writes to two people, each of these writes to two others, and so on. After 20 steps in such a process the number of letters (items) being written is more than one million, and after 30 steps the number becomes greater than one billion.

Fig. 21(c) shows how the size of a human head grows from birth to age 20. At birth it is a little less than 1/4 its full size, and it is growing very rapidly. At the age of 5 the growth begins to slow down appreciably, and at the age of 15 the head is within a few percent of its ultimate size.

We can see what is common to the cooling of a cup of coffee, the rate of increase in a chain letter situation, and the growth of the human head. Each displays an exponential rate of change. In each of the processes, as in the population expansion, growth either increases or decreases exponentially. The change at any step is determined by the value just prior to that step.

Our exponential model thus fits many systems. Such things as the rate at which an automobile coasts to a stop, the growth of plants, and the accumulation of bank interest, can also be represented by an exponential model.

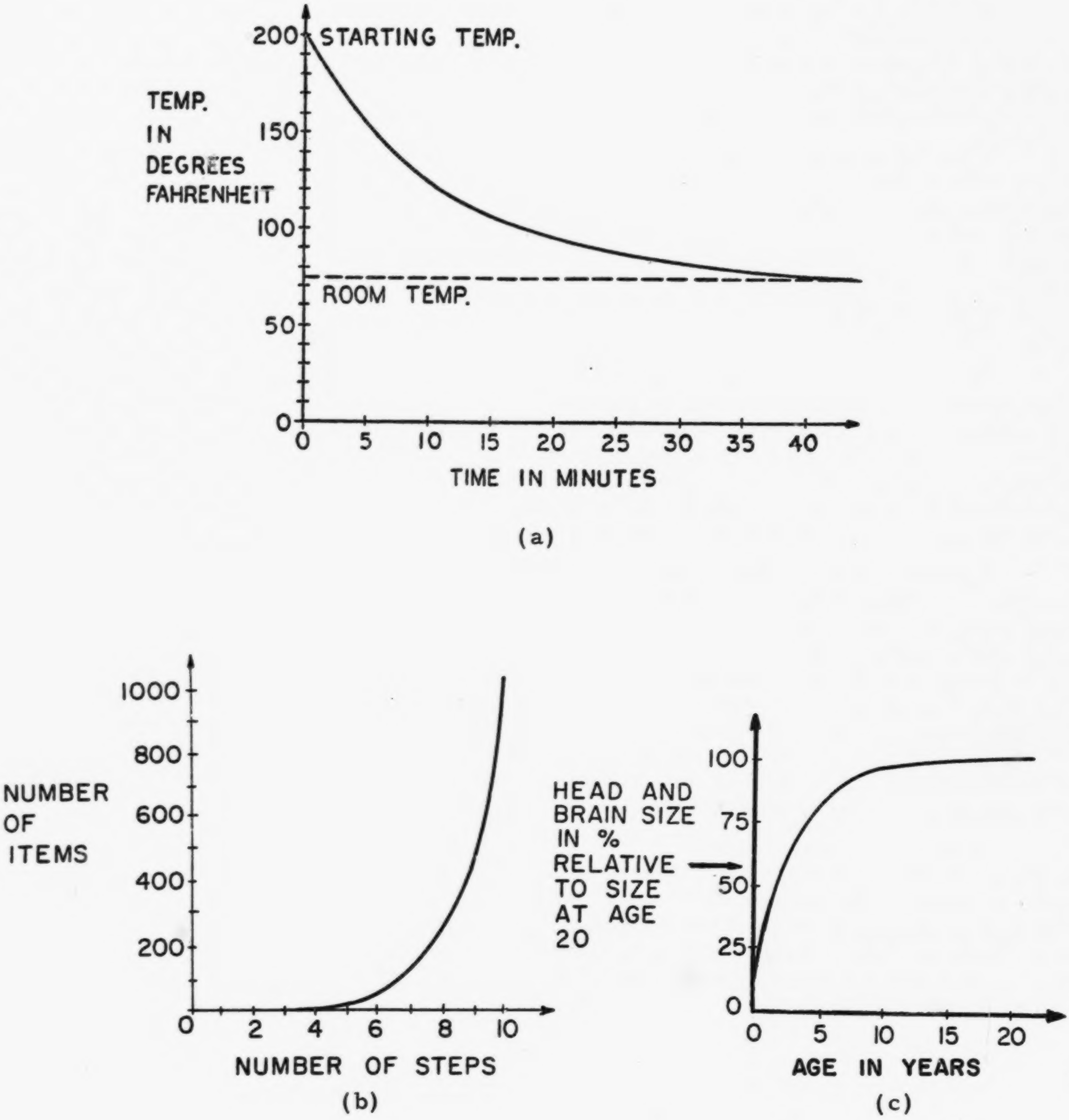

Fig. 21 Three examples of exponential systems.

10. MODEL EQUIVALENCE

Models can be formed with symbols, such as words and numbers, or may be constructed with computers, electric circuits, or hydraulic, mechanical, or chemical systems. A model can be constructed which is equivalent to a conceptual model, so long as the behavior of the model duplicates the behavior of the real world system with an accuracy sufficient for the purposes at hand.

This idea of equivalence can be understood in terms of what engineers call a "black box". They begin by considering that a real world system is a black box whose interior structure is unimportant, only the inputs and the output are essential facts. The relationship between the input and output signals defines what the system does--changes occur at the output of the black box as various signals are applied to the inputs, what is inside this or any other black box is immaterial so long as the input and output are related as in the real world.

If we consider a real nerve cell, with its many complicated stimulus-response relationships, we can devise many black-box equivalents. As long as the output signals change appropriately with specified input signals, it is immaterial whether what resides within the black box is a real nerve cell, a string of words, a graphical plot, a programmed computer, electronic circuits, hydraulic or chemical systems, wheels, gears, and levers, or green cheese (working models made of the last material are uncommon).

Mathematical models are the most general and the most flexible. They provide a compact, precise means for describing, analyzing, and predicting. Very often the mathematical operations required for a particular system become quite complicated, as in the case of the transportation-population example, where large numbers of interdependent equations must be solved simultaneously. For such complex mathematical models computers are often used to manipulate the equations. Their speed, accuracy, and flexibility make possible rapid solutions. Furthermore, computers permit the various numerical factors used in the model as well as their relationships to be changed to permit a study of the properties and to predict the outcome of many different versions of the model.

When a digital computer or an analog computer (which is discussed in the next chapter) is programmed to do the calculations required by a mathematical model, the computer then becomes the working model itself; we call this computer-simulation. By representing the mathematical expression of a real world system, the computer literally is a functioning model of that system. Thus a computer can be a model of a cup of coffee, of world population, or of an air conditioner.

Sometimes a real world system is so complicated that it is not convenient, or even possible, to construct a sufficiently accurate mathematical model or to produce a computer simulation of the situations. In such cases it is often possible to use electrical circuits or to construct mechanical devices or hydraulic or chemical systems which represent the thing to be modeled. Some examples of this alternative are discussed in succeeding chapters.

11. SUMMARY

In this chapter we have considered certain basic aspects of the subject of modeling. The primary objective of the chapter is to focus attention on a theme which runs through a large portion of this course: the point of view that understanding in the scientific and engineering sense comes through simplified versions of reality called "models".

Finding an effective model often begins with observations or measurements of a real situation. Sometimes, it begins with an inspired guess. In any case, the model describes or represents what are considered to be the essential elements of the real situation. Models are useful because they enable us to think rationally about complicated situations, because they permit us to predict future situations and to plan for them; and because they enable us to build and produce man-made devices, systems, and processes to extend man's natural abilities. Finally, there is evidence that human perception and thought are based upon the formulation of models in the human mind.

PROBLEMS

3-1 A more modern version of the six blind men and the elephant is suggested by the following problem. A printed capital letter of the English alphabet is scanned photoelectrically and the resultant signal is converted into digital form and read into a digital computer. Seven subroutines in the digital computer inspect it. The first states that the letter is like a U because it has at least one pocket to hold rain coming from above; the second shows that it is like a K because it has at least one pocket to hold rain from below; the third and fourth find that it is like an A because it has no pockets on right or left; the fifth shows that it is like a V because it has two ends; the sixth shows that it is like an S because it has no junctions; the seventh shows that it is like a D because it has two corners. Combining these seven models of the letter, determine what it is.

3-2 Let us approximate a human body by a cylinder. Since the proportions of the body stay relatively constant as it grows, a tall cylinder will have a larger diameter than a short one. We assume that the height of the cylinder is always 7 times the diameter. Thus, the cylindrical approximation of a 6-foot man will have a diameter of 6/7 foot and a volume of $\pi r^2 h$, or about 3.5 cubic feet. The human body is about 60% water and weighs about the same as an equal volume of water would. Water weighs 62.4 pounds per cubic foot, so the 6-foot equivalent will weigh about 216 pounds.

a. Compute the weights for equivalent cylinders whose heights are 2, 3, 4, and 5 feet. Plot the results, including 6 feet, on a graph showing height versus weight.

b. What kind of curve is this? How does it compare to that of the straight-line-average fit of Fig. 3? Discuss any discrepancies and the validity of the earlier model in light of the new one.

3-3 A paint brush has just been used and the owner wishes to clean it. After the brush has been scraped against the side of the paint can, it still contains 4 fluid ounces of paint. The owner dips it into a quart (32 fluid ounces) of clean solvent and stirs well until the diluted paint solution is uniform. After draining, the brush still holds 4 fluid ounces, part of which is paint and part solvent, since the diluted solution is uniform. The process is repeated with a fresh quart of solvent.

a. How much paint is left in the brush after 5 solvent baths?

b. Prepare a table and plot a curve of the amount of paint remaining after each rinse. What kind of curve is this? Will the paint brush ever get completely clean? Why or why not?

3-4 A man receives his weekly salary of \$150 every Friday and in paying his various obligations spends half of the amount he has in his pocket each day.

a. How much money will he have left on the following Friday?

b. Sketch a graph of his current funds versus the day of the week.

c. If he received \$300 every other Friday, would he be in better or worse shape on the next payday, assuming that his spending habits remain the same?

3-5 You are served a hot cup of coffee at 200°F and a cold container of cream at 40°F, and you do not intend to drink the coffee for 10 minutes. You wish it to be as hot as possible at that time. Assume that the coffee cools as shown in Fig. 20(a) and that the cream container stays at the same temperature.

a. Determine the temperature of the coffee at $t = 5$ and $t = 10$ minutes.

b. If a volume V_1 of coffee at temperature T_1 is mixed with a volume V_2 of cream at temperature T_2, assume that the temperature of the mixture is: $\frac{T_1V_1 + T_2V_2}{V_1 + V_2}$. What will the temperature of the mixture be if 1 fluid ounce of cream is added to 6 fluid ounces of coffee at $t = 10$ minutes?

c. Now assume that the cream is mixed with the coffee at $t = 0$. What is temperature T_o of the mixture at that time?

d. The cooling curve for the mixture is similar to that of Fig. 20(a) except that it begins at the new temperature T_o as calculated in part (c) above and it always lies $\frac{T_o - 75}{200-75}$ of the distance to the given curve from the straight line showing room temperature (75°). What will be the temperature of this mixture at $t = 10$ minutes?

e. Will a hotter cup of coffee result from adding the cream first or later?

3-6 The half-life of radioactive decay is the time in which the amount of the given radioactive material decreases by a factor of two. Radioactive carbon-14 has a half-life of 5700 years, but let us assume that it is 5000 years in this problem to allow simpler calculations. Carbon-14 is created by the action of cosmic rays on the carbon dioxide in the atmosphere, and the amount remains constant with time. Growing plants, and the animals that eat the plants, absorb carbon-14 during their lives, but the process stops when the plant or animal dies. Radioactive decay then causes the relative amount of carbon-14 to decrease. Measurement of the radioactivity of fossils permits an estimate to be made of the time at which they died.

a. What fraction of carbon-14 will remain in a sample after 50,000 years?

b. Approximately how old is a fossil bone in which the amount of carbon-14 is 1.0% of its initial value?

c. Sketch a curve showing the fraction of carbon-14 left in a sample as a function of time.

3-7 Experimental data on the growth of a population of yeast cells are given in the accompanying table.

Time (hours)	Number of cells
0	6
2	10
4	48
6	117
8	234
10	342
12	397
14	428
16	438
18	442

a. Plot a graph of the number of cells versus time in hours. What is the population at 9 hours?

b. The shape of the curve is exponential at first as the cells multiply, but it soon levels off as the supply of food becomes limited. The curve is called a sigmoid. What would you estimate the population to be at 30 hours?

c. Although your estimate may be an accurate one, based on the tabular model above and its graph, it is probably not correct in the real life of a yeast colony. If the table were continued, it would show that the population decreases somewhat as the environment becomes poisoned. During what time intervals is the rate of growth a maximum? A minimum?

3-8 List and discuss some systems like the air-conditioner example which can be described by several different models.

Chapter B-4

MODELS AND THE ANALOG COMPUTER

1. INTRODUCTION

What is the most important property of motion that a drag car racer must understand in order to be first at the finish line? How can we describe the motion of an automobile as it travels from one city to another? What terms do we use to express how fast it is going and the distance it covers in a given time? How does a captain maneuver his ship when he is attempting to moor at a pier? What must he understand about increasing and decreasing speed so that he can move properly for docking? What concepts of motion must an astronaut understand to control his capsule in safety?

All of these situations are dynamic--they involve motion and change. Moreover, they have so many similarities that it is conceivable that we can describe their behaviors with slight variations of the same model. We can invoke the concept of "one model for many systems" which was vividly displayed in Chapter B-3 for the exponential model. We may go further and propose the development of a functional model which, though different in physical form, would be exactly analogous to all of the above situations. Imagine the intellectual and economic advantages of a method by which we can study many different man-made systems with a single functioning model.

These aspects of modeling are of great importance. Our purpose here is three-fold: (1) to illustrate once more the concepts of dynamic modeling, this time with models of moving systems, (2) to illustrate clearly the method an engineer or scientist uses to derive a model for a real situation and (3) to examine how modeling the real world enables the engineer or scientist to note many similarities among different systems and enables him to discover new techniques for analyzing and controlling their behavior. In order to accomplish this objective we will attempt to bring into focus the meaning of inputs and outputs of models and the means by which we describe them; the concepts of position, velocity, and acceleration and their interrelationship, and the distinctions among various types of models.

2. SIGNALS, INPUTS, AND OUTPUTS

The examples in Chapter B-1 indicate that models can be discussed in terms of inputs and outputs. For example, in the trachea-bronchi system of Section 4 of Chapter B-3 the input is the pressure difference and the flow of air represent the output. Such inputs and outputs are described in terms of signals.

A signal is usually the numerical value of an input or an output: Thus: the total population is 15 billion, the temperature of the coffee after 15 minutes is 100°, etc. are input or output signals. Sometimes these values are listed in tabular form (as in Table 3 of Chapter B-3); more often they are given in graphical form (as in the bar graph of Fig. 16 or the exponential curves of Figs. 18 and 19, all of Chapter B-3), or they may be given as equations.

One of the most interesting properties, common to many of the signals that we have discussed, is that they change with time. For example, the coffee temperature decreases; the height of a man increases; the population "explodes." Whether we try to understand the operation of a man-made system or we attempt to design such a system, we are customarily interested in the manner in which the signals into and out of the model of the system change with time.

Two very familiar signals whose values change with time are velocity and acceleration. These signals describe how the position of a body changes with time. Thus, they are known as signals of motion and are important in models of systems of the man-made world in which the parts change their relative positions with respect to each other. The study of motion will develop the meaning and the utility of model construction based on the concepts of input, output, and signals, as well as introduce methods of modeling real world systems which are undergoing complex changes.

3. SIGNALS OF MOTION

What is the meaning of motion? Motion is a concept with which we become familiar at an early age. We walk to school, we run to first base, we swim several lengths of the pool. Thus, one very important aspect of motion is "getting from one place to another". In other words, to achieve motion we must change position.

To describe the position of an object is, however, not a simple task. To describe the exact position of a car, for example, would require several signal values to locate the car on the surface of the earth. Obviously we require a statement of the latitude and the longitude (or the distances east and north of a specified location). In addition, we need the altitude (in case the car is above or below sea level). After these three values are known the car is precisely located with respect to the surface of the earth.

The position signal may also include the orientation of the car at the above location. In which compass direction is the car headed? How much is the car turned through a leaning angle? To what extent is the car pitched (e. g., rear up and front down)? These three additional components of the signal are necessary to describe the orientation of the car which is located at the point in space determined by the first three parts of the signal.

Thus, the complete description of the car's "position" at any given instant of time requires six signal values. Only when all six are known do we have an accurate description of the location of the vehicle. Fortunately, in studying the motion of cars, we are often interested only in one or two of these signal components. For example, if the car is moving along a straight highway toward the north and we are not interested in the rolling or pitching of the passengers, the complete signal is expressed by the one component: the north-south location. The signal at any time is then described by a single number (the distance from an agreed-upon starting point).

One of the primary reasons for the complications in the study of spacecraft motion is that it often requires all six components. All six change with time, and the change of any one (for example: a change produced by the firing of a thrust rocket) results in changes in the other five signal components.

In this section (and indeed in this book), we consider only very simple situations, and, in particular, those which can be studied without the need for large digital computers to perform the necessary calculations. In the remainder of this section, the discussion relates only to a signal described by the position of a body along a fixed route (for example, the north-south highway). Other motions are discussed in Chapter B-5.

In the case where the position is described by one signal value, the numerical representation of the signal requires agreement on the reference position (the point at which the signal is zero) and the direction in which the signal is to be considered positive (the opposite direction then being negative). For example, in Fig. 1, we are interested in travel along the line ABC. If we arbitrarily measure distance along this line from B and arbitrarily choose the positive direction toward A, we can specify the value of the signal in miles from B. A is located then at + 20, B at 0, and C at -15.

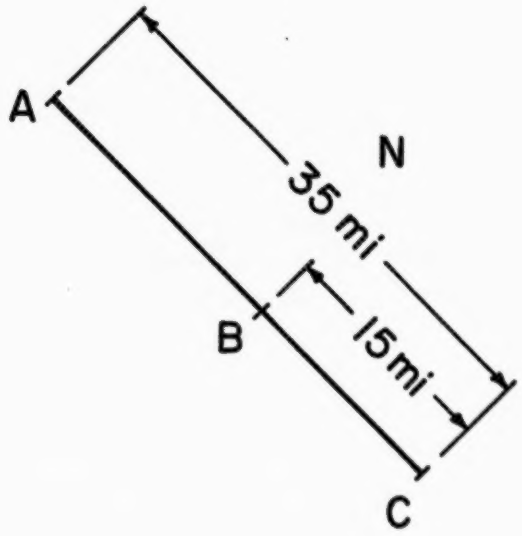

Fig. 1 The position of signal interest

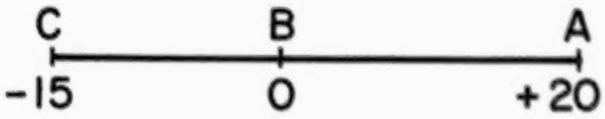

Fig. 2 Redrawing of Fig. 1.

If north is at the top of the page, the figure indicates that the allowable path of the object is northwest-southeast. This orientation is, however, of no interest to us; we are concerned only with the location along this path. Hence, for simplicity it is customary to draw the path as shown in Fig. 2, where A corresponds to a signal value of + 20 miles, C to a value of -15 miles. The position signal (to which we can give any convenient symbol (let us say x) then may vary from -15 to +20; we call this signal "x" the displacement (meaning the distance the body is displaced from the position $x = 0$).

We can now discuss the displacement x as the signal of interest. In general, x varies with time, perhaps as shown in Fig. 3. Here we have plotted x (displacement) versus t (time) where x is measured in meters

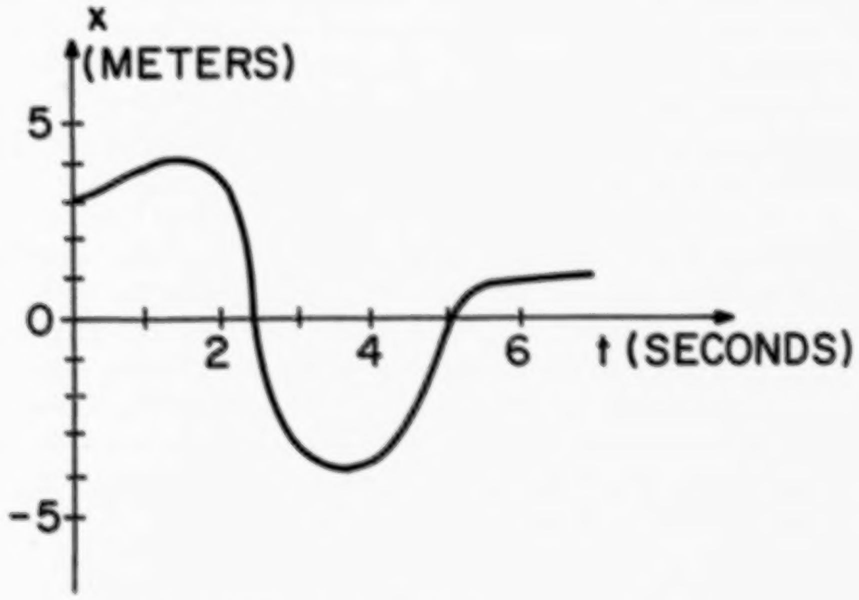

Fig. 3 Displacement signal.

and t is measured in seconds. The fact that the complete signal is described by only one curve (Fig. 3) means that the motion is in only one dimension; towards or away from the point x = 0 along a single line.

If, instead of drawing the curve of Fig. 3, we were to attempt to describe the signal in words, we might say the following: The displacement x starts at time t = 0 at a point +3 meters from the point x = 0. This displacement increases to a maximum of slightly more than + 4 meters at 1.5 seconds after the start, then decreases to zero displacement at t = 2.4 seconds. The object now moves to a maximum displacement in the opposite direction at t = 3.7 seconds. Can you continue this description? The graph is an efficient way to communicate information about the variation of a signal. The axiom "a picture is worth a thousand words" is particularly appropriate when the information is a signal varying with time.

Velocity

While the position signal is completely described by the graph of Fig. 3, it is more informative to describe the system not only in terms of the position of the object at a given time, but also in terms of its velocity at any instant, that is, the speed and direction in which it is moving. For example, if the statement is made that "at t = 2.4 seconds, x = 0", we know only that the curve passes through zero at this time. There is no indication of what the value of the signal is likely to be a fraction of a second later.

On the other hand, a much more meaningful statement is: "at t = 2.4 seconds the displacement is zero and it is changing at the rate of

-22 meters every second"; that is, the object is moving to the left (since the velocity is negative) at such a rate that in one second it will be 22 meters further left if its velocity remains constant. In other words, the velocity indicates how rapidly and in what direction the displacement signal is changing. The velocity indicates quantitatively the very important property of "rate of change of the position of an object".

Definition of Velocity Graphically

We have defined velocity as the rate of change of displacement. If a displacement signal is changing at a constant rate (as in Fig. 4), the velocity is simply equal to this rate. In Fig. 4, for example, the displacement changes by + 1.5 meters (4.5 to 6) in every two-second interval (with the graph for example, from t = 6 to t = 8 seconds). Since this is true for any 2-seconds interval which we may select the velocity has a constant value of

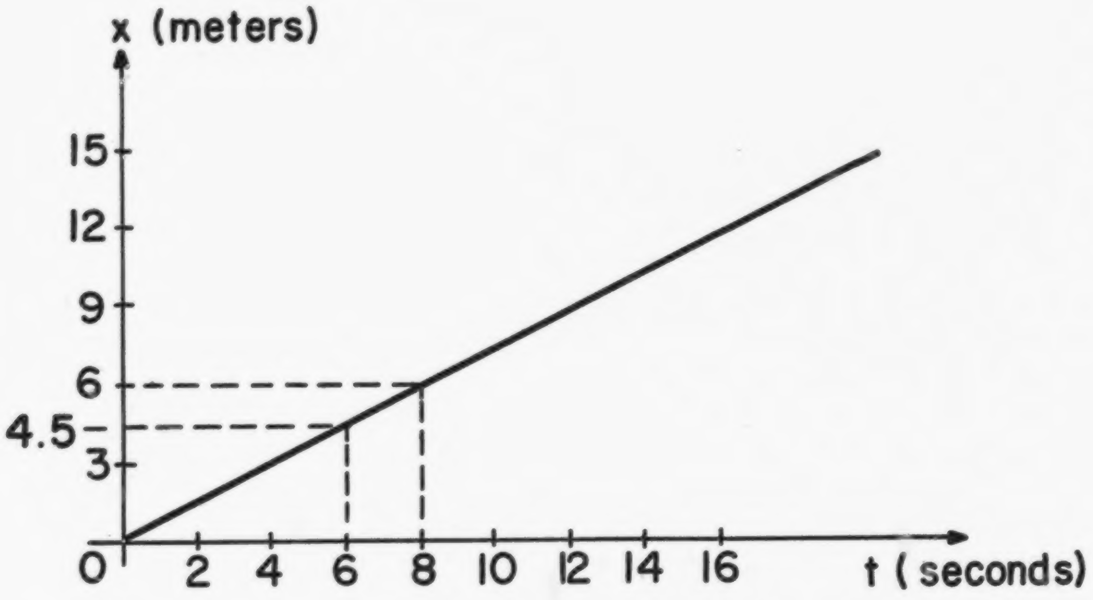

Fig. 4 A displacement signal when velocity is constant.

$$v = \frac{1.5 \text{ meters}}{(2 \text{ seconds})} = 0.75 \text{ meter/second}$$

The velocity in this case is represented by the slope of the straight line. We can write an algebraic equation for x in this example:

$x = 0.75\,t$ (the intercept on the vertical axis is 0, so there is no constant term)

A straight-line type of signal for displacement is obviously not a very exciting case (the velocity signal (slope) is not changing); it corresponds to a car moving in a fixed direction at constant speed along the road. Much more interesting is a displacement that changes smoothly but at a varying rate, as illustrated by the displacement signal shown in Fig. 5. Here the displacement varies through both positive and negative values. We are now concerned with the problem of how to define velocity so that it can be determined from the curve. In paritcular, let us determine the velocity of the moving object at t = 1 second (when x has the value + 3 meters).

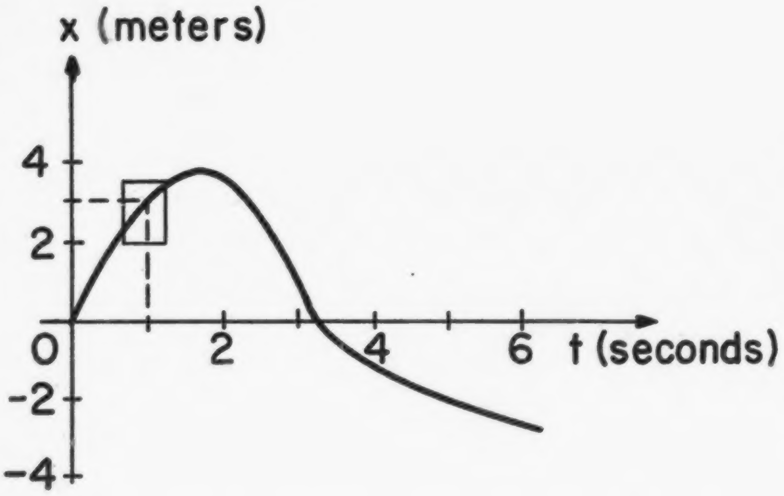

Fig. 5 An example of a displacement signal which varies smoothly at a varying rate.

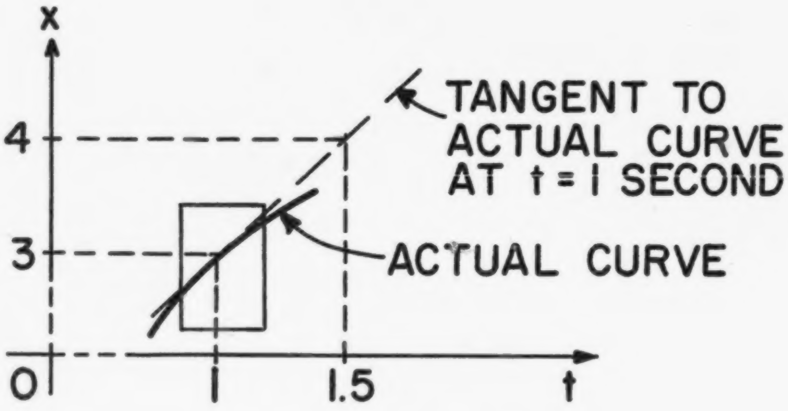

Fig. 6 Magnified portion of x vs t graph near t = 1 second.

If we were to magnify the portion of the curve in the vicinity of x = + 3, t = 1 we should obtain a graph shown in Fig. 6. The actual signal is changing at a certain rate at t = 1 second. If this instantaneous rate of change were to remain constant, the signal variations would follow a straight line called the tangent to the curve, at t = 1 second. In other words, the slope of the tangent to the curve at t = 1 second measures the rate of change of the signal at that instant of time. In Fig. 6, for example, the slope of this tangent line drawn to the curve at t = 1 is

$$\frac{\text{change in x}}{\text{change in t}} = \frac{4 - 3}{1.5 - 1} = \frac{1 \text{ meter}}{0.5 \text{ sec.}} = 2 \text{ meters/second}$$

Hence, the velocity at t = 1 second is 2 meters/second.

In mathematical notation the difference (or increment) between two quantities is generally indicated by the Greek letter Δ (delta). Thus, the difference (increment) between two positions is Δx and between two values of time, Δt. (Note that Δx and Δt are symbols, not products!) With this notation the preceding calculation is written

$$\frac{\Delta x}{\Delta t} = \frac{4-3}{1.5-1} = \frac{1 \text{ meter}}{0.5 \text{ sec.}} = 2 \text{ meters/second}$$

We can restate this process of finding velocity as follows: The value of a displacement signal is first plotted as a function of time. To find the velocity at any time t_1, we construct the tangent to the curve at t_1. The slope of this tangent at any instant of time is the desired instantaneous velocity.

The discussion above is merely a very wordy way of saying that the velocity is the rate of change of displacement. Velocity at any desired time can be determined by drawing a tangent to the displacement curve for the required instant of time, and then calculating the slope of this tangent. For example, in Fig. 7 a displacement signal is given by the solid line. We desire to determine the instantaneous velocity at two different instants: t_1 and t_2. At these instants, the magnitudes of the displacement signals are A and C, respectively. At A, if the signal were to continue to change at the same rate as at t_1, x would follow the path AB. This line is the tangent to the curve at A. Its slope is the velocity at t_1. Similarly, the slope of CD is the velocity at C.

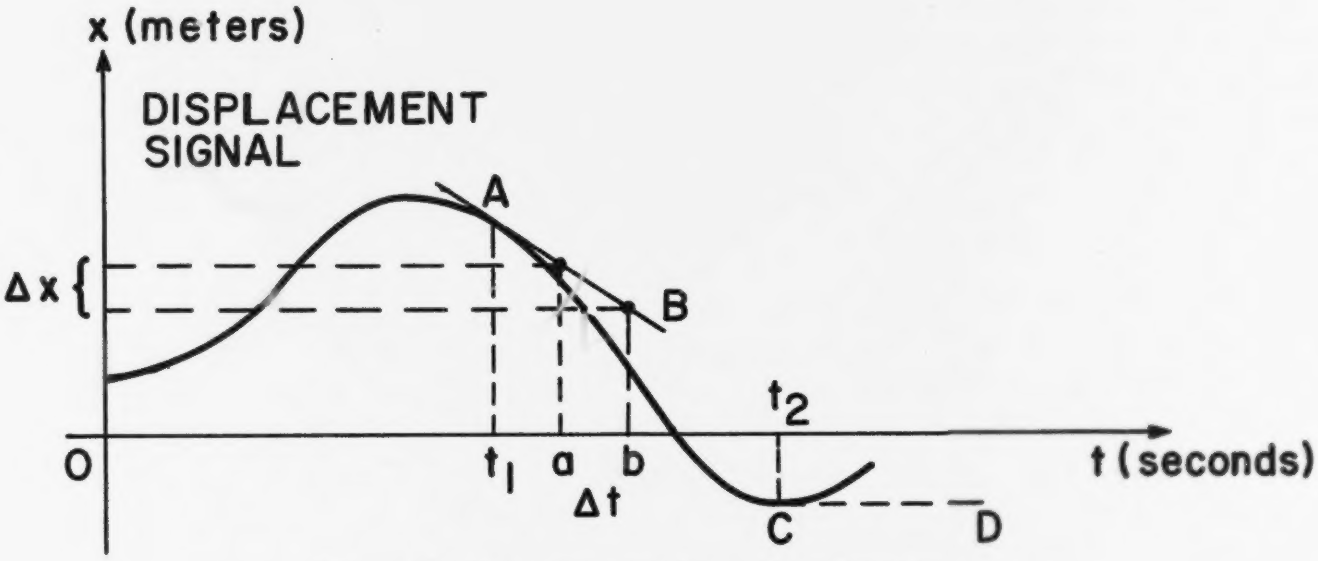

Fig. 7 Determination of velocity.

Definition of Velocity in Terms of Increments

Inspection of Fig. 7 reveals that the velocity at the instant t_1, denoted by $v(t_1)$, is the slope of the tangent AB on the graph. As previously defined, this slope can be written as

$$v(t_1) = \frac{\Delta x}{\Delta t}$$

To compute the numerical value of $v(t_1)$ we must therefore determine a value for Δt and a corresponding value for Δx. To determine these values, we may select any 2 points on the time axis which are reasonably far apart. In our graph these have been indicated as a and b. The difference between them (b - a) is Δt.

If we now project these points up to the tangent AB and then project the points of intersection on AB to the x axis, we can determine the corresponding value of Δx directly on the axis and $v(t_1)$ can then be computed by substitution in the above equation. It is interesting to note that since AB is a straight line, any two points selected for Δt can be used. The ratio of the Δx and Δt values will not change; if we know the instantaneous position as well as the instantaneous velocity of a moving object we can predict the position of the object of a short time later. The next position of the object must be equal to the original displacement to which must be added the displacement of the object during the time interval of motion. If the elapsed time interval is very short (Δt is very small) we may assume that the instantaneous velocity will not undergo any significant change -- that v_1 will remain almost constant. Under such conditions $\Delta x = v_1 \Delta t$ and the new displacement will be

$$x + \Delta x = x + v_1 \Delta t$$

From the graph of displacement versus time we can thus determine all of the instantaneous velocities. If a velocity versus time graph is available we can reverse the above process, and by using small intervals of time (Δt is kept small) we calculate the change in displacement during these small intervals and add these to the displacement at the beginning of the interval to determine the new position of the moving object. Bit by bit we can thus derive a curve which shows the displacement of the object at any instant. The smaller the intervals for Δt the more precise will be our statement of displacement versus time.

We conclude this discussion of velocity with one graphical example of a displacement signal and its corresponding velocity. Fig. 8 (a) shows the plot of the displacement signal, and Fig. 8 (b) shows the corresponding velocity signal. Notice especially that at t_1 and t_2 the slope of the tangent to x is zero (because the rate of change of displacement is zero). Just after the instant t_1, x changes very rapidly downward; the rate becomes less negative and then, just before t_2, it becomes quite large once again.

The correspondence between these graphs may be hard to follow at first. Try actually indicating the slope of the displacement curve in the following way. Lay a ruler against the curve of Fig. 8 (a) at the point where it crosses the displacement axis. Adjust the ruler until it is tangent to the curve. You will notice that it slopes upward to the right. Now change the adjustment to make it become tangent at a point slightly to the right of the first one. Now it slopes upward a trifle more steeply than before. Continue this little game, remembering that each time you set the ruler tangent to the curve, its slope measures the velocity of whatever object is changing its displacement in the manner shown by graph (a). Now if you examine Fig. 8 (b) you will see that it is a graph of the slopes you have been looking at. Where curve (a) slopes up nearly vertically, curve (b) has a high reading. Where curve (a) is momentarily horizontal (zero slope), curve (b) passes through zero. Wherever curve (a) slopes downward to the right, curve (b) is found below the time axis, in the region of negative velocities.

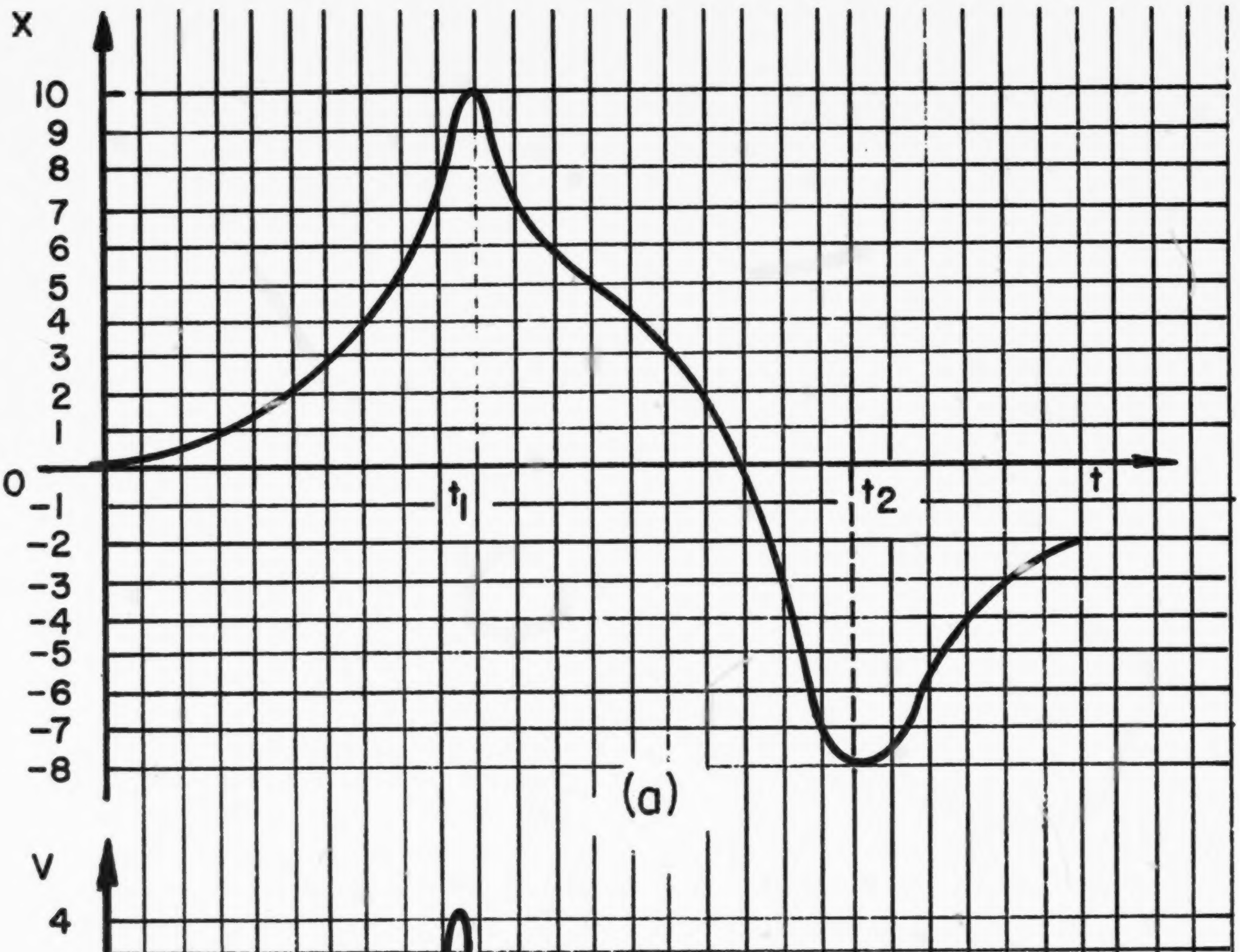

Fig. 8 Displacement signal and corresponding velocity.

In summary, the velocity curve can be accurately constructed if the displacement curve is given. For a succession of values of t, we construct the tangents to the x curve. Calculation of the slope of these tangents gives the v values. Thus, the velocity curve presents no information not already available from the displacement curve. It is also possible to find the x curve if we are given the v curve. Thus, the displacement and the velocity are in a sense equivalent signals: each contains the information needed to find the other.

From an engineering standpoint, we are frequently interested in both signals, x and v. When we travel in a car, x indicates the position and v the rate at which this position is changing. In another sense, x measures past accomplishments toward reaching our destination, v measures the present rate of progress.

Acceleration

We have learned that from a graph of displacement versus time, we can determine the velocity at any instant by drawing a tangent to the graph at any point and determining its slope. The slope represents velocity at that instant of time.

Bodies in motion do not in general move with constant velocity. For example, the velocity of a body moving along a fixed track might vary with time in the manner indicated by the graph in Fig. 9 (a). How fast the velocity of a body changes is a question of considerable importance in many situations. For example, it is important to know how long it takes a jet airliner starting from a standstill to reach its flying speed of 160 miles per hour. Does it reach flying speed before it reaches the end of the runway? Builders of drag-racing cars are very much concerned with how long it takes the car to reach its top speed. How long does it take to slow down a car from 50 miles per hour to a standstill? All of these questions involve concern with how fast the velocity of a body can or does change.

The rate of change of velocity is related to the slope of the velocity-time graph; the greater the slope of this graph, the faster the velocity is changing. If the velocity is given in meters per second by such a graph, and if the time scale is calibrated in seconds, then the slope is measured in meters-per-second per second. The rate at which velocity changes is called acceleration, and is commonly expressed in units of this kind. The acceleration at any instant t is defined as the slope of the velocity curve at that instant, and it is given by the equation

$$\text{Acceleration} = a = \frac{\Delta v}{\Delta t} = \text{slope of a tangent to the v-t graph for the given instant of time.}$$

Thus acceleration is related to velocity in the same way that velocity is related to displacement.

It is clear from the graph in Fig. 9 (a) that $\Delta v/\Delta t$ can be either positive or negative. It is also clear from the graph that a positive

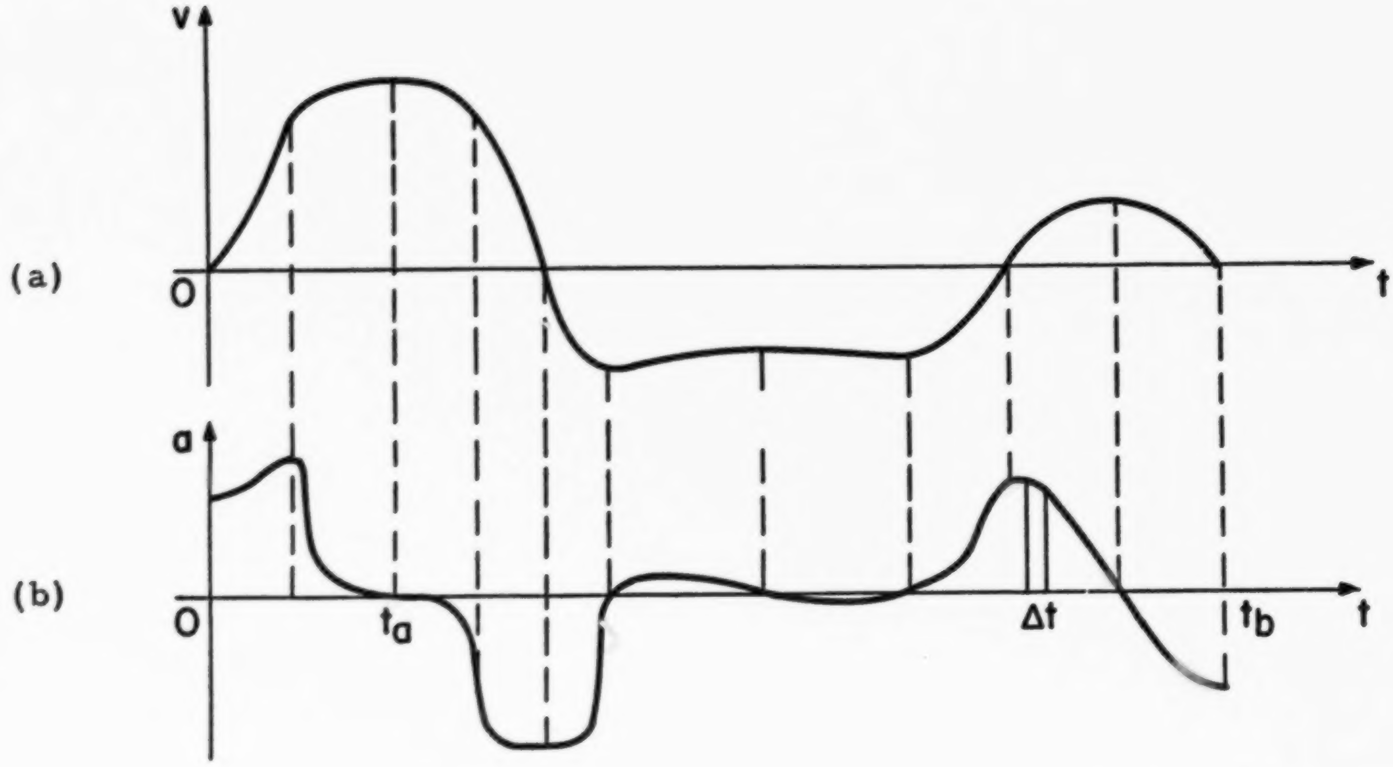

Fig. 9 Graphs of: (a) v versus t for a body moving with non-constant velocity along a fixed route and (b) acceleration versus time corresponding to the v, t curve of (a).

slope means that the body is speeding up, while a negative slope means that the body is slowing down. The term deceleration is sometimes used instead of acceleration to describe the case in which the body is slowing down. However, in science and engineering the single term acceleration is used for both cases; a positive acceleration means that the velocity is increasing and a negative acceleration means that the velocity is decreasing.

Bodies in motion do not in general move with constant acceleration. Thus it is often useful to plot a graph of acceleration as a function of time; a typical graph of this kind is shown in Fig. 9 (b). If a graph of velocity versus time is given, then the corresponding graph of acceleration versus time can be constructed as follows: Calculate the slope of the tangent to the v-t curve at some instant of time. According to the definition this slope $(\frac{\Delta v}{\Delta t})$ is the acceleration of the body at that instant, and it is plotted as one point on the acceleration versus time graph. This process is repeated a number of times until enough points are obtained to permit the acceleration versus time graph to be plotted.

Acceleration is a particularly important signal because one of the basic laws of nature states that the acceleration of a body is directly proportional to the force acting on the body. Consequently, when an airplane or car is brought to a stop, and if the passengers are to stop their motion in the same time interval, they must also be subject to a decelerating force (that is, a negative accelerating force). In an airplane, the force is applied through the seat belts as the passenger moves forward against the belt. In a car without seat belts the force must be applied either through the action of the passenger in bracing himself with his hands and feet, or through contact of the passenger with the dashboard or windshield (a less comfortable technique).

We note now that, as in the relationship between displacement and velocity, the acceleration curve presents no information not already available from the velocity curve. Therefore, if we know the displacement curve we also know the velocity and acceleration curves. We indicate this relationship which permits the derivation of the v and a curves from the x curve by the following shorthand notation:

$$x \rightarrow v \rightarrow a$$

Thus, displacement, velocity, and acceleration are related signals: each contains the information needed to find the others. It is logical then to expect the reverse relation

$$a \rightarrow v \rightarrow x$$

Now that we know how to go from x to a, how do we accomplish the reverse transition? It turns out that the technique for doing this involves nothing more than the simple process of calculating the areas under the a, t and v, t curves, much the same as was done for the population-versus-decades curve of Chapter B-1.

4. THE RELATION: $a \rightarrow v \rightarrow x$

The Relation: $v \rightarrow x$

Figure 10 is a graph of v versus t for a car driven on a highway at a constant velocity of 60 miles per hour at time t = 0. The graph looks extremely dull, but it contains a little more information than you may have noticed at first glance. Not only does it show the velocity of the car at any instant, it also makes it possible to find out how far the car has moved (its displacement) up to that time. Fig. 11 indicates how this is done. Since displacement = velocity x time, and the area of a rectangle = height x base, we can use the area of a suitable rectangle as a model of displacement.* This ancient system still has very great value, be-

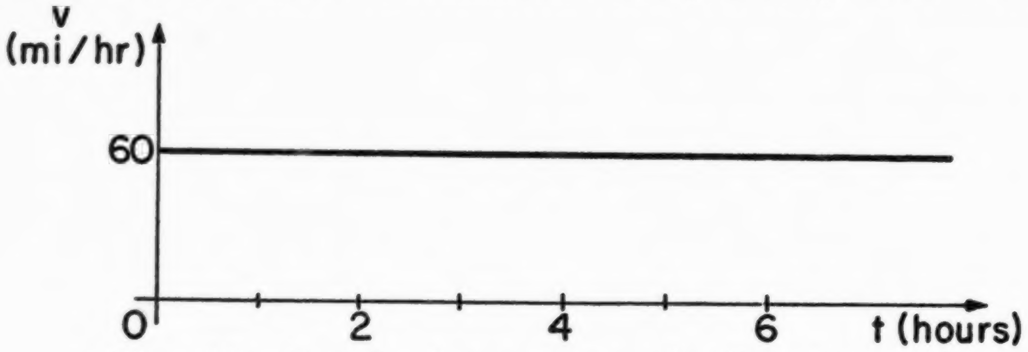

Fig. 10 Graph of constant velocity of a car moving on an interstate highway.

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*This scheme is actually based on the way the ancient Greeks multiplied two numbers. Since they indicated numbers by letters of the alphabet, and the ideas of 0 and of the positional value of digits had not been invented, doing arithmetic was tricky. Showing the product of b and h as the area of the rectangle of base b and height h, however, was quite easy. We still say "the square of 21", as though we meant the area of a rectangle of base 21 and height 21; and in using the Theorem of Pythagoras we do mean just that.

cause we can make an approximation of a curve graph by means of a bar graph made of rectangles. Thus the area under a curve (i.e., down to the horizontal axis, the axis of abscissas) can be at least roughly determined, as in the following example.

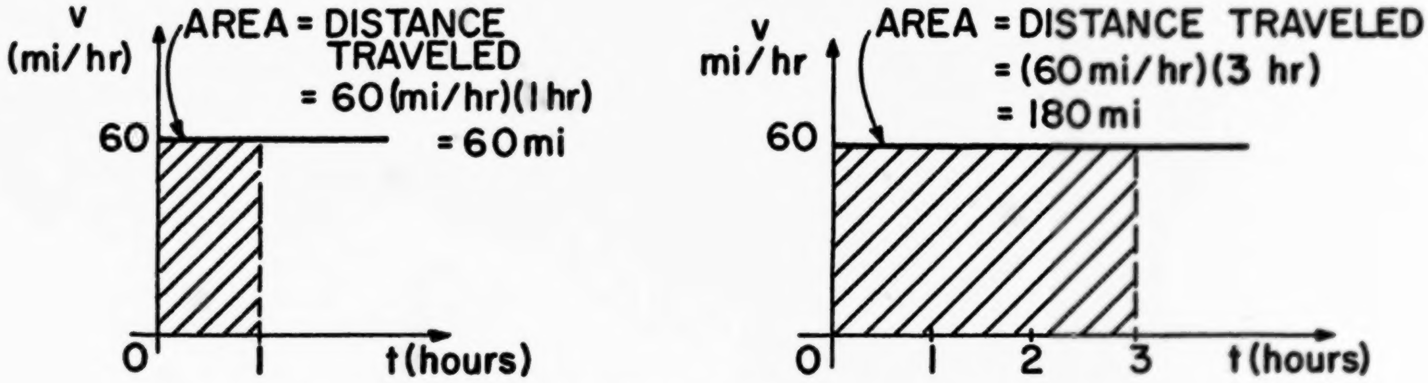


Fig. 11 Relationship between area under curve and distance traveled.

We can seldom drive at a constant velocity (as that shown in Fig. 10) for any length of time. Some one may cut in front of us or we may round a curve or climb a hill and be compelled to slow down. Perhaps we may attempt to pass another vehicle, or the speed limit may increase and we may increase our speed. When driving in the city we change our speed even more frequently because of turns, traffic lights, and traffic jams. Let us consider, for example, how the velocity of an automobile being driven out of Washington, D.C. at 5 P.M. on a Friday afternoon may vary. One likely v, t graph for this situation is shown in Fig. 12. If we wish to calculate how far we have traveled during the first hour we must calculate the area contained under the v - t curve between t = 0 (the start of the journey) and t = 1 hour. It is evident from Fig. 12 that this computation is considerably more difficult than it was for the simple straight-line graph of Fig. 10.

There are several procedures which we may adopt. If we possess a planimeter (a mechanical device for finding area from a graph) we may determine the area with reasonable accuracy. Lacking such an instrument, we may "count squares"; that is, plot our v, t curve on a sheet of cross-ruled graph paper and count the number of squares enclosed by the curve. If we arrange each square so that its area represents a certain unit of distance, then the number of squares under the curve within a given time interval is equal to the distance traveled during that interval.

An alternative and more meaningful approach is to approximate the curve of Fig. 12 by a new curve whose area is approximately equal to the original but which has the property that its area is easy to calculate. Such an approximation is shown in Fig. 13. Here the straight lines are drawn so that the pieces of the original curve lying inside the straight lines have approximately the same area as those

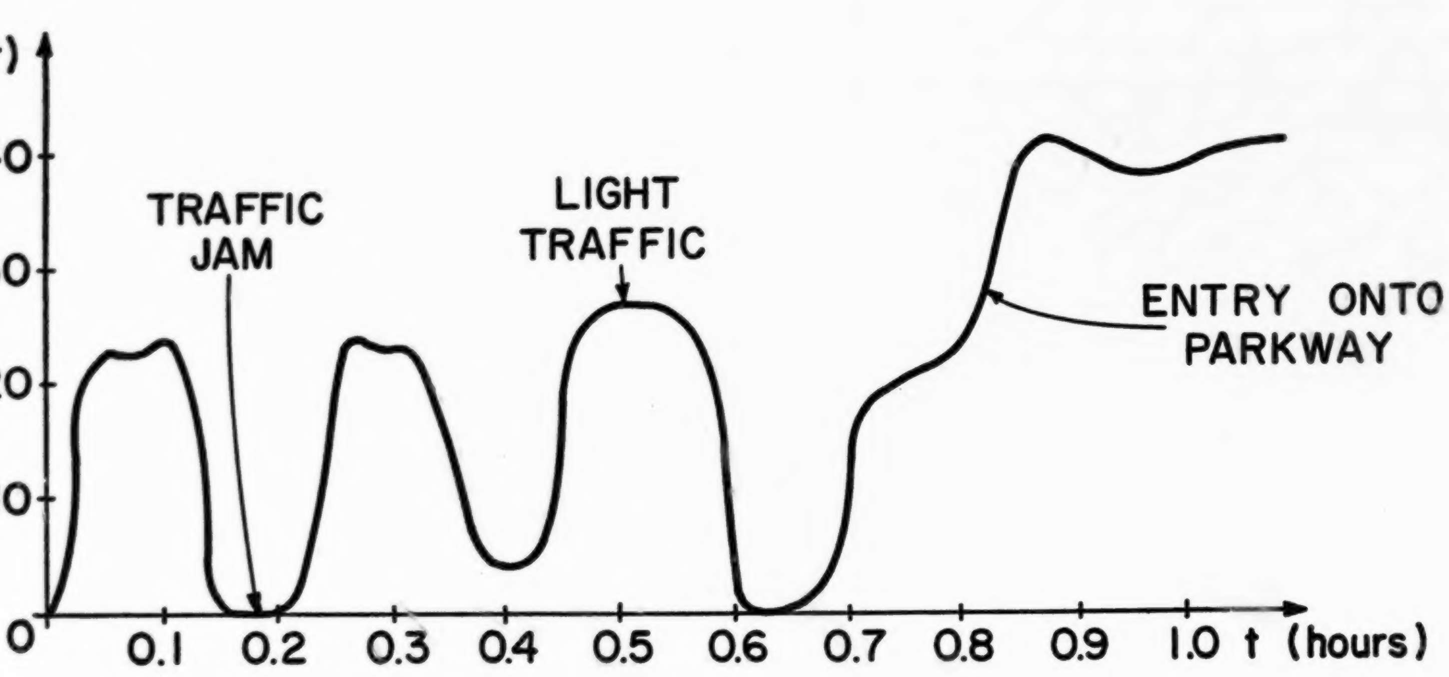


Fig. 12 The velocity of a car leaving Washington D.C. during Friday afternoon rush hour.

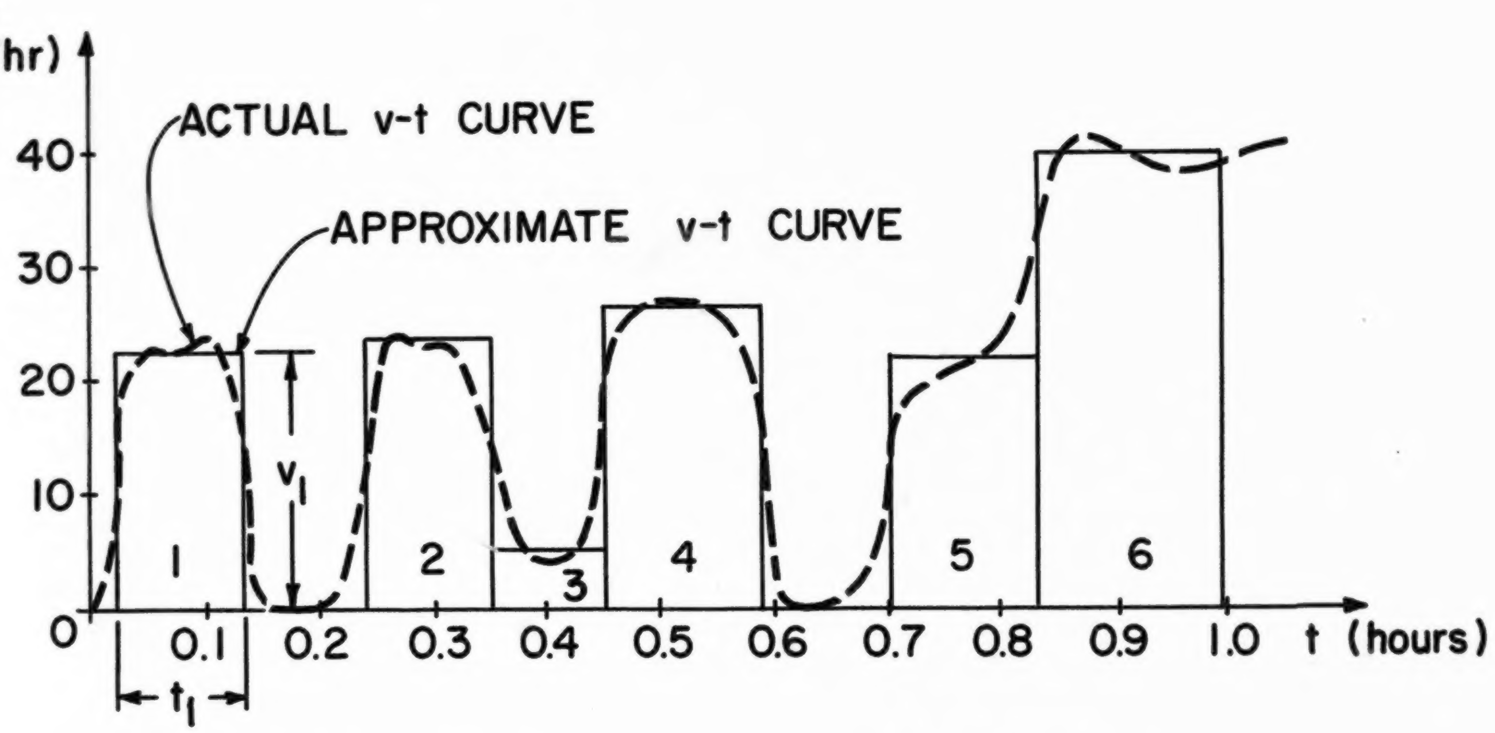


Fig. 13 Approximation to actual v, t curve of Fig. 12.

which are outside of the curve. (This equating of areas is done by "eye"; that is, the lines are placed so that the areas look equal.)

With such an approximate plot, the distance traveled is now the sum of the areas of the set of six rectangles in Fig. 13. This approximate distance (denoted D_a, where the subscript "a" means approximate) is given by

$$D_a = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$

where the A's represent the areas of the individual rectangles. Specifically,

$$A_1 = v_1 \Delta t_1$$

where v_1 is the height (or constant velocity) of rectangle 1 and Δt_1 is its width (or time interval). Similarly, the other areas are

$$\begin{gathered} A_2 = v_2 \Delta t_2 \\ \vdots \\ A_6 = v_6 \Delta t_6 \end{gathered}$$

With this information the equation for D_a can be written as

$$D_a = v_1\Delta t_1 + v_2\Delta t_2 + v_3\Delta t_3 + v_4\Delta t_4 + v_5\Delta t_5 + v_6\Delta t_6$$

Reading the values of the v's and Δt's from the axes of Fig. 13 we see that

$$\begin{aligned} D_a &= (23)(0.11) + (24)(0.11) + (5)(0.095) + (27)(0.135) + \\ &\quad (22)(0.125) + (40)(0.17) \\ &= 18.8 \text{ miles.} \end{aligned}$$

It is important to keep in mind that the above calculation for D_a involved approximating the true v, t curve. If we wish to evaluate the area under the v-t curve more accurately, we may use a greater number of rectangular segments for the approximation. One such improvement is shown in Fig. 14, where the number of rectangles is increased from six to sixteen. In this case, the approximate distance traveled in one hour is given by

$$D_a = A_1 + \ldots + A_{16}$$

or, in terms of the heights and widths of the rectangles:

$$D_a = v_1\Delta t_1 + \ldots + v_{16}\,\Delta t_{16}$$

Obviously, we can achieve greater accuracy if we use 100, or 1000, or 1,000,000 rectangles to represent the actual curve. The sum of the

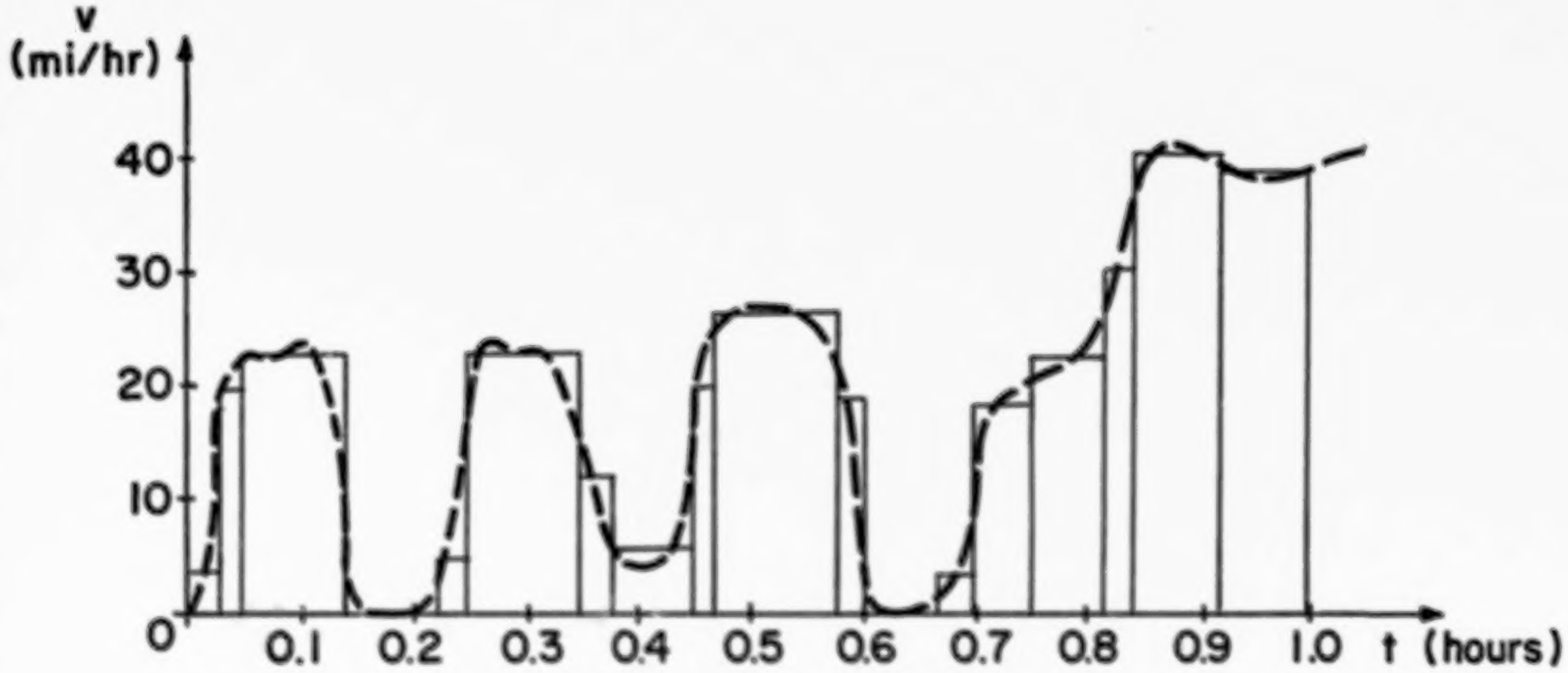


Fig. 14 A closer approximation to the curve of Fig. 12.

areas of a larger number of rectangles is a closer approximation to the area under the actual curve. To simplify the computation where a large number of rectangles are considered, we choose segments of equal width, Δt; the distance D_a can then be written in the form:

$$D_a = (v_1 + v_2 + \dots + v_{n-1} + v_n)\,\Delta t$$

where n is the total number of rectangles and Δt is their width, which is the same for each of the rectangles.

To write a sum of discrete values in less cumbersome form we make use of the symbol Σ, which is the Greek alphabet symbol for the English letter S (for sum). Thus the above statement may be written compactly:

$$D_a = \sum_{i=1}^{n} (v_i \Delta t)$$

We have already explained (Sect. 6, Ch. 3) the meaning of this expression, but it is worth repeating. $(v_i \Delta t)$ means the product of any selected velocity (the one whose identification number is i) multiplied by the fixed time interval Δt. When we write $\sum_{i=1}^{n}$ we mean "substitute all the integers in turn, beginning with 1 and ending with n, in place of i in the expression written to the right. Add the results."

To be more specific, let us assume that the area under the graph is represented by four rectangles, each of width Δt. The value of the heights of these rectangles are v_1, v_2, v_3, and v_4. The sum of the areas of these rectangles would then be

$$D_a = v_1\Delta t + v_2\Delta t + v_3\Delta t + v_4\Delta t = (v_1 + v_2 + v_3 + v_4)\,\Delta t$$

We can use our symbol Σ:

$$D_a = \sum_{i=1}^{4} (v_i \Delta t)$$

To interpret this we simply replace the subscript "i" in the product which appears after the Σ sign with all integral values from 1 to 4 and thus develop four products:

$$v_1\Delta t, \; v_2\Delta t, \; v_3\Delta t \text{ and } v_4\Delta t$$

The sigma sign (Σ) now indicates that these are to be summed and that the sum should be set equal to D_a, the approximate distance through which the object moved, thus:

$$D_a = v_1\Delta t + v_2\Delta t + v_3\Delta t + v_4\Delta t$$

the initial integer for the substitution is always placed just below the sigma sign and the final integer for the substitution is always placed above the sigma sign.

Thus, the symbol $\sum_{i=1}^{n}$ means simply "the sum of the quantity to its right, i ranging in integer values from 1 to n."

Now, the larger n becomes (that is, the greater the number of rectangles which we construct on the graph), the smaller Δt becomes (that is, the narrower we must make the rectangles) and the closer will the value given by $D_a = \Sigma v_i\Delta t$ approach the true value D. In fact, when n increases indefinitely (Δt decreases toward zero), the tops of the rectangles and the actual curve become identical. The total area under the rectangles and the true curve of velocity will then be equal and $D_a = D$.

A special symbolism is used to represent this situation where $D_a = D$. The symbol sigma (Σ) which represents a sum of a limited number of products of the same form, ($v_i\Delta t$ in this example) is replaced by another symbol $\int$, and the symbol Δt, which represented a small but finite time interval, is replaced by dt, which represents a time interval so small in value that a practically infinitely large number of rectangles are produced under the graph. As noted above, this infinitely large sum of infinitely small rectangles represents the true area under the graph, which in this example is the true displacement.

$$D = \int v \, dt$$

It should be noted that this is a general statement. It must be interpreted properly. It merely states that the distance the object has moved can be determined by dividing the area under the v - t curve into infinitely thin slices, of width dt. For each of these slices we determine the value of the associated velocity v and then calculate the area vdt. The statement now indicates that the sum of this large number of vdt products represents the total area or the total displacement.

Obviously the total displacement will depend on the time during which motion occurs, and this must be indicated in some manner. It is therefore customary to include the time limits within which we seek the displacement by showing these limits near the $\int$ symbol as follows:

$$D = \int_{t\,=\,0\text{ sec}}^{t\,=\,2\text{ sec}} vdt$$

Here the time limits indicate that we are concerned with the displacement of the moving object between the time zero and time equal to 2 seconds.

The process expressed by the equation $D = \int_{t=0}^{t=2} vdt$, a process of finding the area under a curve, is called integration, and the area thus calculated is called an integral. With such terminology the expression "the area under the v- t curve is the value of the displacement" can be restated as, "the integral of velocity with respect to time is the displacement."

The Relationship Between Discrete and Continuous Models

In the graphical model of the growth of population, the total number of people counted over a period of 100 years could be determined from the sum of all the bar areas. This summation process is similar to the method used to determine the approximate area under the v-t curve. In the latter case we computed the area under a continuous curve by converting the curve into a discontinuous or discrete series of small rectangles, then calculating and adding all these areas.

There is thus a close relationship between models which are represented by discrete values -- such as population census at fixed dates -- and those which are represented by a continuous variation of numerical values -- as in the velocity of a moving vehicle. We can approximate continuous signals by discrete signals and vice versa. But in the graphs which display discrete time-changing signals, the areas under the graphs are found by cumulative summation (Σ) while the areas under a continuous curve are precisely determined by the related process of integration ($\int$). The man-made world can thus be described in terms of either discrete or continuous graphical models, but despite difference, these models are related and may be interchanged to simplify mathematical treatment, provided the limits within which these substitutions are valid are kept in view.

Initial Displacement

Returning now to the determination of x from v, we recall that by computing the area under the curve of Fig. 13 we determined the distance D which the car traveled from the time it started its journey. If we now wish to state its position one hour after the start, we must indicate the initial position of the car. The value of D represents only the change from the original position. Suppose the car had been driven 900 miles from a position in Florida two days previously and we wish to indicate its present position relative to its starting position in Florida. Obviously, we would add 900 miles to the computed area D to obtain the answer. Thus, the displacement information determined from the area under the v-t curve is not sufficient in itself to indicate the present position of a moving body: we must combine knowledge of the distance traveled with knowledge of the position from which we started. Mathematically speaking then,

$$\begin{aligned} x &= x_o + D \\ &= x_o + \int_o^t v dt \end{aligned}$$

where v is velocity, x is the total displacement from the reference or initial position, and x_o is the displacement of the vehicle from the initial position before this last leg of the journey began. (In more general terms, x_o is known as the initial condition.) This equation can be considered a model for describing how a vehicle on a fixed route changes position.

In keeping with the concept set forth in Section 9, Ch. 3, that one model may apply to different systems, let us consider a system completely different from a moving car but in which the ideas of the above equation apply directly.

In a chemical plant producing hydrochloric acid, the chemical engineer must keep account of the total amount of acid in the plant's storage tank, not just the amount which has flowed in since the start of the day's work. The storage tank is shown in Fig. 15. In this figure, the acid flows from the filler pipe into the tank at a rate of q gallons per hour. The volume of acid already present in the tank is V_o gallons. The variation of q with time for a typical morning is shown in Fig. 16. (The dip at 10:15 A.M. resulted from a malfunction of processing equipment, and the noon-time dip is caused by reduced production because of lunch.) At 8:00 A.M. on this day, the volume of acid stored in the tank (the residue from the preceding day's production) is 1700 gallons. In other words, the initial volume (or initial condition) for the day is V_o = 1700 gallons. Using the ideas expressed by the equation of the last section the engineer can determine the volume of acid V in the tank at any time, from the relation

$$V = V_o + \int_o^t q dt$$

where V is analogous to x, V_o is analogous to x_o and q is analogous to v. In other words, q is a rate of change of a quantity in time, as is v, and V is the

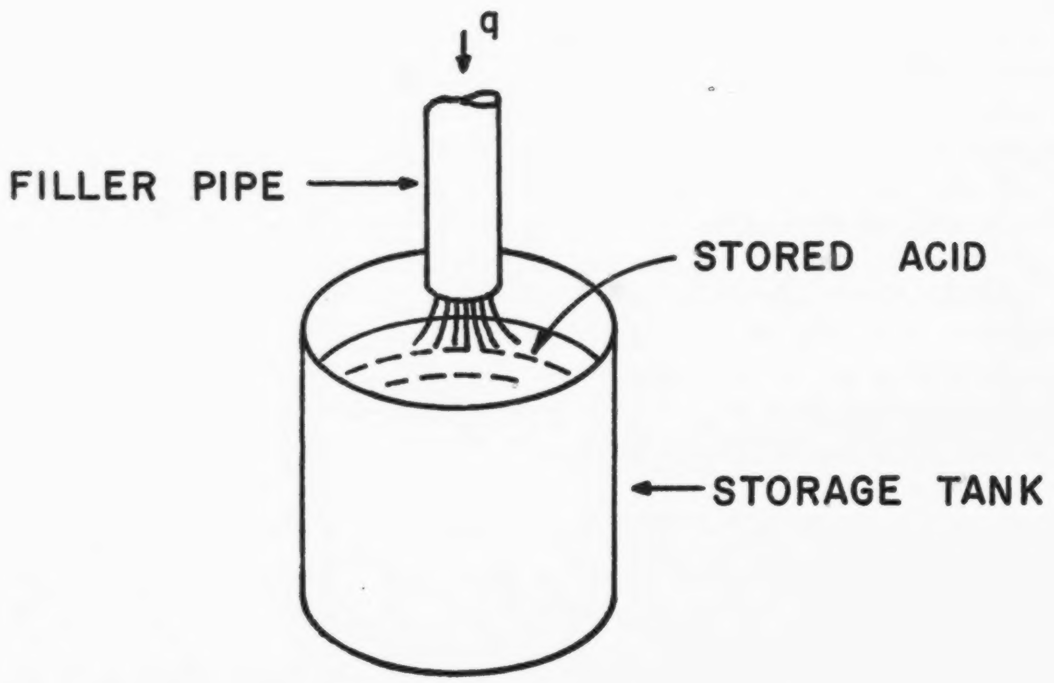

q
FILLER PIPE
STORED ACID
STORAGE TANK

Fig. 15 Hydrochloric acid storage tank.

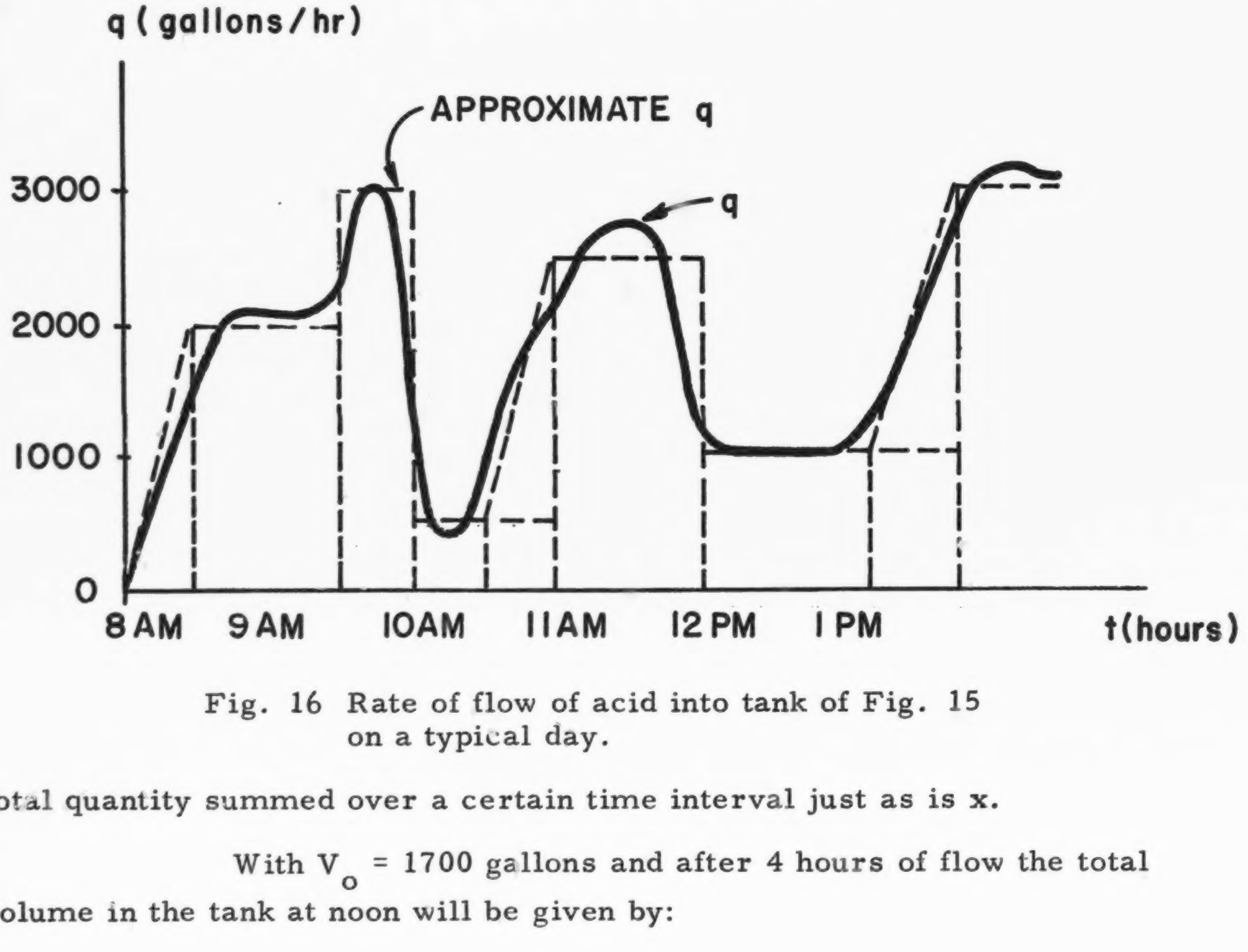


Fig. 16 Rate of flow of acid into tank of Fig. 15 on a typical day.

total quantity summed over a certain time interval just as is x.

With V_o = 1700 gallons and after 4 hours of flow the total volume in the tank at noon will be given by:

$$V = 1700 + \underbrace{\int_0^4 q\,dt}_{\substack{\text{area under} \\ q\ -\ t \text{ graph}}}$$

where t = 0 represents 8 A.M. One possible approximation of the area under the q - t graph is shown by the dotted lines in Fig. 16. Calculating this area from the graph yields:

$$\int_0^4 q\,dt \approx 7500 \text{ gallons}$$

or

$$V \approx 1700 + 7500 = 9200 \text{ gallons}$$

(Notice in Fig. 16 that triangular areas were used in two instances as part of the approximation. Although only rectangular areas were used before to show the transition from a true curve to an approximate curve to derive the total area, any conveniently shaped area can be used for computing the approximate area under the curve.)

The Relation: a → v

When examining the relationship between velocity and displacement it was shown that if the v-t graph is given, the x-t graph can be constructed by calculating the area under the v-t curve. In a similar way, if the a-t graph is given, the v-t graph can be constructed by measuring the area under the a-t curve. To demonstrate this fact, $a = \Delta v/\Delta t$ is rewritten as

$$\Delta v = a\,\Delta t$$

Now if Δt is small enough so that a is constant during this interval of time, then this equation states that the change in velocity Δv during the small time interval is equal to the area of the approximating rectangle of width Δt under the a-t curve [a typical rectangle of this kind is shown in Fig. 9 (b)]. Similarly, the change in velocity during an adjacent interval of duration Δt is equal to the area of the approximating rectangle associated with that particular interval. Continuing this procedure, the total change in velocity in the interval between any two instants of time, t_a and t_b, in Fig. 9 (b) is determined by dividing the entire interval $(t_b - t_a)$ under the curve into tiny rectangles of equal width Δt and summing the areas of these rectangles. The result can be written as

$$\begin{aligned} \Delta v = v(t_b) - v(t_a) &= a_1\Delta t + a_2\Delta t \ldots + a_n\Delta t \\ &= (a_1 + a_2 + \ldots + a_n)\Delta t \\ &= \text{area under a, t curve from } t_a \text{ to } t_b \end{aligned}$$

where a_1, a_2, . . . a_n are the heights of the n rectangles.

When Δt is made infinitely small this equation can be written

$$
\begin{aligned}
v(t_n) &= v(t_a) + \text{area under a-t curve from } t_a \text{ to } t_b \\
&= v(t_a) + \int_{t_a}^{t_b} a\,dt
\end{aligned}
$$

where $v(t_a)$ is the "initial velocity."

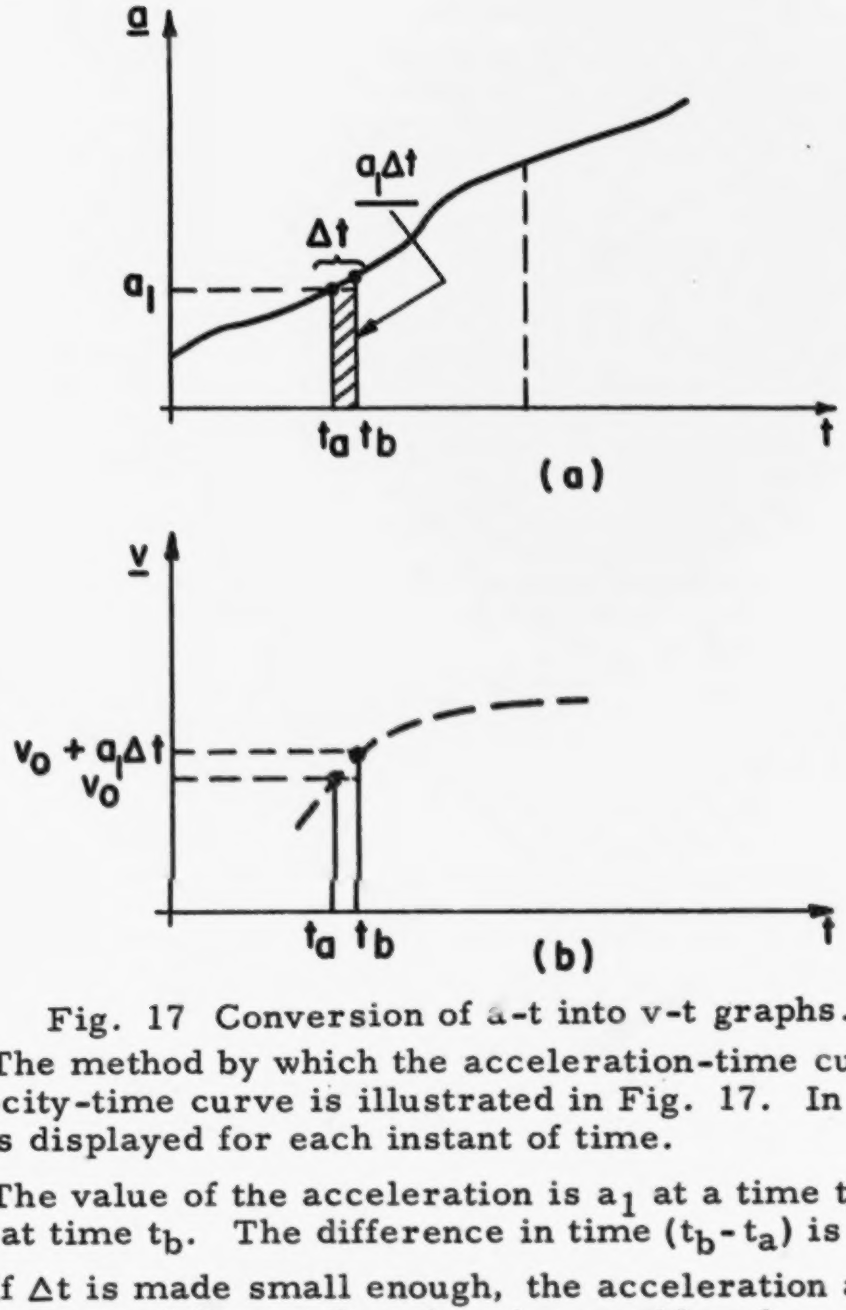


Fig. 17 Conversion of a-t into v-t graphs.

The method by which the acceleration-time curve can be used to develop a velocity-time curve is illustrated in Fig. 17. In Fig. 17 (a) the acceleration is displayed for each instant of time.

The value of the acceleration is a_1 at a time t_a, then it changes to another value at time t_b. The difference in time $(t_b - t_a)$ is represented by Δt.

If Δt is made small enough, the acceleration a_1 will not change any significant amount, and it may therefore be considered as practically constant throughout the entire interval Δt. Under these conditions the product of $a_1 \Delta t$ represents the shaded area in the graph.

Now, $a_1 \Delta t$ also represents the product of an acceleration and time, which is physically representative of the change in velocity of ΔV, during the time interval Δt. If the velocity at time t_a is known to be some value v_o [as given in the graph Fig. 17 (b)], the velocity at the later time t_b must be equal to $v_o + \Delta v$ or $v_o + a_1 \Delta t$, as shown on the graph.

With the velocity at instant t_b thus determined the process can be repeated. An instant of time not too far from t_b is selected, so that the acceleration within that interval is practically constant. The area of the small rectangle is then calculated to determine the velocity change during the interval, Δv. If Δv is now added to the velocity at t_b the sum will represent the velocity at the new instant of time. This process is then repeated a number of times for different values of t_b, until enough points are obtained to permit the entire v-t graph to be drawn.

If the value of the displacement x is known for a single value of t, such as at t_a, the area under this velocity-time curve can be used to obtain the corresponding graph of displacement versus time, by the same procedure. Thus if we know the graph of a versus t, and if we know the values of v and x at some instant of time, (i.e., if we know the initial conditions) then we can apply the area process to the a-t curve to obtain the corresponding v-t graph, and we can apply the area process to the v-t curve to obtain the corresponding x-t graph. That is, we can use information about acceleration to calculate first the velocity and then the displacement of the vehicle.

Thus, there is a two-way transition possible among displacement, velocity, and acceleration, which is expressed simply as

DISPLACEMENT ↔ VELOCITY ↔ ACCELERATION

5. A MODEL OF MOTION

In deriving a model of a system we first must understand in general terms how the system operates. The several systems (dragster racing toward finish line, car traveling from city to city, and ship maneuvering toward a pier) mentioned in the introduction above are similar in their principles of motion. In order to change velocity, an acceleration control of some sort is used. This is a pedal linked to the engine's carburetor in the case of the dragster and car; it is a throttle which controls the engines of the ship. In operation, it is the position of the control which determines vehicle acceleration, which in turn determines how far the vehicle will travel in a given time. The farther we depress the accelerator pedal of our car, the faster it goes and the greater the distance it travels in a given time.

A model for the description of such vehicle motion must involve a mathematical relationship between the position of the acceleration control (which we label with the letter u) and the vehicle's displacement x. The block diagram of Fig. 18 shows the interrelation between the input and output. How is x determined by u? What happens as a result of an input signal u applied to a vehicle?

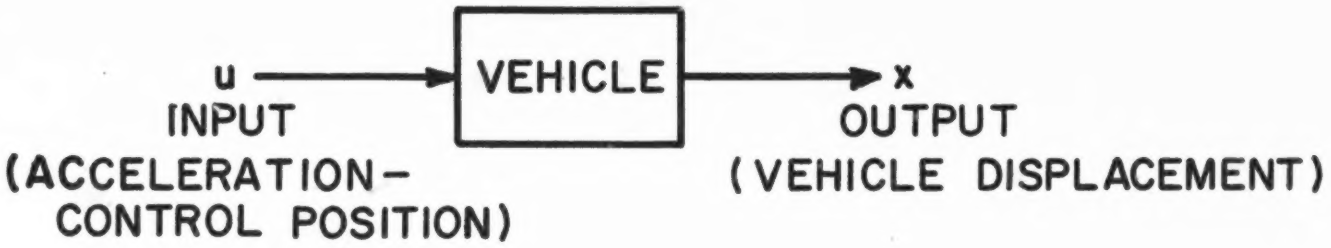


Fig. 18 A simple motion system.

Let us consider what happens as the vehicle operator moves the control.

1. We stated above that the position of the acceleration control causes acceleration of the vehicle. The exact relationship between this acceleration and the position of the control may, in some cases, become quite complicated. In a practical sense, however, we can assume that the acceleration of the vehicle at any moment is directly proportional to the position of the control. Acceleration controls are usually designed in this manner. Thus, quantitatively, we can write that acceleration a is equal to a constant multiplied by u:

$$a = Cu$$

Fig. 19 shows the relationship which exists within the system. Since C is a constant, the equation is regarded as a scaling operation (that is, the input is changed in magnitude by the value of C). Therefore, C is named "scaling coefficient"

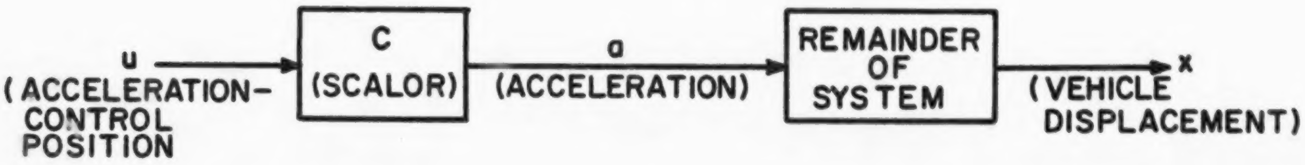

u
(ACCELERATION-
CONTROL
POSITION
C
(SCALOR)
a
(ACCELERATION)
REMAINDER
OF
SYSTEM
x
(VEHICLE
DISPLACEMENT)

Fig. 19 The vehicle motion model after step (1).

or scalor and the block diagram representation of this operation is indicated by a block labeled "scalor" in the figure.

2. We learned in Sec. 4 that the vehicle's velocity is simply the area under the a, t curve. We can indicate this process of finding the area under the a, t curve with a block labeled "area-finder", or an "integrator". The integrator thus converts the acceleration signal at its input into an output which is a v-t signal. "Time" comes into the system because the integrator is a dynamic instrument. That is, the signal at its output changes from moment to moment exactly as the velocity of a real vehicle does, just as long as the acceleration control is not set at 0.

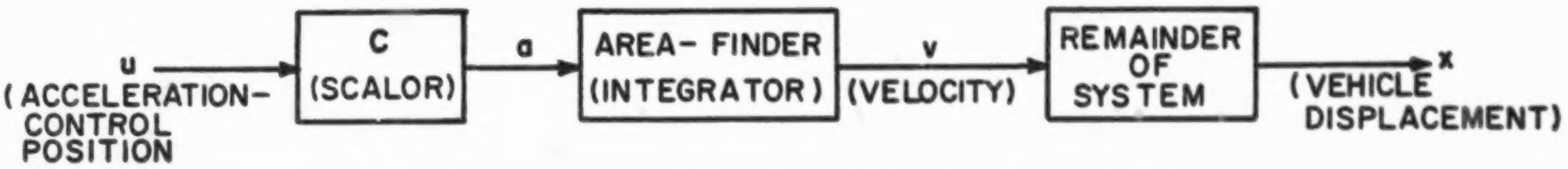

u
(ACCELERATION-
CONTROL
POSITION
C
(SCALOR)
a
AREA- FINDER
(INTEGRATOR)
v
(VELOCITY)
REMAINDER
OF
SYSTEM
x
(VEHICLE
DISPLACEMENT)

Fig. 20 The vehicle motion model after step (2).

3. Since displacement is the area under the v-t curve, we may apply the v-t signal to another "area-finder" or integrator and produce a signal which represents displacement at each instant of time. The complete model of the vehicle acceleration system would then be represented in block diagram form as shown in Fig. 21.

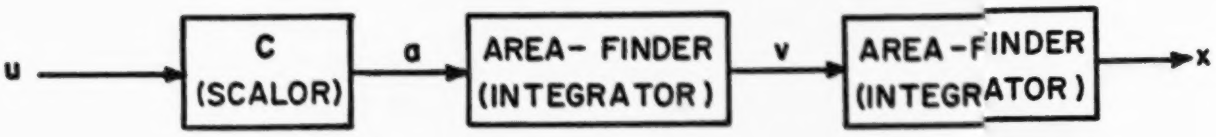


Fig. 21 The complete model of vehicle motion.

Now that the model has been derived, let us emphasize several points. We derived the model by working from a general rather than specific knowledge of the system's operation. The model compels us to represent this knowledge in a quantitative fashion. In the description of the relation between the control position and acceleration our original comments were qualitative and simply pointed out that "an acceleration results from the position of the acceleration control". In our model, we need to determine the exact mathematical relationship (for example, in the model above, $a = Cu$). The model guides us in the analysis we make to determine the detailed properties of the system.

The model of Fig. 21 is this a concise and brief description of of the system. The model is really equivalent to the several paragraphs of general discussion with which we began this section. Indeed, the model is an attempt to produce a precise system description which cannot be misunderstood -- in contrast to any more verbal statement which may be subject to misinterpretation and misunderstanding. Thus, the model is a special language to describe a system, and is the starting point for studying its characteristics. Finally, the construction of a model in a step-by-step fashion indicates the physical laws that are required to understand the system's operation. In the particular case of Fig. 21 there were two such laws:

1. The relation between acceleration-control position and acceleration, which described the way the control was designed.

2. The relation: $a \rightarrow v \rightarrow x$.

In the model of Fig. 21 we included no information about the initial displacement x_o and the initial velocity v_o (i.e., the initial conditions). If these values were not equal to zero, then the complete (and more general) model of vehicle motion would appear as shown in Fig. 22. The "adder" blocks in the figure are so named because they permit the introduction of the initial conditions v_o and x_o into the equations of motion.

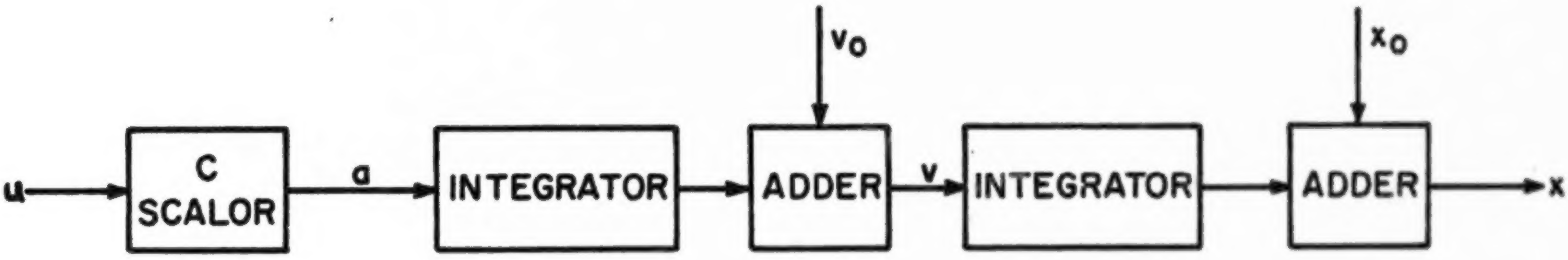


Fig. 22 The complete model of vehicle motion including nonzero values of initial velocity and displacement

The initial velocity v_o is added to the change in the velocity calculated by the first integrator. The initial displacement x_o is added to the change in the displacement calculated by the second integrator.

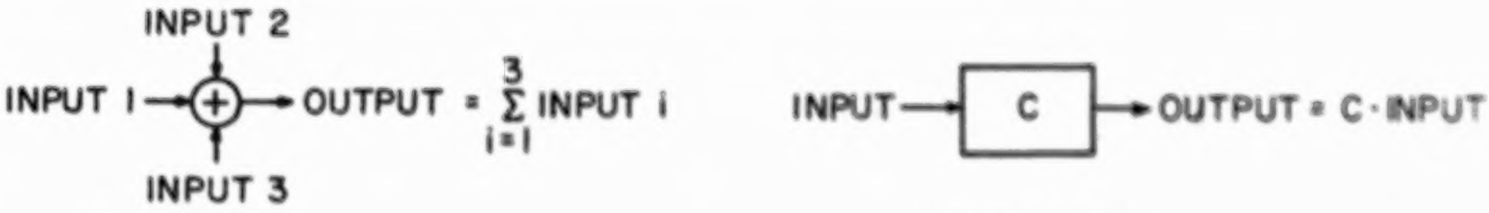
6. THE ANALOG COMPUTER

Once a model has been developed it can be used to investigate many properties of the system (or systems) from which it was derived. This investigation can take place with pencil and paper, or experimentally if we can build a functioning model which represents the system under consideration. Instead of merely calculating possible outputs, we can operate the functional model and observe its behaviour with different input conditions.

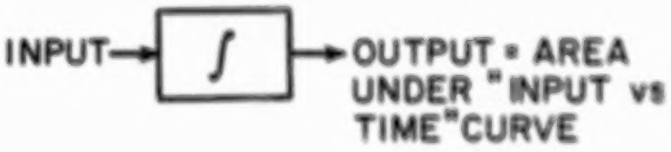
The analog computer can act as such a functional model provided we program it so that its inputs and outputs are analogous to those of the system under study. Analog computers take many forms: mechanical, hydraulic, and electronic but the last is the most common. Some basic components are: the adder, the scalor, and the integrator. These components automatically perform the calculations we would ordinarily make on paper.

The symbols for the basic analog computer components are shown in Fig. 23. In terms of these components, the vehicle motion model in Fig. 22 appears as shown in Fig. 24. If we program an electronic analog computer according to Fig. 24, we can simulate the vehicle motion as follows: The acceleration-control position is simulated by a manual control which varies the electrical input. The vehicle displacement x, which is the output of the computer, is displayed on a meter. We can start with the meter needle set at zero and then attempt to change the input control to move the needle from its initial position of zero to any pre-determined final value position (e.g., we might choose +1 on the meter to represent the distance we wish to travel).

What is meant by "operating" the system on the analog computer?


INPUT 2
INPUT 1
OUTPUT = $\sum_{i=1}^{3}$ INPUT i
INPUT 3
INPUT
C
OUTPUT = C·INPUT

(a) ADDER (b) SCALOR


INPUT
∫
OUTPUT = AREA UNDER "INPUT vs TIME" CURVE

(c) INTEGRATOR

Fig. 23 Analog computer components.

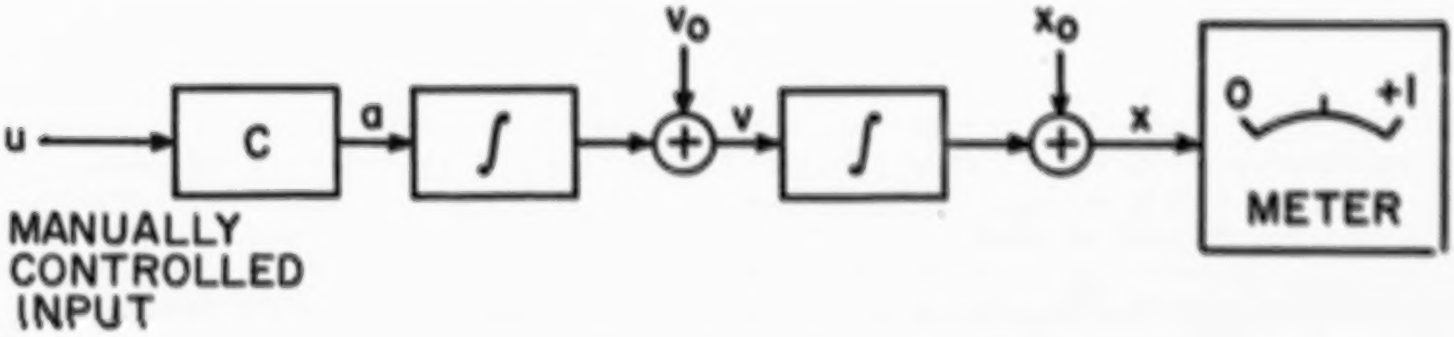

u
C
a
∫
v_0
v
∫
x_0
x
0
+1
METER
MANUALLY CONTROLLED INPUT

Fig. 24 Simulation of vehicle motion model on the analog computer.

For example, how would we simulate maneuvering a ship next to a pier? In reality, the ship's captain observes the place to which he must move his ship and adjusts the acceleration control to make the displacement of the ship change as smoothly as possible from its initial value to the final desired value. To "maneuver" the analog model then, the operator should observe the meter reading and attempt to adjust u, the input, so that the meter changes from 0, the initial position, to +1, the final position of the pointer. Exceeding the value + 1 simulates the ship's crashing into the pier.

Thus, a simple manipulation of an electrical control can imitate the operation of a vehicle described by our model, and we can gain valuable insight into the workings of the real system without actually building it.

7. SUMMARY

This chapter concentrates on an important aspect of dynamic modeling: the description of models which involve motion or which change in time. During our study we learn that model inputs and outputs are described in terms of signals which, in turn, are numerical values usually presented in graphical form or as mathematical equations. We introduced two important motion-describing signals of this kind: velocity and acceleration. These signals of motion permitted us to study the modeling of simple systems which varied from time to time.

Velocity is the rate of change of displacement with respect to time and is equal at any instant of time to the slope of the tangent on the x -t curve at that instant. Similarly, acceleration is understood to be the rate of change of velocity with respect to time and is equal at any instant of time to the slope of the tangent on the v-t curve at that instant.

The concept, introduced in Chapter B-3, that area under a curve has an important meaning was further developed by a study of the relation

$$x \longleftrightarrow v \longleftrightarrow a$$

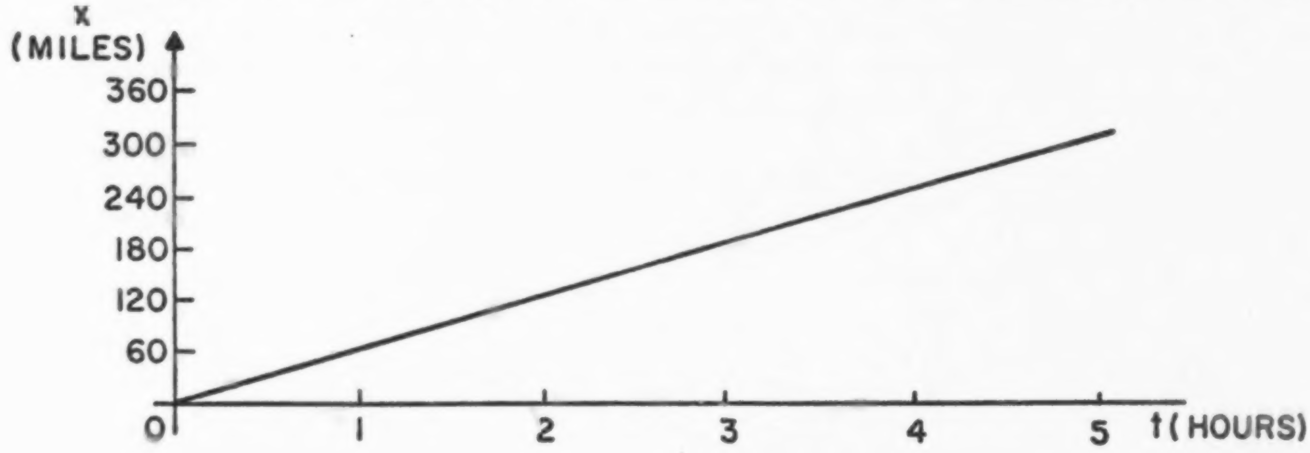
We move from left to right (x towards a) in this expression by finding the slopes of the tangents to the curves and we move in the reverse direction by finding areas under curves. Knowledge of the variation in any one of these signals gives us the basis upon which we can compute the other two variable quantities.

The concept introduced in Chapter B-3, that one model may frequently apply to many different systems, permitted the derivation of a simple model of motion which could be used to the study of any moving vehicle. We applied this model to situations such as a car moving along a highway and a ship maneuvering toward a pier.

Finally, the introduction of the analog computer gave us a means for developing a functional model of motion. Programming the electronic analog computer with electrical inputs and outputs that simulate those of the system under consideration enabled us to produce a laboratory simulation of the real system, and eliminated the need for the construction and study of full size prototypes.

PROBLEMS

4-1 A graph of an automobile trip via an interstate turnpike is given below. Draw a graph of the velocity of the car during this trip.

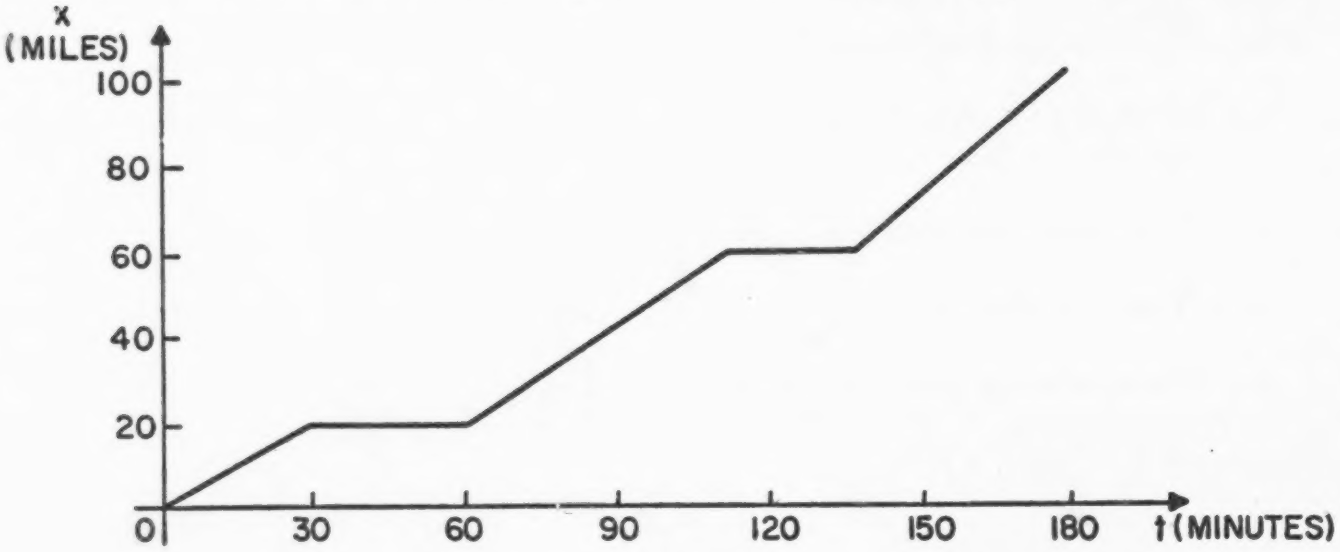


4-2 A distance-time curve for a 100-mile auto trip is shown below. Determine the velocity:

a. 120 minutes after the start.

b. 30 minutes before the end.

c. When the car is midway between the starting point and the destination.



4-3 It is found during an acceleration of a racing car that it is 15 feet from the starting point at the end of the first second, 60 feet from it at 2 seconds, 135 feet in 3 seconds, and 240 feet in 4 seconds. Plot these data as a smooth curve and determine the velocity at 2 seconds and 4 seconds.

4-4 A river has a current velocity of 10 mi/hr. A motor boat on this river moves through the water at 30 mi/hr.

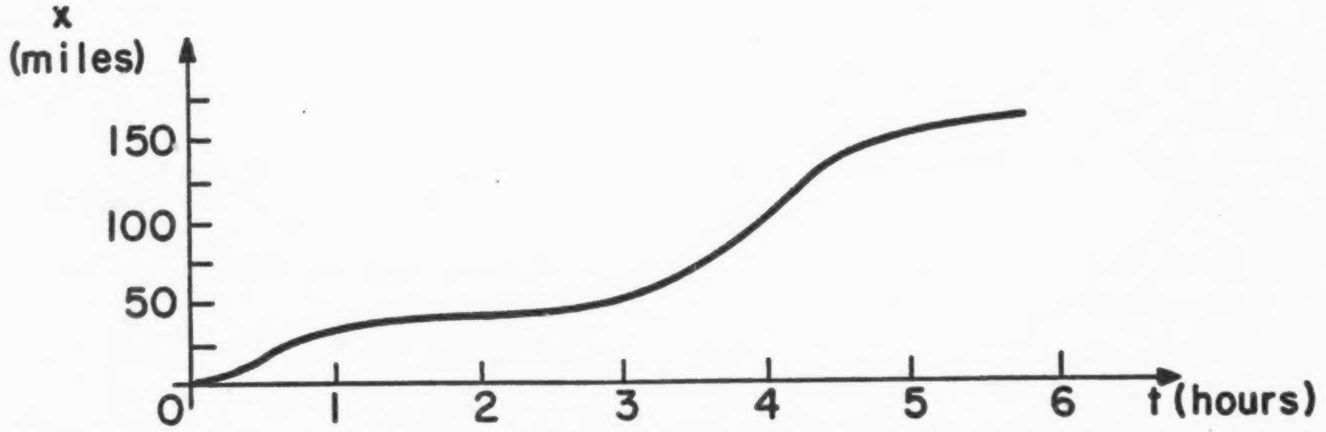
a. What will be the actual velocity of the boat when going upstream?

b. What will be the actual velocity of the boat going downstream?

4-5 A graph of an automobile trip via a variety of roads and highways is given below.

a. Draw two graphs of the velocity of the car during this trip, taking slopes at 30-minute intervals and at 60-minute intervals. What accounts for the difference in the two graphs?

b. During which period of time was the velocity the greatest? the least?



4-6 The velocity of an automobile at various instants after starting time is given in the graph.

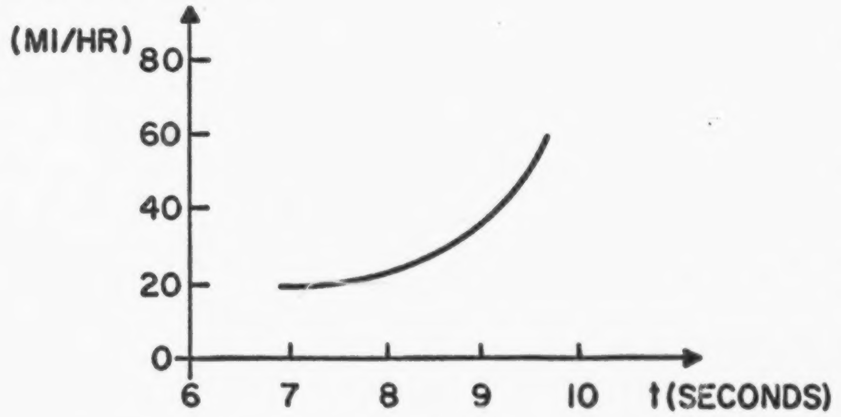
a. Draw an acceleration versus time curve for this motion.

b. With a graphical construction determine the acceleration of the car at t = 8 seconds after the start.

c. Describe the motion of the car between times 7.0 and 7.5 seconds.

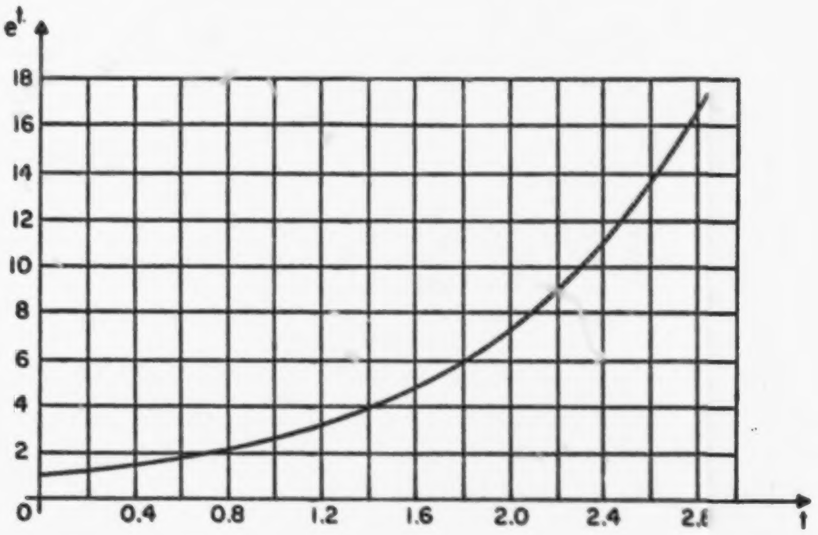
d. How far did the car travel between the times 7.0 and 7.5 seconds?

e. Using the graph, determine the area underneath the curve as accurately as you can. How far did the car move between 7.0 and 9.0 seconds?



4-7 If the earth is about 1.5×10^{11} meters from the sun, and is orbiting in essentially a circular orbit, what is the velocity of the earth in its orbit around the sun?

4-8 Show that the slope of the exponential curve given below is the same at any point as the value of the curve itself at that point. Show also that one plus the area under the curve between 0 and any time t is the same as the value of the curve itself.



4-9 Plot the distance-time curve, $d = \frac{12\,t^2}{t+3}$ from t = 0 to t = 5 seconds, where d is in meters. Determine the slope at 1-second intervals and plot the velocity-time curve. Using this curve, determine the area under the curve in each 1-second interval and plot the resultant distance-time curve. How does it compare with the original curve?

4-10 The mileposts on the Garden State Parkway start at 0 at the southern end. You have passed milepost 15 on this parkway at 12:01 P.M. Your velocity is a steady 60 miles per hour, north.

a. At what time should you pass milepost 25? 135? 165?

b. Which of the predictions made above is most likely to be correct?

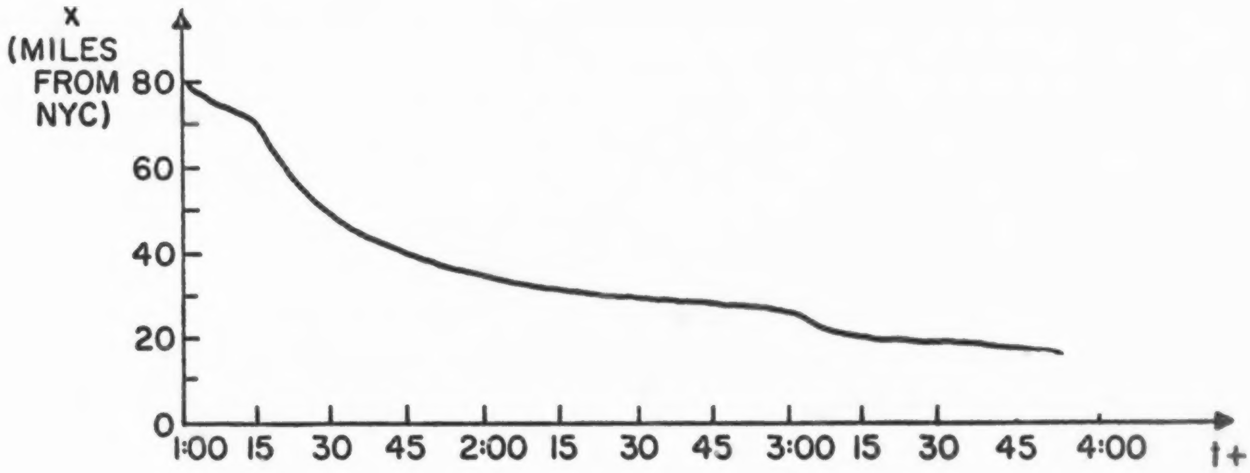
c. Which is least likely to be correct? Why?

4-11 A radar operator in the weather bureau at New York City is tracking an approaching storm. The following graph indicates the distance from New York with respect to time.

a. At 1:15 PM he predicts the possible arrival of the storm at NYC as 3:05. On what did he base this prediction?

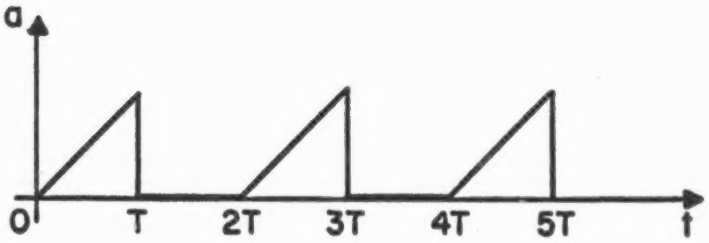
b. If he made his prediction at 1:30 what would the new predicted time be?

c. When he makes a new prediction at 2:20, what will be the new estimated time of arrival of the storm?



4-12 An airplane landing at an airport touches down on the runway at 150 miles per hour and decreases its speed linearly to zero in a thirty-second interval. What must be the minimum runway length for the safe landing of this plane?

4-13 An acceleration signal generated by a spaceman repeatedly opening and closing the jet valve on the gun that propels him is shown below. What do his velocity and displacement curves look like?



4-14 Suppose a body starts from rest with a uniform acceleration of magnitude a. Show that when the body has attained a velocity v, it will have gone a distance d that is related to v by the formula

$$d = \frac{v^2}{2a}$$

(Hint: Sketch the acceleration, velocity, and displacement curves. Derive formulas for velocity and displacement in terms of t. Eliminate t.)

4-15 One important application of finding areas under curves (i.e., integration) used frequently by engineers is the determination of the average value of some signal of interest. By definition, the average value of any "signal s" (denoted $\bar{s}$) is:

$$\overline{s} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s dt$$

This equation says that the average value of s equals the area under the curve of s versus t, in the interval between $t = t_1$ and t_2, divided by the length of the interval, $t_2 - t_1$. (This is a generalization of the averaging technique which you have learned in your mathematical course.)

The current in a Geiger-Mueller tube is used to monitor the level of radiation coming through the shielding around the core of a nuclear reactor. This current varies with time as shown in the table below. We wish to determine the average value of this current in order to determine the average radiation level. What is this average current? (It is advisable to plot the given data to help in the calculation.)

t(seconds)	current (microamperes)
0	66
1	80
2	76
3	78
4	80
5	71
6	59
7	50
8	53
9	55
10	52

Note: the ampere is the internationally accepted unit of current.
A microampere is one millionth of an ampere.

4-16 Just before touchdown a plane is moving at 160 mi/hr. The plane uses 1.5 miles of runway to stop.

a. How many seconds does it take to stop the plane? (Assume uniform acceleration).

b. What was the average acceleration in mi/hr · sec. or ft/sec^2 of the plane?

(See problem 4-15 for definition of "average").

4-17 A train moving across the country passes a series of telegraph poles, equally spaced 30 meters apart. A passenger on the train amuses himself by determining the average number of poles he passes per second at 15 minute intervals. The table displays his observations.

Time	Average number of poles passed
10.00	1.0 poles per second
10.15	1.3 " " "
10.30	1.1 " " "
10.45	1.0 " " "
11.00	1.2 " " "
11.15	1.4 " " "
11.30	1.2 " " "
11.45	1.0 " " "

a. Calculate and graph his average velocity at each instant of observation.

b. What was the maximum acceleration achieved by the train?

c. From the velocity versus time graph, determine how far the train moved between 10:15 and 11:00 A.M.

d. What was the average velocity of the train between 10:00 a.m. and 11:00 a.m.?

(See problem 4-15 for definition of "average".)

4-18 A baseball moving at 60 feet per second toward a batter is hit in such a way by the batter that after the hit it is moving at 80 feet per second in the opposite direction. The bat-ball interaction time is about 2×10^{-2} seconds.

a. What was the change in the velocity of the ball?

b. What was the average acceleration of the ball in ft/sec^2?

c. If the acceleration was constant, what was the ball's velocity 1.0×10^{-2} seconds after the interaction started?

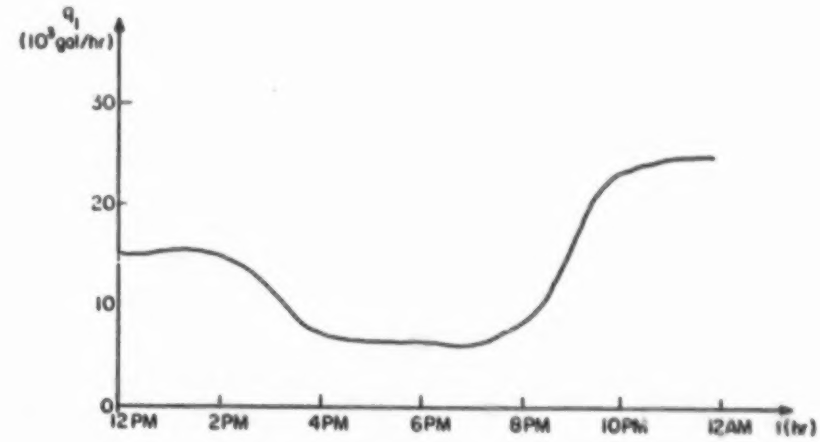
d. After the hit, how long would it take the ball to travel back to the pitcher's mound (60.5 ft).

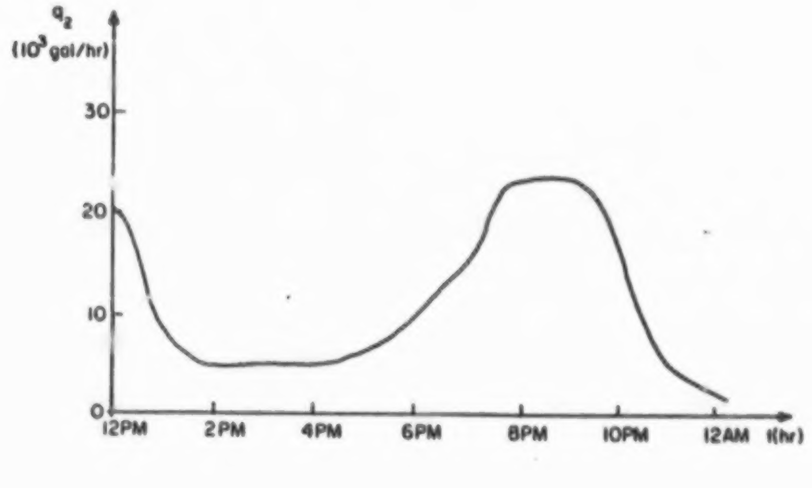
(See problem 4-15 for definition of "average".)

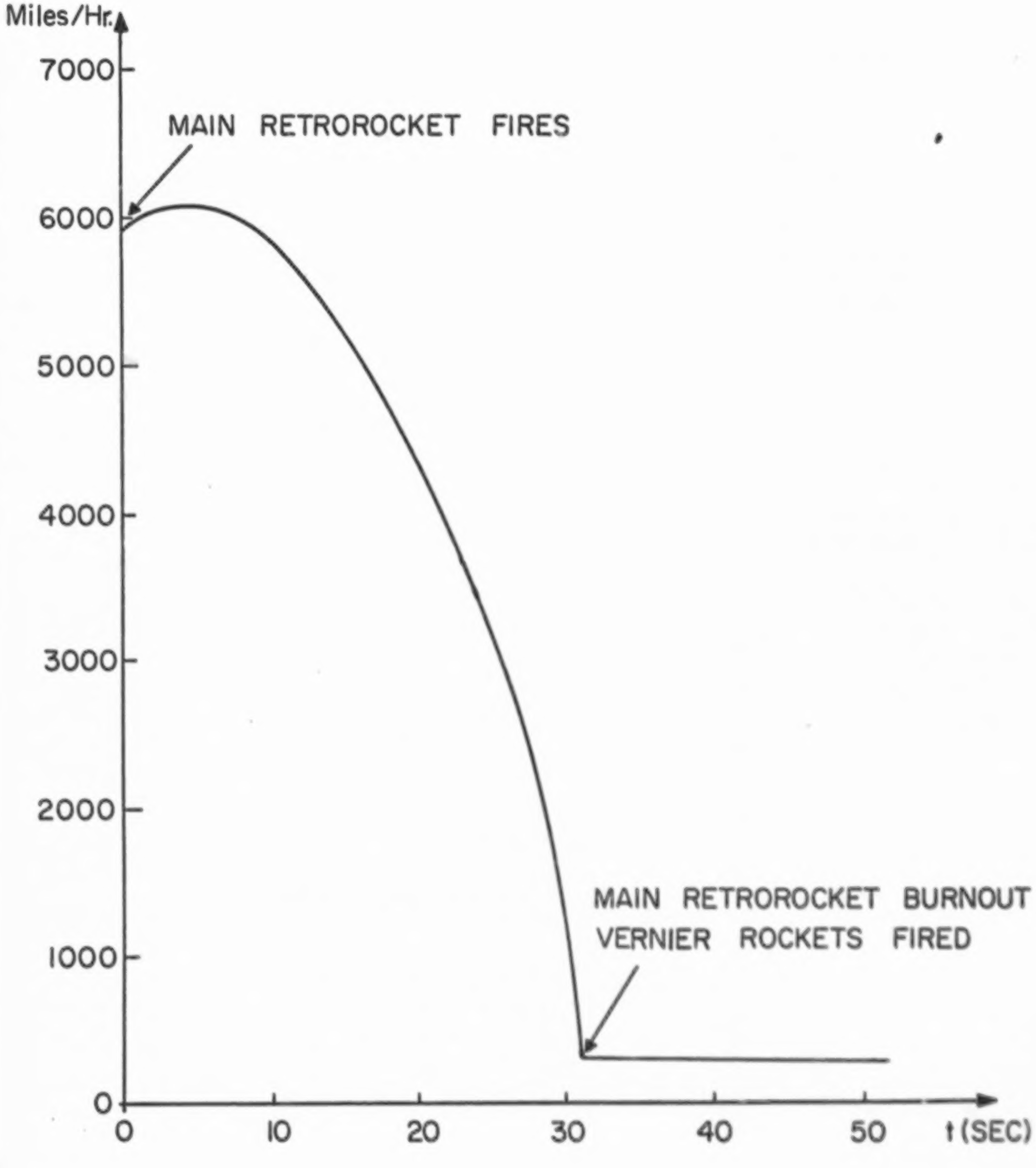
4-19 Assume that all of the turnstiles in a ballpark have counters which are connected to a central device which gives a reading of p, the number of people per minute entering the park. At noon the reading is zero when the gates are opened. The value of p increases linearly (i.e., as a straight line) from zero to 400 people per minute in the first half-hour, remains at 400 for one-half hour, and then drops linearly to zero by 2:00 p.m., thirty minutes after the game has started. (a) What was the total attendence? (b) How many people missed the start of the game?

4-20 In June of 1966, Surveyor I landed softly on the surface of Earth's moon and began taking a historic series of photographs. The velocity of Surveyor, as it approached the lunar surface, was telemetered to tracking stations on Earth. This velocity varied as shown in the figure on page B-4.36. The main retrorocket fired at an instant we have defined at $t = 0$. At this time, Surveyor was 52 miles above the lunar surface. (a) What was its altitude when the main retro-rocket burned out (at $t = 30$ seconds)? (b) What was its altitude at $t = 50$ seconds?

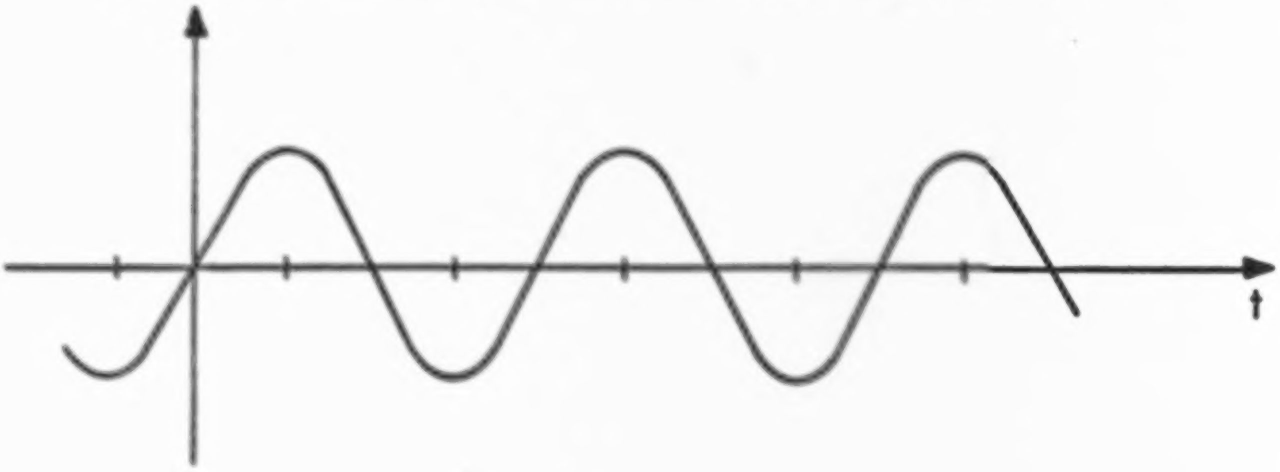
4-21 Dix Hills, New York, stores water for its residents in a large elevated storage tank. Water is poured into the tank from underground wells to replenish the supply as it is used. This added water flows in at a rate q_1. The residents drain off water from the tank at a rate of q_2. If q_1 and q_2 (in gallons per hour) vary as shown in the figure, determine the volume of water in the tank at 10 P. M. if the volume at noon (i. e., the initial volume) is 16,000 gallons.






Miles/Hr.
7000
6000
5000
4000
3000
2000
1000
0
MAIN RETROROCKET FIRES
MAIN RETROROCKET BURNOUT
VERNIER ROCKETS FIRED
0
10
20
30
40
50
t (SEC)

4-22 Shown in the figure is a velocity curve that represents the motion of many physical systems, some being the motion of a pendulum, the swaying of a bridge, and the movement of electronic charges in the lamp on your desk. Determine the displacement and acceleration curves associated with this velocity curve. What is the most significant comment you can make regarding your results?



Chapter-B 5

PATTERNS OF CHANGE

1. THE IMPORTANCE OF CHANGE

For a dull evening, nothing surpasses being paired with an individual who can talk about one subject and without any original or provocative ideas on that topic. The danger of falling asleep while driving is especially great on a modern thruway with little traffic, only gradual curves, and no billboards to arouse our interest. Every college student is painfully conscious of the effect created by a teacher who lectures (with great authority, perhaps) in a monotone--his voice never raised or lowered, and with only an occasional pause for breath. Finally, no one is anxious to live in a community devoid of change; a town where no roads or parks are built, where there is no change in inhabitants, and where nothing "exciting" ever happens. In spite of the occasional, whimsical plea of the older generation for a return to "the good old days," very few people really want a reversion to a lower average standard of living, a shorter life expectancy, less personal mobility, disenfranchisement of women, and so forth.

Change is essential in a interesting, exciting, and challenging life; the promise of change in the future stimulates and drives the individual to personal accomplishement.* In the same way, situations of interest in applied science are normally characterized by change. The interesting problems in the last two chapters are those in which we wish to calculate the position of the vehicle when the speed is changing according to a given curve, or in which the rate of population growth is changing because of the discovery of life-saving medicines or new and improved health measures.

Indeed, in science we can even go further and state that the really interesting situations are those in which change occurs and in which the change is unpredictable. This feature of unpredictability adds enormously to the depth of interest. For example, no one watches a television screen very long when a test pattern is constantly portrayed. In a broader sense, no one would watch TV if he could predict precisely what pictures would be shown for the entire program duration. Interest is aroused because the future changes are uncertain, unpredictable.**

The engineer even goes further than the above general statements and states that the amount of information contained in a message depends on the probability.

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*We are not saying, of course, that change is always desirable, particularly when it may occur much too rapidly. One frequently wishes certain experiences could linger on and on, that the passage of time could be slowed.

**There is an interesting example of the importance of unpredictability. Television stations frequently broadcast sports events (football games, prize fights, etc.) a day or several hours after the actual occurrence. If one has learned the results from the newspapers or news broadcasts, interest is sharply diminished; the more detailed knowledge one has, the less the interest in the delayed telecast.

of that message. As an example of how information and probability are related, we consider a message or signal which consists of a sequence of binary numbers

0 1 1 0 0 1 0 1 0 0 0 1

Each number can have either of two possible values, 0 and 1. If the two values are equally probable, each number is said to carry one bit of information. A sequence of twelve numbers represents 12 bits of information.

If we are using an alphabet with more than two symbols (e.g., the English alphabet of 26 letters plus a space, punctuation marks, and numbers), each letter in the message carries an amount of information which depends on the probability of that letter occurring: the smaller the probability, the greater the information. Thus, in a message using the English language, the letter E represents much less information than an X or Z, since E is a common letter (its probability is high).

Actually, we go further and define the information in terms of an algebraic equation. If the probability of a particular letter is p_i, the information represented is

$$\log_2 \left(\frac{1}{p_i}\right)$$

or the logarithm of one over the probability. This particular logarithm is to the base 2; in other words, what power must 2 be raised to in order to give $1/p_i$? For example, if the probability of the letter E is 1/8, the information contained in the E, when it appears, is

$$\log_2 8 = 3 \text{ bits}$$

since 2 must be raised to the power 3 (i.e., cubed) to give 8. For other probabilities, we can evaluate $\log_2 \left(\frac{1}{p_i}\right)$ by use of the log tables or a slide rule.*

In other words, the engineer can determine a numerical value for the amount of information per second in typical messages: telephone conversations, television pictures, computer outputs, and so forth. Such a measure of information permits the system designer to determine what type of equipment is required to transmit the signal, whether the equipment is being used economically, and how the quality of the communication system can be improved.

In this and the next chapter, we are not, however, interested in quantitative measures of information. Rather, our interest focusses on changing signals: signals in which the nature of the change is the primary feature of interest. In particular, we wish to consider a few of the types of change which occur most often in the man-made world, how these changing signals can be used to understand system operation, and finally some of the ways in which systems can be designed to control these changing signals.

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*For calculation purposes, it is useful to know that

$$\log_{10} X_{10} = 3.32 \log_2 X$$

since tables are most of ten given in terms of the logarithm to the base 10.

Problem

In writing or speaking, we ordinarily do not want to maximize the information content. For example, if this book contained a maximum amount of information per page, every letter, word, sentence, figure and so forth should come as a surprise to the reader. Obviously such a book would be totally unreadable and incomprehensible; any reasonable book reads smoothly as the reader is led gently from one thought into the next. Such a smooth transition can be achieved only by a large amount of redundancy--extra words and sentences to lead gradually into a new idea.

The redundancy of the English language can be illustrated very simply as follows. Write a logical sentence of at least ten words. For each letter (one by one), flip a coin. If the coin comes up heads leave the letter untouched; if tails, erase the letter. Approximately half the letters are now erased. Now give the sentence with erased letters to a friend and ask him to fill in as many of the letters as possible.

The usual success in completing the sentence demonstrates that, even in a simple sentence, half the letters are really unnecessary. In a paragraph or page, even more letters could be omitted, and indeed in a full novel or a textbook such as this, a few pages missing would normally not be particularly serious.

2. PREDICTION

In the preceding pages of this chapter, we mention a variety of systems or situations in which the most important characteristic is that a signal of particular interest changes as time progresses.* We call such a situation a dynamic system.

For example, the structure that is the Empire State Building is a dynamic system: when it is hit by high winds, the building oscillates, with the top waving back and forth over a distance of several inches. In this case, we can determine a model of the system (the building) by observing the motion during a known wind; from this model, we can then estimate the motion to be expected in case of much stronger winds which occur only very infrequently. In other words, we do not have to wait until we have tested the building with every possible wind before deciding it is safe for occupancy.

Very often we can use the signals from dynamic systems even more simply--for example, in order to predict the system signals into the future. The population discussion of Chapter B-3 is an example: there we use the population curve to predict the world's population a decade or more into the future. This use of the signals of dynamic systems for prediction is so important in modern technology, we devote this section to two specific examples, which are particularly important in this chapter because of our overall concern with changing signals.

Example 1: U.S. solid waste

If the United States is considered as one vast system (containing as elements the manufacturing plants, the transportation vehicles, the people, and the natural and man-made devices), one of the important signals which can be measured is the solid waste. Such solid waste material includes all the organic and inorganic material which we throw away: the six million cars which are scrapped every year, the appliances discarded, the refuse from construction and building demolition, and the garbage generated by individuals. The United States, with less than 10% of the world's population, creates appreciably more than half of the world's rubbish. Our highly advanced technology and the associated high standard of living lead to a national problem of increasingly serious magnitude: how can we dispose of this solid waste economically and without dangerously fouling the environment.

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*Throughout the remainder of the book, we use the term signal to refer to any variable of particular interest--e.g., the speed of a car or the pressure in a sound wave. The signals of interest are those which vary as time elapses. Thus, a signal is the time variation of a physical quantity.

The magnitude of the problem is vividly portrayed in Fig. 1,* which shows the quantity of solid waste produced per year in the United States since 1920. The significance of this particular signal is perhaps clearer if we note that in 1965 more than four pounds were produced each day for each person in the country. Furthermore, the rate of increase is appreciably greater than the rate of population increase.

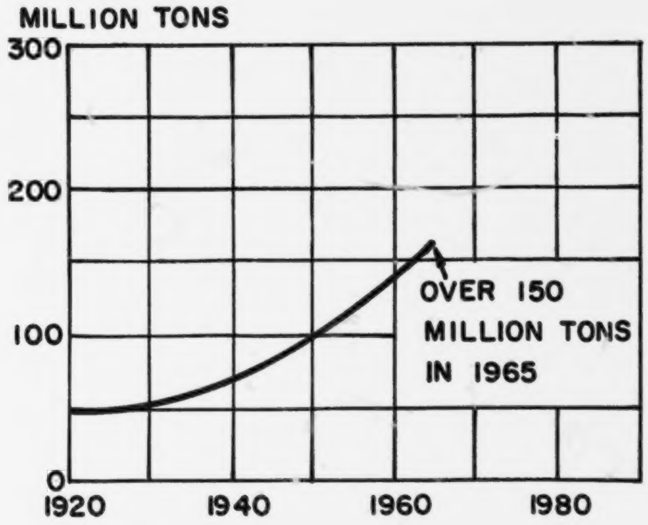

Fig. 1 U.S. solid waste

The importance and urgency of the problem derive from two principal factors:

(1) In most cities, available land for dumps is being rapidly exhausted. At the same time, the nature of the solid waste is changing: a few decades ago, the rubbish was primarily garbage and ashes; today it includes vast quantities of metals, plastics (e.g., non-returnable containers), and other new products, many of which can not be economically burned without contributing to air pollution.

(2) We know very little about the effects of environmental pollution on the physical and mental health of the individual. To what extent are mounds of junked automobiles or polluted air, and the corresponding changes in man's environment, responsible for the increases observed in mental illness, in urban unrest, and in such physical illnesses as lung cancer. Even data such as shown in Fig. 2 are not

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* Figures 1, 2, and 3 are taken from the report, "A Strategy for a Livable Environment," published for the U.S. Department of Health, Education, and Welfare in June, 1967.

easily interpreted since individuals living in the city may smoke more heavily, may lead lives under greater nervous tension, and so forth. The problem of evaluating the importance of environmental pollution is further complicated by the realization that major effects on the balances of nature and the characteristics of man are unlikely to become evident for a generation or more (when it may well be too late to reverse the established trends).

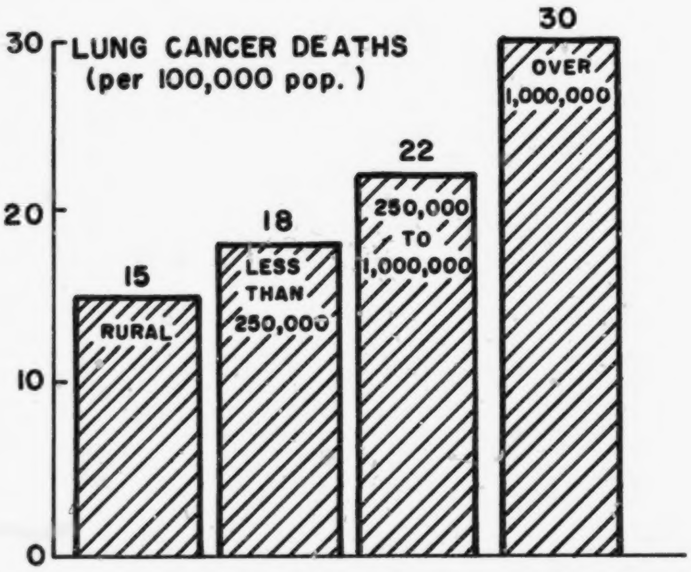

Fig. 2 Annual deaths from lung cancer as a function of the size of the community.

Any logical approach to a national attack on this problem of solid waste disposal requires that we predict the future extent of the problem Figure 1 shows the past history of the system: this particular output signal as a function of time over the years 1920 to 1965. In order to use the data of Fig. 1 to predict the system signal at least a few years in advance. This need for prediction arises for two reasons:

(1) Data are usually available only some time after they are valid (in problems of this broad a nature, a year or two may be required). Thus, the curve of Fig. 1 runs only to 1965, even though it was published nearly two years later.

(2) Design and construction of the facility require several years. A system for which planning is started in 1968 may not be operational until 1974; if properly designed, it should be useful for at least six years thereafter, so that the 1968 planning must be based upon the needs of 1980. *

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* This development lag (the time between conception of the idea and operation of the facility) plagues transportation planners. In the typical U.S. city, the public and its political representatives commonly refuse to accept a new airport (for example) until existing airports are hopelessly snarled. With the typical seven-year lag between authorization of a new airport and operation, and the concurrent growth in air travel, the air transportation system is hopelessly overloaded. By the end of the seven-year interval and the opening of the new facility, we are again close to overloading and a repeat of the entire cycle.

The data of Fig. 1 can be used in two different ways to predict the solid waste production in the future. First, we can merely extend graphically the curve, as shown by the dashed portion of Fig. 3. According to this prediction, there will be 250 million tons of solid waste produced per year by 1980.

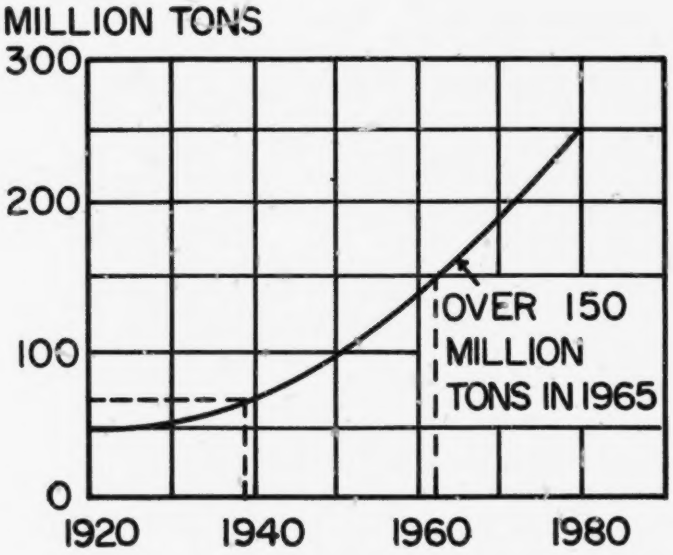

Fig. 3 Graphical prediction of U.S. solid waste production.

Instead of a graphical approach, we can attempt an algebraic or analytical prediction. We notice from Fig. 4 (which is just Fig. 1 redrawn here) that from 1950 to 1962 the production increased by 50% (from 100 to 150 million tons). Similarly, in the 12 years from 1938 to 1950 the increase was 50% (from 67 to 100). It seems that every 12 years the production increases by 50%.

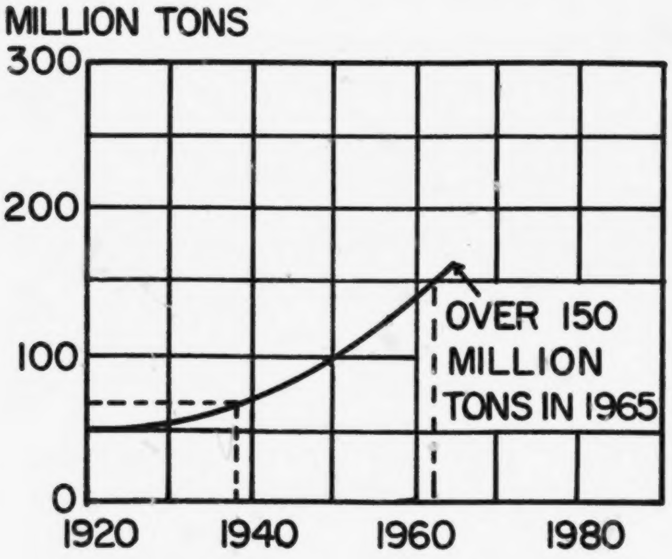

Fig. 4 U.S. solid waste production

This property (that the multiplying factor is the same for every time interval of equal length) is the characteristic which defines the algebraic curve or function called the exponential. Thus, we know that, if we let y represent the solid waste production, we can describe the curve of Fig. 4 by the equation

$$y = A\left(\frac{3}{2}\right)^{t/12} \qquad (1)$$

where A is adjusted to give the correct value in any one year.* If, for example, we choose 1950 as the time when $t = 0$ (i.e., we measure time from 1950), A is 100 and

$$y = 100 \left(\frac{3}{2}\right)^{t/12} \qquad (2)$$

The fact that this equation is valid can be demonstrated by consideration of the three specific years 1938, 1950, and 1962:

1950 Here $t = 0$ $y = 100 \left(\frac{3}{2}\right)^0 = 100$

1938 Here $t = -12$ $y = 100 \left(\frac{3}{2}\right)^{-1} = \frac{100}{3/2} = 67$

1962 Here $t = +12$ $y = 100 \left(\frac{3}{2}\right)^1 = 150$

Equation (2) is an analytical or algebraic description of the data of Fig. 4. This relationship can be used to predict solid waste production in the future. For example, the equation states that in 1980 ($t = 30$ since time is measured from 1950), production will be

$$y = 100 \left(\frac{3}{2}\right)^{30/12} = 100 \left(\frac{3}{2}\right)^{2.5} = 100 \left(\frac{3}{2}\right)^2 \left(\frac{3}{2}\right)^{1/2} = 280 \text{ million tons.}$$

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* In very general terms, an exponential is characterized by the fact that the fractional change over a given time interval is the same, regardless of when that time interval occurs. Thus, the exponential is described by three parameters

The time interval (we call this T)

The factor of change in that time interval (let us call this a)

The value at any one time

In these terms, the mathematical expression for the exponential is

$$A\,(a)^{t/T}$$

As an example, Professor Smith's total savings double every seven years. His savings S are then represented by the equation

$$S = A\,(2)^{t/7}$$

The quantity A can be found if we know his savings at any one time. For example, at age 42 ($t = 42$) he had saved $20,000. Then substitution gives

$$20,000 = A\,(2)^6$$

$$20,000 = A\,(64)$$

$$A = 312.5$$

At any age t, his savings S are given by the equation

$$S = 312.5\,(2)^{t/7}$$

Why are the two answers different: the 250 predicted graphically and the 280 predicted algebraically? In both cases, we have made a guess. When we worked graphically, we tried to extend the curve beyond the region for which data are available. We attempted to find a smooth extension which continued past trends, but clearly several different extensions are possible. The further into the future we go, the larger the error is apt to be. When we worked algebraically (and found the equation), we assumed that the given data can be represented by an exponential -- an assumption which is only an approximation.

Which of the two answers should be used? This question can not be answered, but one might conjecture that, if past trends continue, the solid waste production will fall somewhere between 250 and 280 million tons annually by 1980. Certainly unless specific industrial and governmental measures are taken to modify these trends, we should plan a disposal system for 1980 which can accomodate at least 250 million tons annually.

Example 2: Evolution of a new product

Example 1 illustrates the use of a system signal for prediction merely by extending the signal into the future. We can also use system signals in a somewhat different manner: if we know the dynamic behavior of the system in the past, we can predict the behavior in the future under similar circumstances.

Figure 5 shows the way in which profit or loss vary with time after an electronics company decides to produce a new product (for example, an electronic instrument for automatic measurement and control of the level of anesthesia during an operation in a hospital).* During the very early stages there are research costs as the feasibility of the instrument is investigated and the possible future market evaluated. Next come the development costs associated with conversion of the research idea into a device which can be used by typical engineers

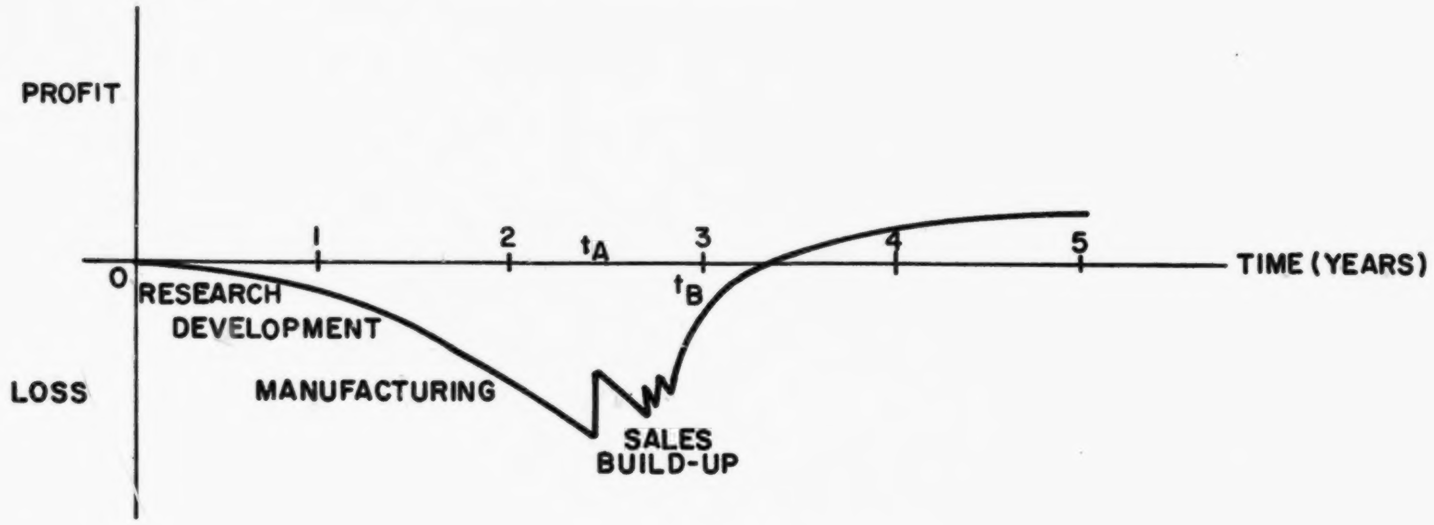

Fig. 5 Profit-loss history of a new product.

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* Quite similar curves describe many other business operations. For instance, when a new restaurant is opened, there is a long period of loss during construction and in the early states of operation while a clientele is being developed. Several years may pass before the total operation begins to show a net profit.

or technicians, which is sufficiently reliable and dependable, and which can be manufactured at a cost which permits an ultimate profit.

When the decision is made to manufacture the product, a major increase in investment is required. Not only must raw materials be purchased, but manufacturing facilities must be acquired, detailed plans made for the equipment, production workers trained, operating and maintenance instruction manuals written and printed,* and sales personnel trained. During this period there are still no sales to yield income to offset, at least partially, the rapidly increasing investment.

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* Such manuals institute a major output of the American printing industry. For example, for one major military system, the various manuals total 1,500,000 pages--the equivalent of 5000 three-hundred-page books, each replete with figures, charts, and tables.

Finally, at time t_A in Fig. 5 the first sales are made, but typically it is some time later before sales build up to the point where there is any noticeable decrease in the total financial investment in the product. If the product is succesful, sales then grow rapidly during a period when additional costs are primarily steady, representing raw materials and manufacturing and sales personnel.

The importance of data such as shown in Fig. 5 is based upon several factors. First, if the company management understands the characteristics of this curve and the factors which are represented, intelligent decisions can be made early in the process. Clearly, the business manager who terminates manufacture of a new product when the total loss reaches a pre-determined amount may be stopping the process just before the rise into the net-profit region. Understanding of the curve for a specific company and a particular type of product permits the manager to detect unanticipated deviations from normal performance (e.g., unusually high costs because of unexpected needs for special equipment).

Furthermore, understanding of the data of Fig. 5 is the basis for logical decisions about where to focus the investment of resources. In today's rapidly changing technology, many products have a life span from conception to absolescence of perhaps five years. In other words, five years after the product is conceived (or invented), an improved version will be available from the company itself or a competitor. If this life span is largely pre-determined by the way in which technology is changing, the importance of minimizing t_B in Fig. 5 is obvious. If this time to full production and sales could be cut from 3 years to 2.5 years, the subsequent life of the product would be increased from 2 to 2.5 years (possibly with a corresponding 25% increase in total profit).**

** An appreciable portion of the delay up to full production results from the problems of establishing manufacturing procedures: how to place the various components within the equipment, how to design to ensure dependable operation, and how to allocate tasks to the various production workers. The complexity and the importance of these problems combine to provide a major economic incentive to develop procedures whereby computers can be used to accomplish major portions of this decision-making. Consequently, we find computers being used more and more in industry to plan the production line, the layout of the equipment, and the organization of various processes of machinery, soldering, and so forth.

Concluding remarks

The two examples illustrate two somewhat different uses of system signals. In the first case, the signal is used directly for prediction -- essentially by extension of the past, known signal into the future. In the second example, the signal is used as a basis for developing an understanding of the system -- and hence for prediction in a new situation (the development of a new product).

In both cases, we use the signals to understand the dynamic system and its behavior. In neither case can we hope to learn all the details about the internal operation of the system; rather we use the signals to attempt to discover gross or general properties of the system. Such understanding then provides the basis for logical design and decision-making.

3. TYPES OF SIGNALS

Signals can be used to describe significant properties of dynamic systems, as we saw in the preceding section. A system is dynamic if the observed signals change with time in a manner which depends on the internal construction of the system. In applied science, we are particularly interested in those dynamic systems in which the signals actually represent motion: the swaying of the Empire State Building in a high wind or the bouncing of a car along a road. In this section, we wish to focus our attention on such motional systems and to investigate what types of signals occur in such cases. As a specific example, we consider the problem of a car bouncing on a bumpy road. We use this system because

(1) It serves as an illustration of many problems involving motion,
(2) We can model the system on a simple analog computer,
(3) We can make measurements in the laboratory.
(4) In the next chapter we wish to consider similar problems in greater detail.

Model for the automobile ride

To describe the frequently-encountered phenomenon of back-and-forth (or oscillatory) motion, we start with the problem of finding a suitable model of an automobile suspension system. The model must be simple, so that we can analyze it, understand it, and deal with it quantitatively. The model should include the essential features of the problem, so that no effect of crucial importance is omitted. An acceptable model should describe the motion of the passengers in the car under various conditions of operation.

The automobile can display various types of motion. For example, a passenger can be jostled up and down, or from side to side. In an actual road situation, all of these must, of course, be given consideration. But because we are attempting to discover how such motions come about in the simplest possible fashion, let us single out only one of these motions for further study: namely, the up-and-down motion.

What is the origin of the forces which give the mass of the car the motion that can make the passenger feel uncomfortable? These forces must somehow enter the model. We consider a wheel of the car going over a bump as in Fig. 6. On a level road at any instant, there is a force, F_1, from the road acting on the tire (this force is necessary to balance the gravitation force acting on the car). The axle of the wheel is then at a distance, d_1, above the road. When the car

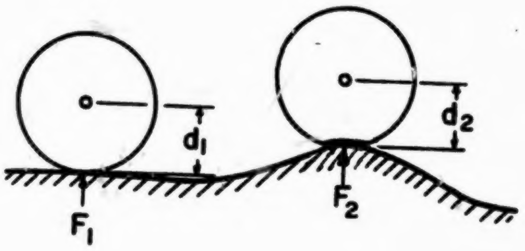


Fig. 6 Forces on a tire going over a bump.

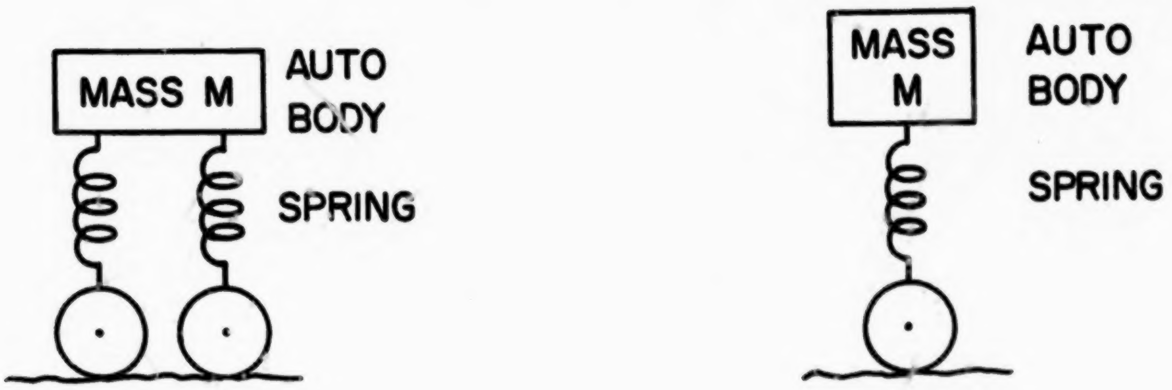
passes over a bump, the tire is flattened to some extent. As a result, the distance d_2 between the axle and the road is smaller than d_1, and the elevation of the axle is higher.

Not only is the tire compressed, but, with reasonable sized road bumps, the spring which connects the car body to the axle of the wheels may also become slightly compressed. Since this compression of both tire and spring is responsible for the effect which we are anxious to study and to model, let us examine it in greater detail.

With the car motionless on the road, we observe that the car body is in equilibirum under the action of two sets of forces: the force due to gravity acting downwards and the force due to the springs and tires acting upwards. If several people enter the car, the car body again achieves a state of equilibrium but under changed conditions; the greater downward force of gravity, which occurs because of the increased weight of the passengers, produces a compression of the springs and the tires, until the upward force arising from the increased compression of these components becomes equal to the new downward force which arises from this added weight of the passengers. An increase in the compression of the spring and tires is thus equivalent to an increase in the upward force on the car body. Therefore, the bump in the road which momentarily compresses the tires and the springs produces an additional and sudden application of an upward force to the body of the car.

Let us develop a very simple model of the up-and-down (vertical) motion of a car. In Fig. 7(a) the car is reduced to an equivalent bicycle. Its mass is lumped into a rectangle and each of the two rear wheels and springs as well as the two front wheels and springs is replaced by a single wheel and spring. This model can retain the general vibratory motions of the car in the vertical direction, but in addition, rotational motion about an axis perpendicular to the page is also possible (e.g., the back up and the front down). In other words, this model can pitch. The model now differs from the actual car since roll from side to side is no longer possible through action of the springs.

The model of Fig. 7(a) is more complex than it need be since it permits pitching motion. To simplify the model further, we replace it with the model illustrated in Fig. 7(b), where the car now has the appearance of a unicycle. Spring action now produces neither pitch nor roll, but only vibration in a vertical direction.



(a) Equivalent bicycle (b) Equivalent unicycle

Fig. 7 Simplified models of a car suspension system.

We combine the entire mass of the car into the box above the spring and also combine the four tires into the one wheel of the unicycle. But how can we combine the four springs into a single spring? If we test springs made of various materials we find that in all cases, the magnitude of a force(Δf)required to extend or compress a spring from a given length is proportional to the extension or compression (Δd) of the spring. The proportionality factor k is called the spring constant. Thus,

$$\Delta f = k \ \Delta d$$

This fact seems to have been observed first by Robert Hooke in 1678 and is called Hooke's Law.

Let us return to our model of the up-and-down motion of the car. The car actually has four springs, each with the same spring constant k_a. If the car moves up or down, each spring deforms and exerts a force. Thus, if we are to have a unicycle with a mass equal to that of the entire car and with a single spring which can replace the four springs on the car, the unicycle must have a spring which is four times as stiff as each of the car's springs. In other words, the spring must exert four times the force of each of the car springs for a given deflection. Hence, in our model the spring constant must be $4k_a$ (For simplicity, we now call this "total" spring constant K).

We can simplify our model even further. The wheels and tires are not particularly important in the study of the up-and-down motion of the car. As long as we include their "springiness" in the spring of the model, we can eliminate them. Thus our model is no longer a unicycle. It is simply a mass M on a spring K as shown in Fig. 8., where $K = 4k_a$.

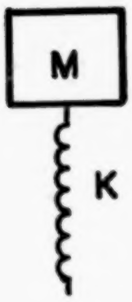


Fig. 8 Simplified model of the up-and-down motion of a car.

What does Fig. 8 have to do with an automobile? It certainly does not have the appearance of an automobile. However, its appearance is not important. It is simply a model of a special feature of the automobile that we wish to study. It is an example of a dynamic model -- a system which we have isolated from a complex situation for special study. It is by no means the complete system with which an automotive engineer must deal, and its value depends on whether or not it helps in understanding and designing an automobile to give a comfortable ride. The model of Fig. 8 may appear to be over-simplified. On the other hand, such bold idealizations lead to progress in apparently unmanageable or complicated situations. For example, we might use such a model to make the first choice of K in designing the tires and springs. The mass M of the car and passengers is approximately known; the designer selects K to give a ride which is comfortable. He then has a general picture of the kind of springs that have to be used in the detailed design.

We have idealized the combination of actual springs in the car, the "springiness" of the tires, and the fact that there are four tires and springs; all of these effects have been modeled by a single spring which is characterized by a single numbe K, the so-called spring constant. If we measure force versus distance for an actual car, we find that Hooke's Law is not followed precisely; fortunately, however, the approximation does permit us to analyse the system and to obtain quantitative relation ships among the various factors entering into the design.

Signal generated by this model

To begin our quantitative study of the model, let us draw the model with reference to a coordinate system, as shown in Fig. 9. We have assumed that the shape of the road has been given in the form of distance above the horizontal (y_r) as a function of the distance x along the road.

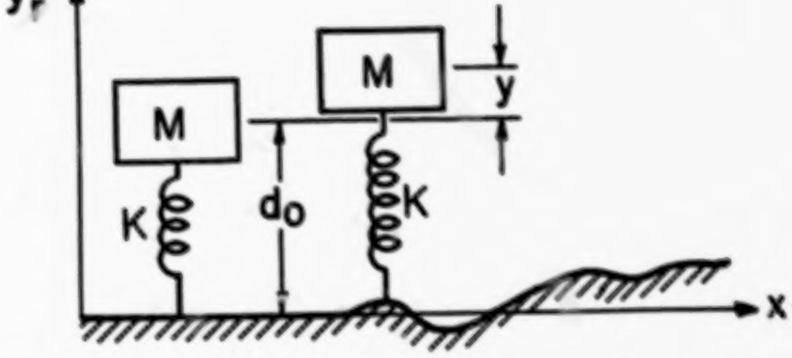


Fig. 9 Model of suspension system showing spring and coordinate system.

The distance d_o is the rest or the static position of the mass M (automobile body) above the horizontal when the car is stationary. The displacement of the body from that rest position is labeled y in Fig. 9, and it is the fluctuation in y that the passengers feel when they are riding in the car. Thus, the automobile designer is interested in the behavior of y over a variety of road shapes.

Let us examine what happens in a particularly simple situation; namely, a flat smooth road with only one bump, as shown in Fig. 10. We assume that the model car moves to the right at a uniform speed and passes over the bump. The resultant displacement of the mass is indicated in Fig. 11. (for the moment, we merely state this answer; we are not interested in the proof that it is the correct answer). The mass, or car body, is set into oscillation by the bump, and this oscillation persists for many complete up-and-down cycles. In other words, the oscillatory motion is periodic: it repeats itself agin and again. The time taken to complete one cycle is called the period. This motion is such that each cycle is exactly the same as any other.

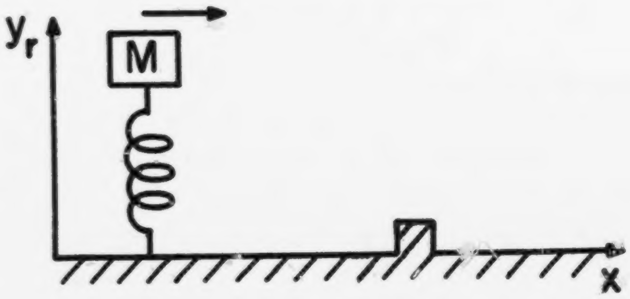


Fig. 10 A simplified road surface.

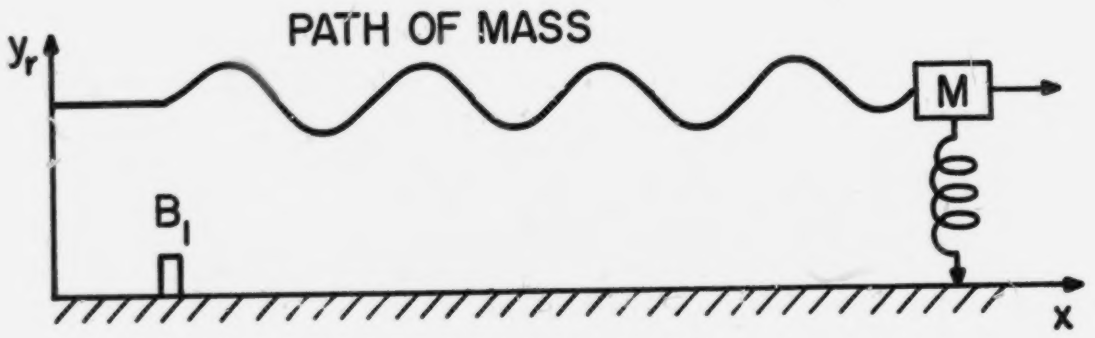


Fig. 11 Vertical displacement caused by motion over a narrow bump B_1.

This particular motion--the periodic back-and-forth or up-and-down motion of a body--is so important in the man-made world that we wish to digress now to discuss the characteristics of this motion in more detail. In the next chapter, we return to the consideration of automobile bouncing system; our purpose in introducing this section in this chapter is, thus, twofold: (1) to introduce the oscillatory signal and show one type of physical system in which it appears, and (2) to provide the background for the analysis of the bouncing motion in the next chapter.

4. SINUSOIDAL SIGNALS

The curve representing the motion of the mass bouncing on a spring (Fig. 11) has a particular significance; namely, it occurs in many situations. For instance, in Fig. 12 such a curve is traced out by the end of a swinging pendulum as paper from a roll is pulled beneath it at a uniform speed.

Basically, this curve results whenever a particular situation exists. The crucial condition for generating this particular motion is that as the mass departs from its rest position, there must be a force tending to restore the mass to its rest position. Furthermore, this restoring force must be proportional to the displacement from the rest position. In the case of the mass-spring system, this force is supplied by a Hooke's Law spring. In the case of the pendulum, the restoring force is supplied by gravity and satisfies the proportional condition provided the amplitude of the oscillation is small.

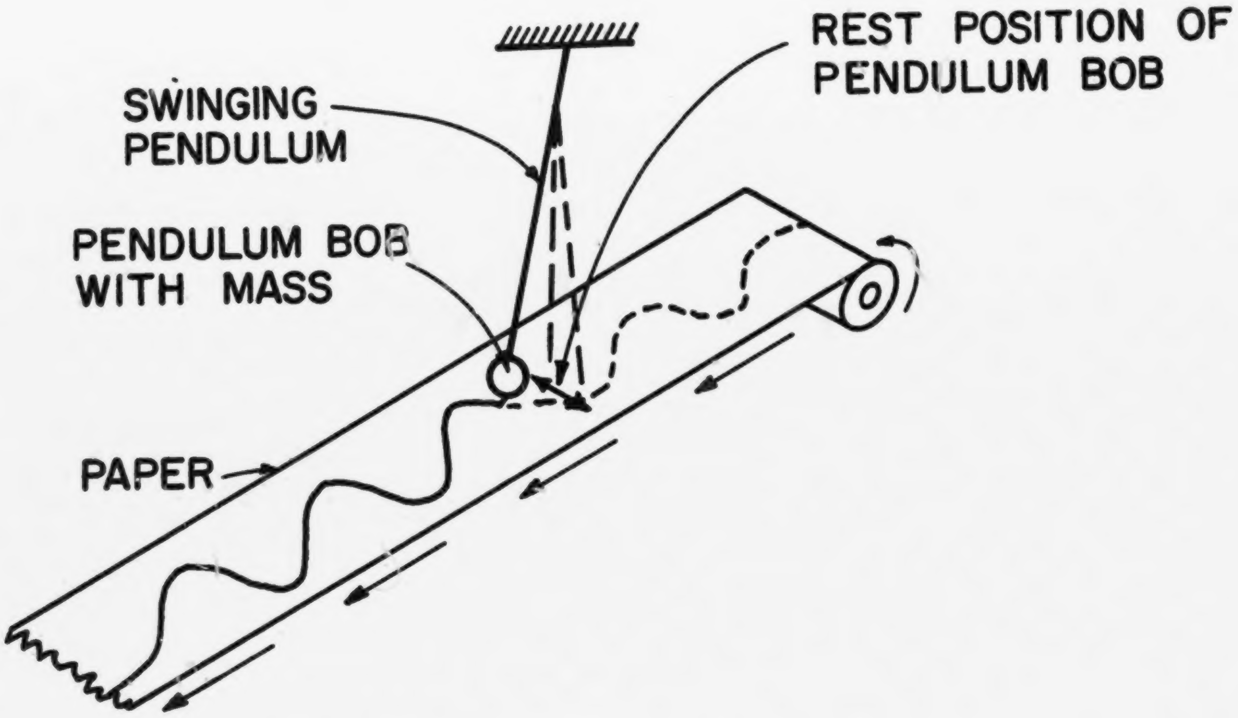


Fig. 12 Tracing out the oscillatory motion of a swinging pendulum.

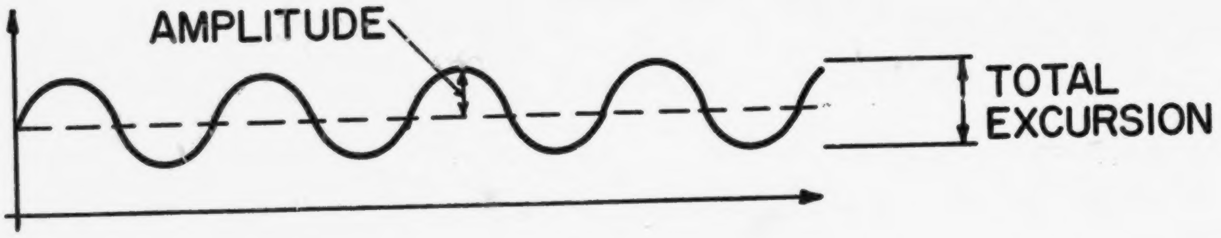


Fig. 13 Definition of the excursion and amplitude of a sine wave.

Because we encounter motion of this sort so often, it has been given a special name, simple harmonic motion, or alternatively, sinusoidal motion. The snaky curve itself is called a sine wave. *

Sine waves have several properties which are particularly important for us. Their size is usually stated as half the total excursion, as indicated in Fig. 13, and is known as the amplitude. Sine waves of various amplitudes are shown in Fig. 14.

The frequency of a sine wave is the number of "up-and-downs" or full cycles completed in one second. To make this clear, we note that, after the mass encounters the bump, it bounces up and down, and we can count the number of oscillations that take place in each second. This number is the frequency of the oscillation.

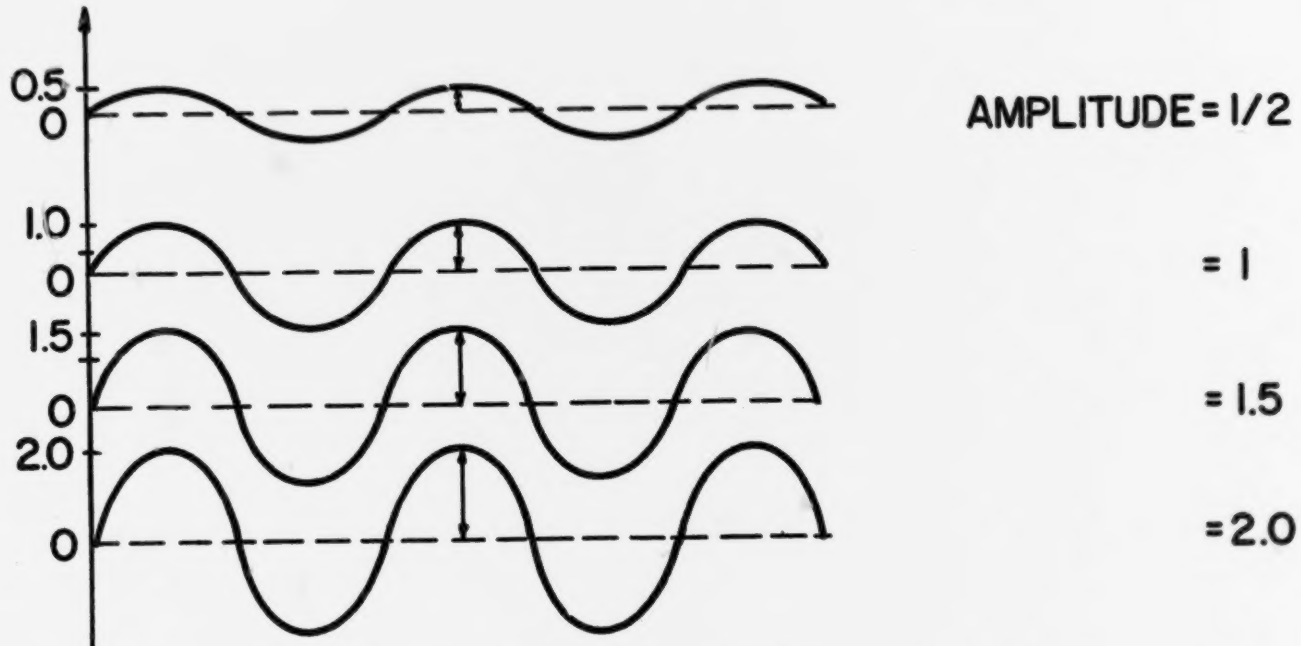


Fig. 14 Sine waves of various amplitudes.

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*Actually, this is called a sine wave because the signal is related to time by the trigonometric sine function. In particular, the signal y (which may be any physical variable -- e.g., velocity, pressure, temperature, voltage) is given by an equation of the form

$$y = A \sin (\beta t + \theta)$$

where $A, \beta,$ and θ are constants (numbers): e.g., $y = 3 \sin (2t + 0)$. For any value of time t, we can then calculate the corresponding value of the signal y by using trigonometric tables. In this equation, both Bt and θ are measured in radians.

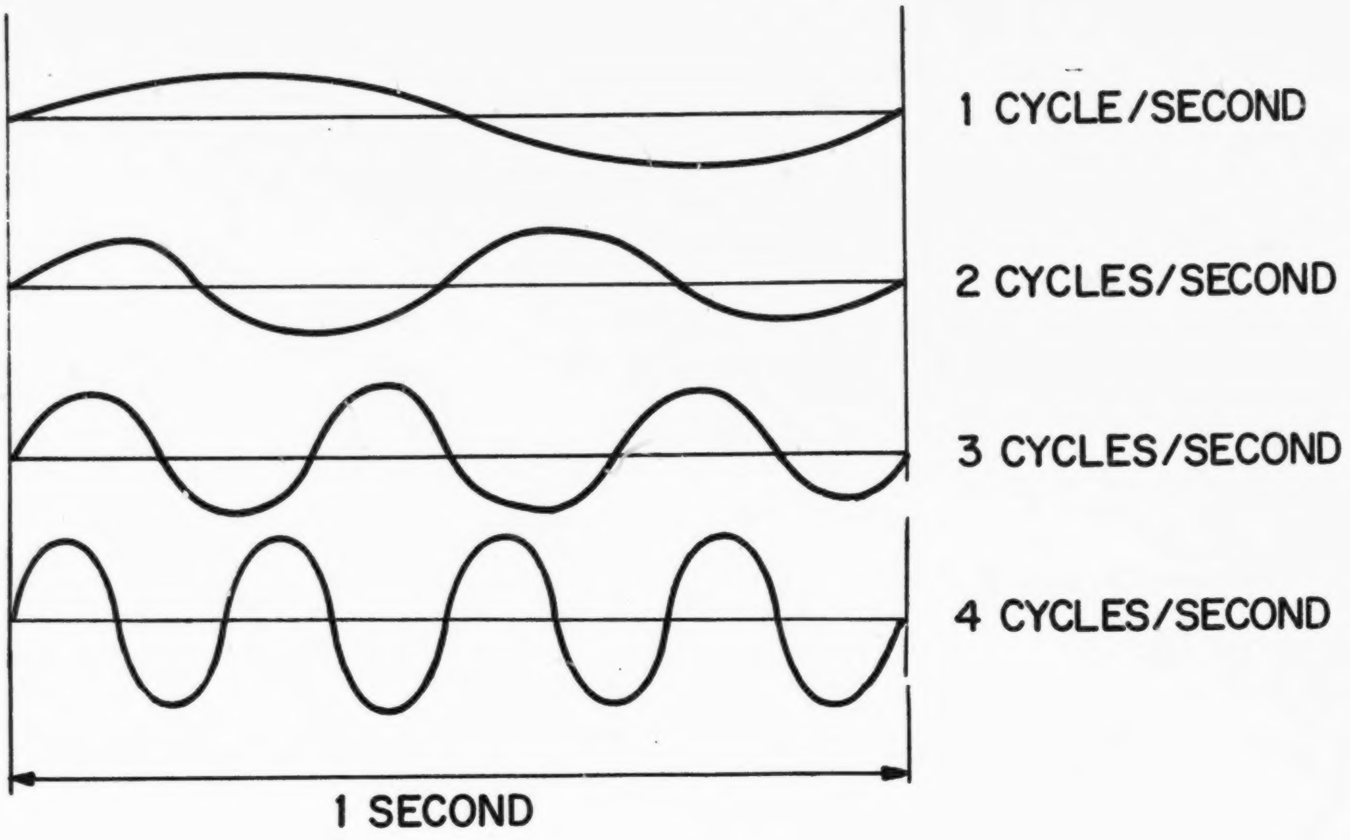


Fig. 15 Sine waves of various frequencies.

Sine waves of various frequencies are shown in Fig. 15. The frequency (denoted by the letter f) can be found by counting the number of cycles or periods in one second (the easiest way to do this is to count the number of crests or troughs).

Sine waves can be combined in a rather simple fashion. In the case of the mass-spring combination, we can consider what would happen if there were two bumps on the road instead of one, as in Fig. 16. If bump B_1 were absent and another bump were present further along the road (Fig. 17), the car motion would be identical to that shown in Fig. 11, except that it would be shifted so that its beginning would coincide with the position of the second bump B_2 as shown in Fig. 17. (If B_1 were larger than B_2 by some factor, say 1.5, the amplitude of the motion resulting from B_1 would be 1.5 times larger). The total motion must be some combination of the motions caused by B_1 and B_2 separately. Indeed the total motion resulting from B_1 and B_2 together is the sum of the individual motions. This sum is shown in Fig. 16.

In this example, it happens that the amplitude of the excursions is larger than that produced by either B_1 or B_2 alone. This is not true for all situations; if B_2 could be moved slightly, we can find a location for B_2 such that the motion ceases altogether. The motion curve then appears as in Fig. 18.

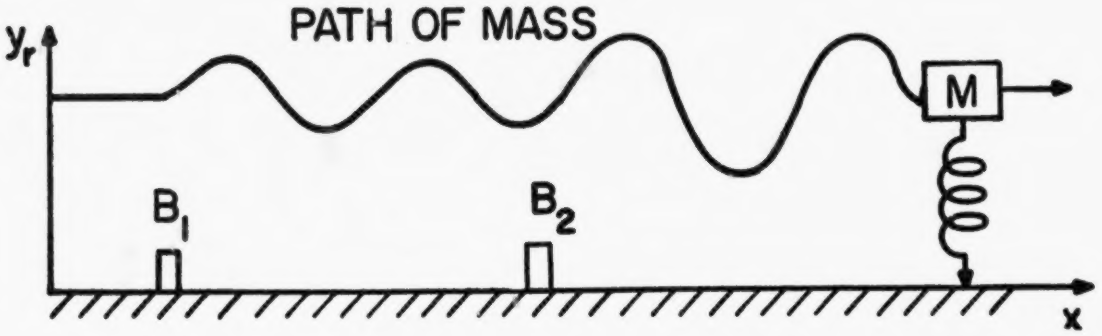

PATH OF MASS
y_r
M
B_1
B_2
x

Fig. 16 Total motion caused by both B_1 and B_2.

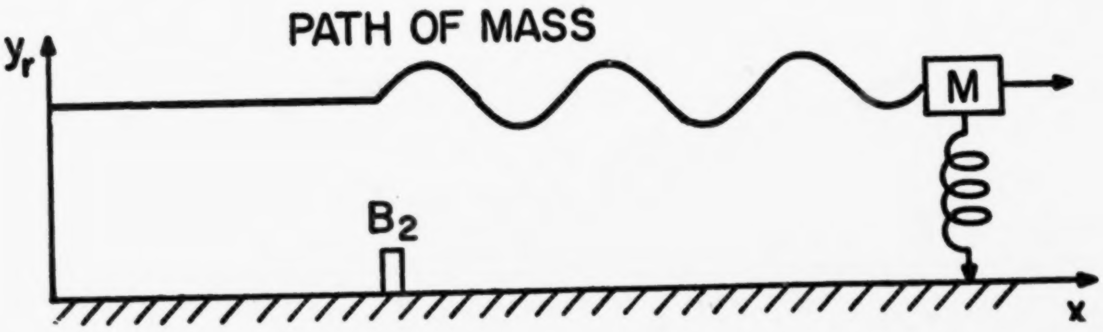

PATH OF MASS
y_r
M
B_2
x

Fig. 17 Motion resulting from bump B_2 alone.

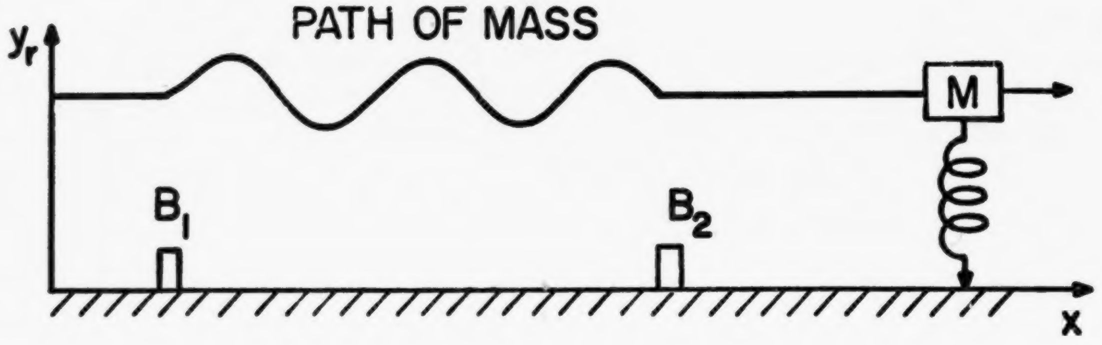
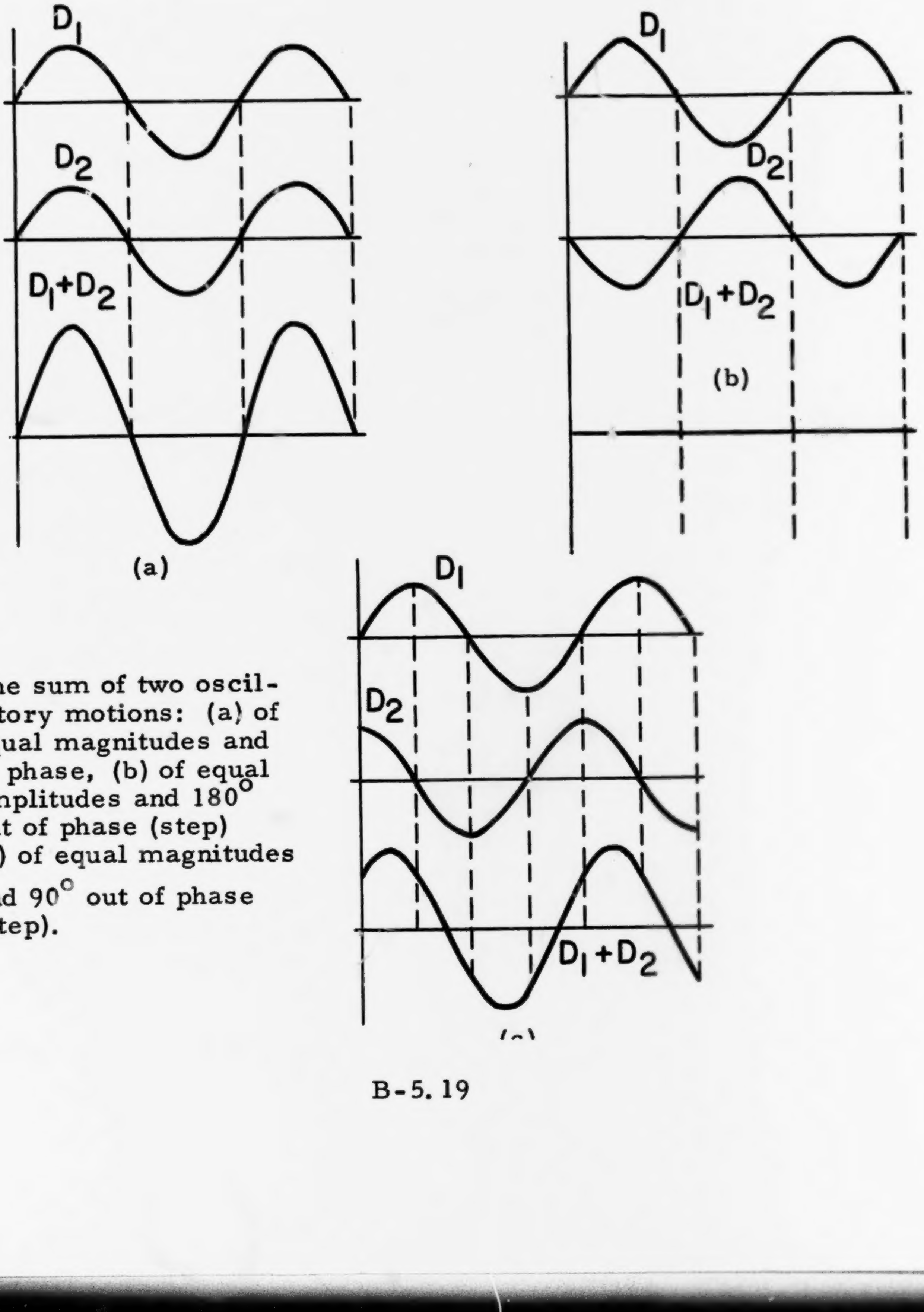

PATH OF MASS
y_r
M
B_1
B_2
x

Fig. 18 Resultant motion ceases when B_1 and B_2 are identical and properly spaced.

Thus, the resultant motion depends on the distance separating B_1 and B_2, as well as on their relative magnitudes. This can be seen clearly in Fig. 19 where two oscillatory motions are added with different relative phases or different times of peak values. If the vibratory motions are in phase (i.e., if they start together), their sum is also oscillatory with the same period, but its amplitude is the sum of the amplitudes of the individual motions. If they are 180° out of phase (i.e., if they begin one-half cycle apart), their sum is zero since they are the negative of each other. If they are 90° out of phase (i.e., if they start a one-quarter cycle apart), their sum is oscillatory with the same frequency or period, and the amplitude is greater than the amplitude of any one but less than the sum of the amplitudes. The sum of two harmonic oscillations, with equal periods but different phases and different amplitudes, is always harmonic and of the same period. In other words, the sum of sine waves of a given frequency is another sine wave of the same frequency, but of different amplitude.

This additive method is valid for finding the total motion no matter how many bumps there are or what may be their relative positions.



(a)

(b)

(c)

Fig. 19 The sum of two oscillatory motions: (a) of equal magnitudes and in phase, (b) of equal amplitudes and 180° out of phase (step) (c) of equal magnitudes and 90° out of phase (step).

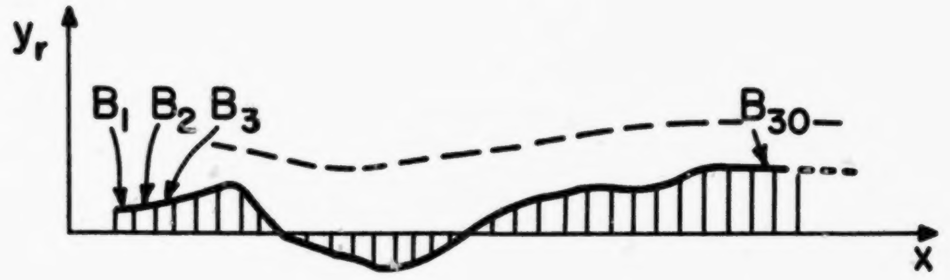


Fig. 20 An illustration of describing any road surface as a series of separate bumps.

We consider any road as if it were made of a series of "elemental" bumps all exactly alike except for height, as indicated in Fig. 20. Thus, the additive property permits us to compute by simple addition the motion for any road surface, once we know the motion for an elemental bump.

In this section, we have introduced the idea of a sinusoidal signal: a signal which varies periodically and is called simple harmonic motion when the signal represents motion,

(1) Occurs when a mass oscillates on a spring or a pendulum oscillates back and forth with small motion or in a wide variety of similar physical situations (e.g., the motion of the top of the Empire State Building in a wind or other examples described in the next chapter).

(2) Represents periodic motion: each cycle is an exact replica of every other cycle.

(3) Has the unusual property that the addition of two sinusoids of the same frequency always results in another sinusoid of the original frequency.

Other origins of sine waves

Actually the sine wave is a signal which occurs again and again in nature or in the man-made portion of the world. Perhaps the most familiar example is in music, where a pure note is a sinusoidal signal (if a sound signal, the sinusoidal variation is in air pressure since transmission of sound corresponds to travel of pressure waves through the air). In this sense, the frequency corresponds to pitch; actually, the pitch is a subjective quality determined by how one reacts to the sound heard, but pitch is determined by the frequency: the higher the frequency, the higher the pitch.

Musical notes are normally not pure sine signals, but usually contain harmonics (additional notes at frequencies which are multiples of the fundamental frequency). For example, the A above middle C is normally about 440 cycles per second; it's second and third harmonics are then 880 and 1320 cycles per second. Typical musical instruments playing middle A have the sine components shown in Fig. 21. The exact frequency of A depends on how the conductor chooses to have the instruments tuned.

A more complex combination of sine signals exists in ordinary speech. Here

Fig. 22 shows the various frequency components which are present when a human being speaks the successive sounds required for the phrase, "Speech we may see." Just as in Fig. 21, the darkened areas represent the frequencies present as a function of time along the horizontal axis. Whereas the musical instruments yield sounds which contain largely the primary note (440 cycles per second) and the harmonics, the speech signal contains (during any one sound) a very large number of sine components--many different frequencies existing at the same time.

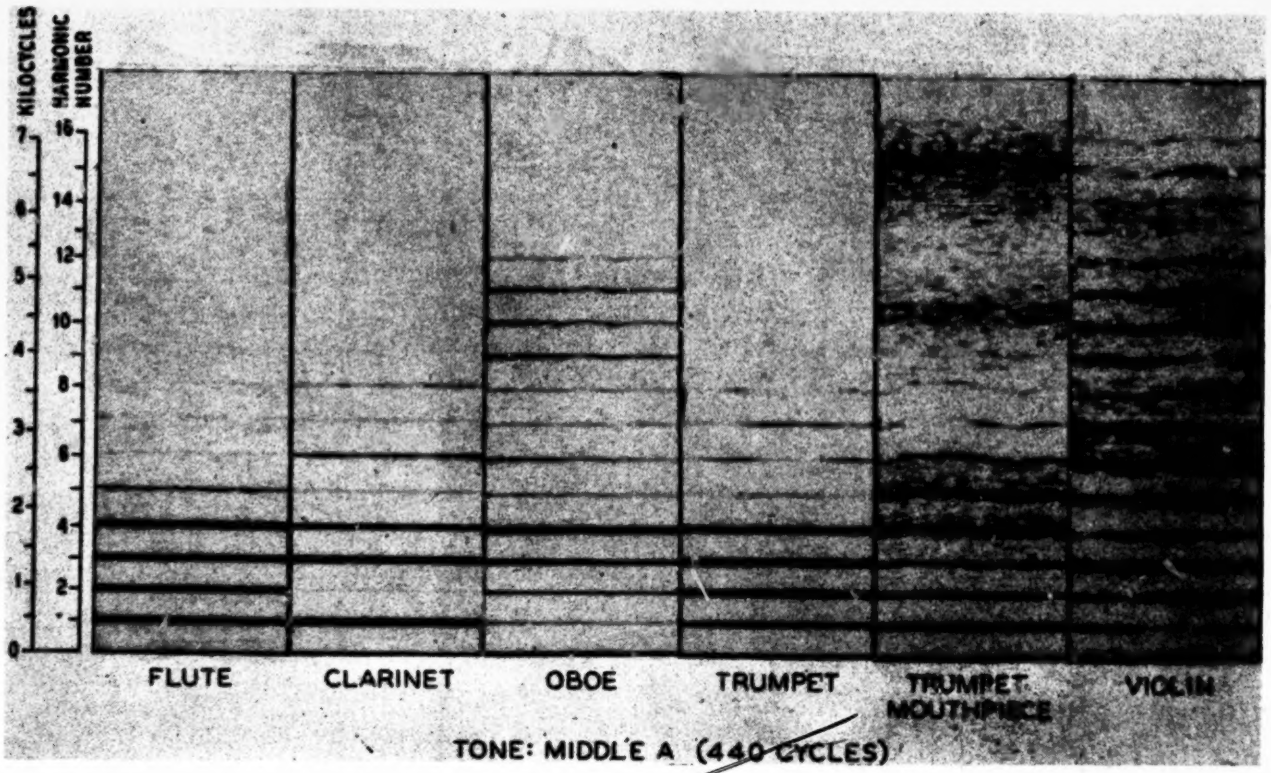


Fig. 21 A spectrogram of the sounds of some common instruments. Depending on their shape, musical instruments emphasize certain harmonics. (Reproduced from W. A. van Bergeijk, J. R J. R. Pierce, and E. E. David, Jr., "Waves and the Ear," Doubleday and Co., Inc., Garden City, N. Y., 1960).

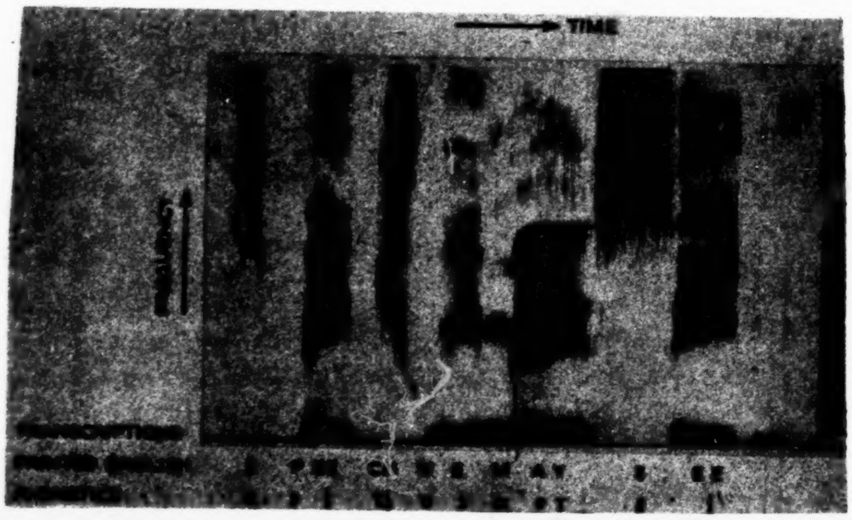


Fig. 22 A spectrogram of human speech, depicting the frequency distribution of speech power for successive time intervals. (Reproduced from W. A. van Bergeijk, J. R. Pierce, and E. E. David, Jr., "Waves and the Ear," Doubleday and Co., Inc., Garden City, N. Y., 1960).

Fig. 22 illustrates the basic idea that any signal can be represented as the sum of a group of sine signals of different frequencies.* In many actual cases, the number of different components required to represent the actual signal is so large that no one would ever attempt to determine the amplitude and frequency of each component. Knowledge that such a decomposition can be made, however, is a major help in the design of communication and control equipment (as we see in more detail in the next chapter).

5. SIGNALS RELATED TO SINUSOIDS

The sinusoid is a particularly simple signal to generate and to analyse; in the next chapter we consider various systems in which sine signals (or close approximations) occur during normal operation. This particular type of signal is also significant because many signals of importance are derived from sinusoids.

As we noted previously, both speech and music are sums of sinusoidal signals for a short period of time. For example, a particular speech sound can be analysed or decomposed into a set of sinusoids lasting for the duration of the sound. The radio and television signals broadcast by commercial stations can be represented as sums of various sinusoids.

These particular examples are rather complex because speech (for example) involves a continually changing pattern of sounds. We can illustrate the importance of the sinusoid by considering other, simpler signals. In the following paragraphs, we consider radar and sonar as two important, yet relatively simple illustrations.

Radar ranging

Radar* operates on the following principle: a radio signal of very short duration is transmitted; the sending equipment then is turned off, and the receiver detects echoes returning from targets which reflect radio waves (e.g., metallic targets). Since radio signals travel at contact speed (the velocity of light, which is approximately 186,000 miles per second), the time between transmission of the signal and reception of the echo measures the distance from the radar set to the target (Fig. 23).

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*This concept is termed the Fourier theorem. Fourier, a French applied mathematician (1768-1830), made far-reaching contributions to problems on heat conduction; his work has provided a foundation for modern communication engineering. At the age of 12 he was already the author of superb sermons used by leading churchmen. Fourier became a teacher of mathematics at the age of 26. He was instrumental in revolutionizing science teaching by not using detailed notes, by standing while lecturing, and by encouraging questions and comments from the students. He became an important advisor of Napoleon and travelled with him on the Egyptian invasion. While in Egypt he acquired the conviction that desert heat was healthy: the reason he spent so much of the time in his later years swathed as a mummy and in rooms unbearably hot.

*Radar is a word derived from radio detecting and ranging. Initially developed during the 1930's to test radio reflections from the ionosphere (the layer of charged particles over the earth), radar became a vital weapon of World War II when it was refined for detection and location of ships and airplanes.

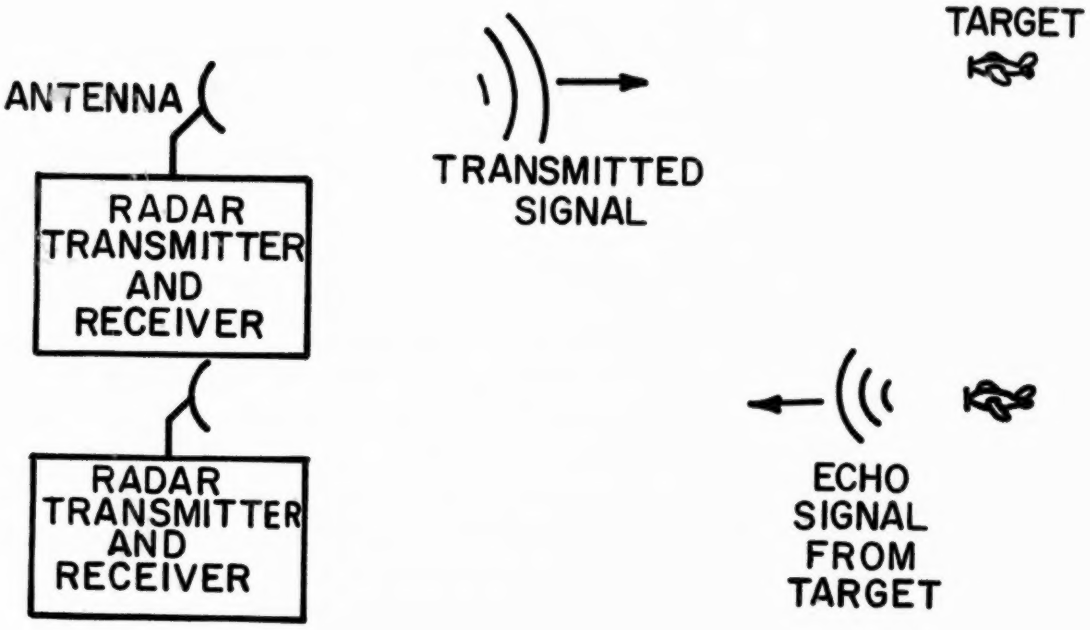
ANTENNA
RADAR TRANSMITTER AND RECEIVER
TRANSMITTED SIGNAL
TARGET
RADAR TRANSMITTER AND RECEIVER
ECHO SIGNAL FROM TARGET

Fig. 23 Radar system operation.

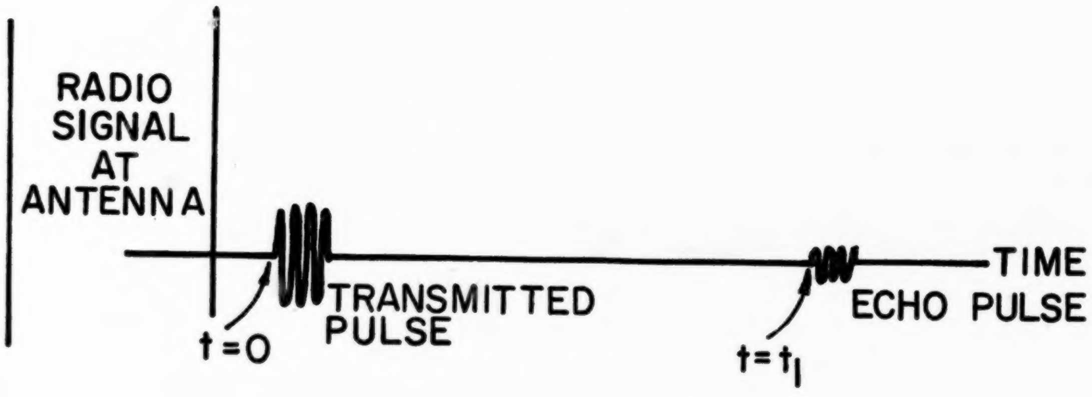
RADIO SIGNAL AT ANTENNA
TRANSMITTED PULSE
t=0
TIME
ECHO PULSE
t=t₁

Fig. 24 Antenna signals.

The operation is shown in a different way in Fig. 24. At a particular instant of time, which we call $t = 0$, a short-duration sinusoidal signal is transmitted from the antenna. At a later time (t_1), an echo signal appears at the antenna. This echo is, of course, very much smaller than the transmitted signal, but it possesses the same shape. During the time from $t = 0$ to $t = t_1$, the radio signal travels from the antenna to the target and back again--a total distance equal to twice the range of the target.

The numerical relationship between the time duration t_1 and the range of the target can be derived from the known velocity of light. Since light or radio waves travel at 186,000 miles per second, an echo appearing one second after the transmitted signal ($t_1 = 1$ second) corresponds to a target at a range of 93,000 miles. An echo appearing one microsecond (10^{-6} second or one millinoth of a second) after the transmission then means the target is at a range of $93,000/10^6$ or 0.093 miles or 1/10.75 miles. Thus, every 10.75 microseconds corresponds to one mile of range.

The relationship

$$10.75 \text{ microseconds/mile}$$

can be used to convert any measured time t_1 into the corresponding range. For example, an echo delay (t_1) of 90 microseconds means a range of

$$\frac{90 \text{ microsec}}{10.75 \text{ microsec/mile}} = 8.4 \text{ miles}$$

Radar direction finding

The above discussion indicates how radar is used to find the range of the target--by measurement of the time that passes before the echo is received. Location of the target also requires determination of its azimuth* and elevation angles (i.e., its direction from the antenna and its altitude). If radar is being used to locate a ship or object on the surface of the earth, it is only necessary to determine the azimuth angle of the target, since the elevation is known.

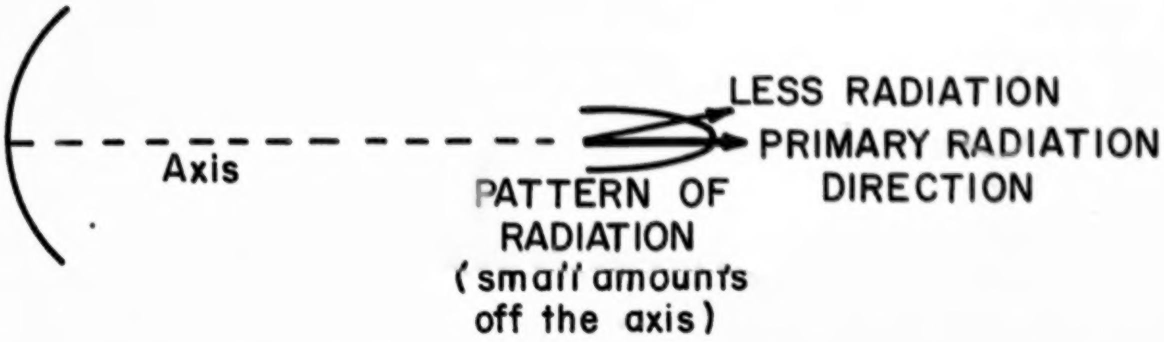
LESS RADIATION
PRIMARY RADIATION DIRECTION
Axis
PATTERN OF RADIATION (small amounts off the axis)

Fig. 25 Radar antenna radiation or transmission.

The basic technique for target-bearing measurement involves use of a highly directional antenna which rotates continuously. The directional property is commonly achieved by using of a spherical or parabolic antenna "dish", with the signal transmitted primarily along the axis of the antenna (Fig. 25). As the antenna rotates, short pulse signals are transmitted frequently, so that in any direction several pulses are sent. When an echo is received, we know that the target is in the direction of the antenna at that particular moment.

There are various techniques for more accurate determination of the target. For example, once a target is detected, we can stop the regular rotation of the antenna and switch to a mode of operation in which the antenna axis is wobbled or moved slightly around the target to locate carefully the direction in which the echo is the strongest. With these techniques, angular accuracies of much less than one degree can be achieved in the determination of the angular position of the target. *

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*The azimuth angle measures the bearing of the target on the surface of the earth. For example, a measured azimuth angle of 35° means that the target bearing is 35° clockwise from our reference direction (which can be due north or, if the radar is on a ship, can be the direction in which the ship is moving). Thus, the azimuth angle merely locates the direction of the target along the earth's surface; the elevation angle measures how much above the earth's surface.

*The same techniques can be used for measurement of the elevation angle (and hence the altitude) of the target. The antenna is varied up and down to sweep through the various vertical angles.

Radar measurement of range rate

If the radar system is being used to aim guns shooting at the target, we need to determine the velocity of the target as well as its position. This velocity has two components: the rates of change of the range and of the azimuth angle (again for simplicity we consider tracking a surface target, so elevation is not of interest). Either of these rates of change(i. e., derivatives) can be determined by measurement of the signal over a period of time and observation of the way it is changing. In the case of range rate of change, however, there is an alternative approach which is more accurate.

In 1842, Doppler* pointed out that, if we move toward or away from a source of light, radio waves, or sound, the observed frequency depends on the relative motion of the observer and source. In particular, if we move toward the source or the source moves toward us, the observed frequency of the signal increases (the whistle on a train coming toward us is higher in pitch than with the train stationary).

The reason for this apparent change in frequency is clear if we consider the transmission of sound from the train to our ear. If both the train and we are stationary, the train whistle sends out a sequence of air pressure peaks. Each of these pressure maxima travels toward us at the speed of sound (about 1100 feet/second). Our ear receives or senses the peaks as they pass by. Since we are stationary, the times from transmission to reception for each peak are the same.

If we move toward the train, each successive pressure maximum is sensed by our ear a little sooner than it would be if we were further away and stationary. Hence, as we move toward the train, the successive maxima occur slightly closer together. Our motion toward the source tends to make the successive peaks in the sinusoid occur more rapidly: hence the frequency seems to be higher.*

In Doppler radar, a rate of change of range results in a shift in frequency between the transmitted and the received signal at the antenna. The shift is given by the formula

$$\Delta f = f\,\frac{2v_r}{c} \qquad (4)$$

Here Δf is the shift in frequency (in cycles per second), f is the frequency of the sinusoid transmitted during the short pulse (Fig. 24), c is the velocity of light and v_r is the rate of change of range (c and v_r are measured in the same units).

---

*Christian Johann Doppler (1803-1853), an Austrian physicist and mathematician, who was interested in the color (frequency) of light emitted by the stars. The Doppler principle permits astronomers to measure the rotation of stars and planets (one side is moving away from the observer, one side toward the Earth, so the colors are different in the light received from the two sides).

*Actually, the optical or radio Doppler effect is not identical with the acoustical or sound phenomenon. In the acoustical case, it makes a difference which (observer or source) is moving, motion of the medium affects the observed frequency, and there is no effect of motion at right angles to the line from observer to source. These differences are of secondary importance, however, and we can focus on the similarity between the two phenomena.

The Doppler shift (the change in frequency) is very small compared to the transmitted frequency. If the target and antenna are moving toward each other at 50 miles/hour (two ships moving full steam together), Eq. (4) indicates that

$$\frac{\Delta f}{f} = 2\,\frac{50 \text{ miles/hour}}{(186000 \text{ miles/sec})\,(3600 \text{ sec/hr})} = \frac{1}{6,696,000}$$

If the frequency of the transmitted sinusoid is 3×10^9 cycles/second (a not unusual value for such a radar system), the frequency shift

$$\Delta f = 449 \text{ cycles/second}$$

The received signal is 449 cycles/second higher in frequency than the transmitted signal. If in the receiver we "beat" the incoming signal against a sample of the outgoing signal, we hear a note at about middle A on the musical scale. In actuality, the receiver compares electronically the two frequencies, and generates an electrical signal which indicates, according to Eq. (4), the rate at which the range to the target is changing.

Thus, radar can be used for sensing or determination of the target's

Range	Elevation angle
Azimuth angle	Rate of change of range

In addition, the Doppler effect can be used to select (from all the targets spotted by the radar) only those targets which are stationary; more commonly, we are interested in just the moving targets, and we eliminate from the receiver output any stationary targets. For example, in flight control near a city airport, we want the radar screen (similar to a television screen) to show only the moving targets (the aircraft), not the tops of tall, stationary buildings or nearby hillsides.

Bat navigation

One of the most impressive sights in the natural world is the exodus of thousands of bats from Carlsbad Caverns, New Mexico at sundown. After spending the daylight hours sleeping in the caves, the bats emerge at the mouth of the caverns, circle a few times, and then head off in search of insects for food. The awesome feature of this mass departure is the uncanny success of the bats in avoiding collisions, as the air is darkened with the high-density traffic flow. To the human observer the situation seems comparable to that which would occur in the center of a major city if all traffic were able to move at 60 miles/hour in all desired directions with no resulting accidents.

The success of the bat both in navigating among obstacles and in the capture of insects in flight depends upon an extremely refined sonar system.* During normal cruising in search of food, a bat emits short pulses about every 0.1 second. During the pulse of transmission, the sound signal is a sinusoid which varies in frequency from about 100,000 cycles/second down to 40,000 cycles/second (Fig. 26). The signal sent out is obviously not strictly a sinusoid, but rather is what is called a frequency-modulated signal: i.e., the frequency is modulated or changed during the transmission. In this case, the frequency is steadily decreased.

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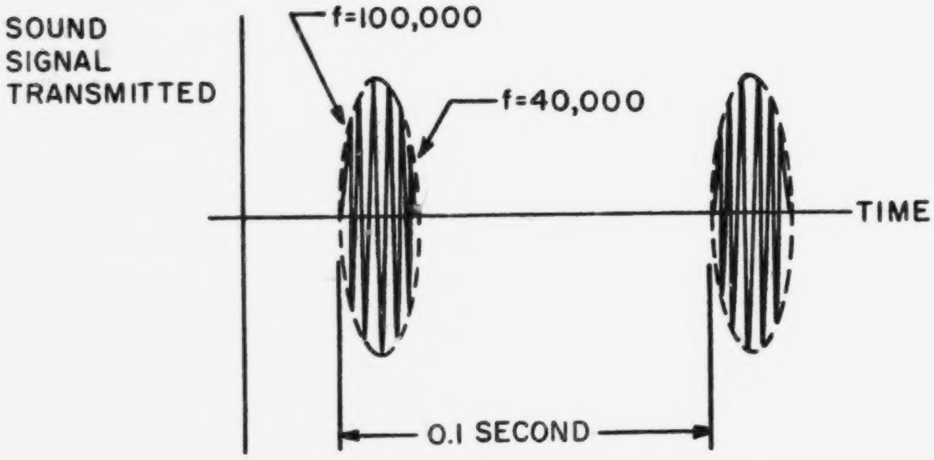
*Sonar (an acronym derived from sound navigation and ranging) was highly developed during World War II for underwater detection (of submarines and of surface ships by submarines). In contrast to radar, sonar uses sound waves, commonly at frequencies slightly above the normal human range (e.g., at 18,000 cycles/second).



Presumably, a changing frequency is used so that the bat can recognize the echo from his own signal (particularly when there are thousands of other bats in the vicinity). The frequency change also could permit the bat to recognize precisely which part of the transmitted pulse generated a particular part of the echo.

When the bat discovers a target (an insect), the pulses are transmitted more frequently (as often as 200 pulses per seond just before the capture), the frequency of the transmitted sinusoid is lowered (varying from 30,000 to 20,000 cycles/second during a pulse), and the pulses are shortened. Once the target is captured and consumed, the normal cruising operation is resumed.

In the case of the bat, the signals are generated by the vocal chords, and the ears serve as receivers. The location of the target with respect to the straight-ahead axis (i.e., the angle to the right or left) is determined by the comparison of the echo signals received at the right and left ears, just as a human being determines the direction of a source of sound. If the object is in front and to the right of the bat, the echo is received at the right ear slightly earlier than at the left.

Conclusion

In this section, we consider various modifications of sinusoidal signals: the radar signal which is a burst or pulse of a sinusoid for a short duration, and the bat's acoustical signal, which is a pulse of a "sinusoid" of continuously varying frequency. In both cases, the sine signal is a basic starting point for the understanding of an important class of problems from the natural and man-made worlds.

6. CONCLUSIONS

This chapter is intended to serve as an introduction to the next few chapters. In this chapter we consider a few types of signals which occur in physical systems; in the next chapters, we study the ways in which such signals interact with systems and can be modified by systems. This entire set of chapters is concerned with dynamic systems: equipment and devices in which change is used to create particularly desired results.

The most important signal (or form of change) in both natural and man-made systems is the sinusoid: the signal which changes according to the equation

$$y = A \sin(\beta t + \theta)$$

where

y is the signal (a position, temperature, velocity, voltage, etc.)

A is the amplitude (in the same units as y)

$\beta = 2\pi f$, where f is the frequency in cycles/second

t is time in seconds

θ is the phase angle, measuring the starting time, in radians.

We say such a signal has a sinusoidal variation with time: as time progresses, the signal varies periodically back and forth from 0 to +A to 0 to -A to 0, etc.

The sine signal is important for two primary reasons:

(1) Many signals are either sinusoids or sums of sinusoids for an appreciable period of time -- e.g., speech, the vibration of a building or car, and so forth. In the next chapter, we study several systems characterized by sinusoidal variations. Perhaps the most common sinusoidal signal is the voltage appearing at the electric outlet: in the United States, normally a sinusoid at 60 cycles/sec with an amplitude of about 162 volts (when we say 115 volt signal, we refer to the amplitude divided by $\sqrt{2}$).

(2) Many signals are formed from sinusoids -- the radar and sonar signals discussed in the preceding section, the television or radio signals transmitted from the station to home receivers, and so forth.

The preceding sections of this chapter are primarily focussed on an introduction to such sinusoidal signals.

Not all signals are sinusoids

Many signals which occur in the real world are not sinusoidal in character and have very little relation to sinusoids.* While the basic objective of the chapter is to stress the significance of the sine signal, it is perhaps worthwhile to mention three examples of non-sinusoidal signals.

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*Fourier's theorem, mentioned in Sec. 5, states that any signal can be considered as the sum of sinusoids of different frequencies. If there are thousands of components, such a decomposition is not much help, however.

(1) The price of a given stock on the New York Stock Exchange is not a simple sum of a few sine waves. If it were, we could look up the price over a long period of time in the past, evaluate the different sinusoidal components, and use this characterization to predict price variation in the future. While numerous "applied mathematicians" and "engineers" have in the past written articles arguing for such a possibility, it is perhaps sufficient to note that these individuals are still working for a living in more mundane occupations; none seems to be independently wealthy.

(2) Figure 27 shows the acceleration signals as a function of time when a stationary car with two passengers is struck from the rear by another car moving at 20 miles/hour. Four separate, acceleration signals are shown:

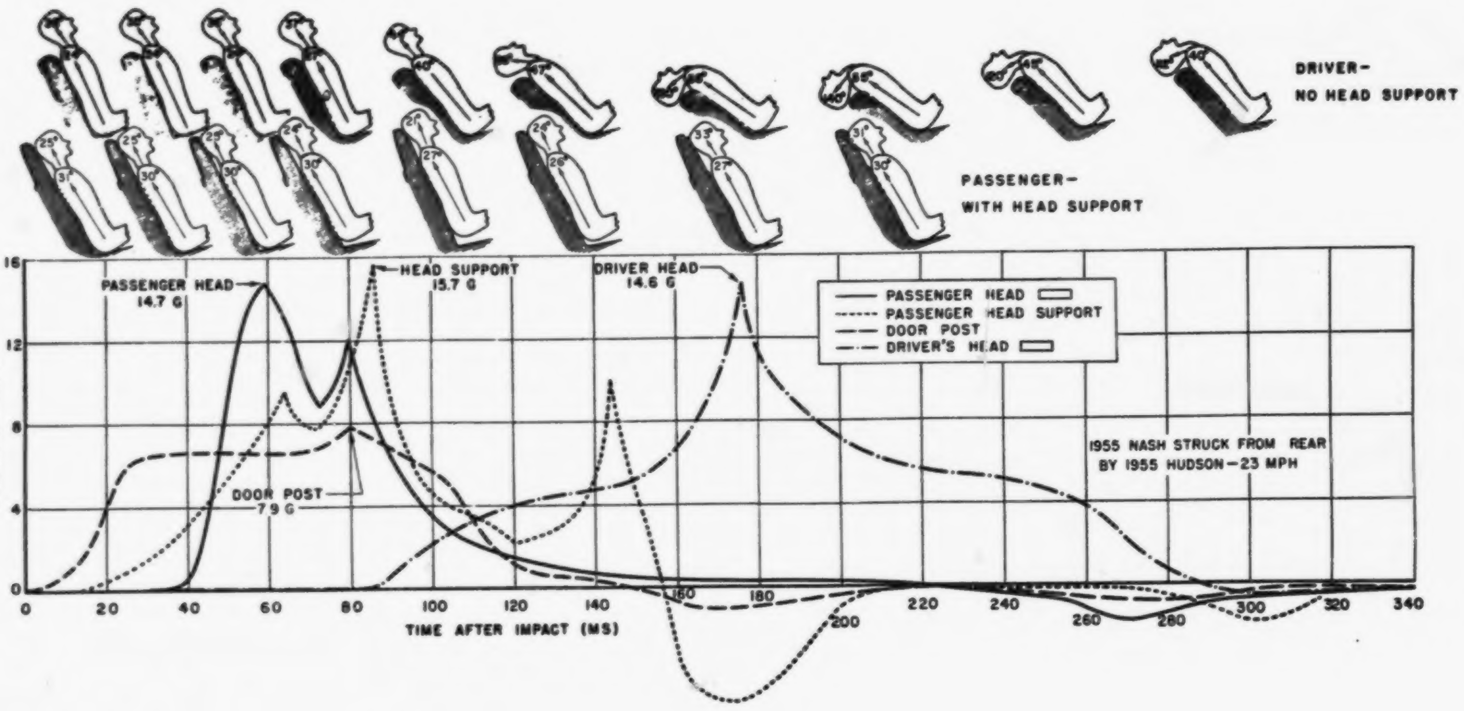

Fig. 27 Kinematics of the supported and unsupported head in a 20 mph rear-end collision. (from "Traffic Safety, A National Problem," proceedings of a Symposium of the National Academy of Engineering, The Eno Foundation for Highway Traffic Control, Saugatuck, Connecticut, 1967).

(a) The head of the passenger, who has a head support

(b) The head support

(c) The door post (a part of the main frame of the car)

(d) The head of the driver, who has no head support

The interesting feature of the four curves is the relative timing after impact. The passenger's head moves approximately with the head support and generally with the car body: hence the severe whiplash effects are minimized. For the driver with no head support, however, the most severe acceleration (14.6g or 14.6 times gravity) of the head occurs well after the shoulders and the rest of the body have accelerated; the driver's head is thrown violently backward, with the resulting whiplash injuries. Thus, the supported head accelerates in synchronism with the shoulders; the unsupported head is accelerated much later by forces transmitted through the neck. When a head rest is not used, severe binding and shear stresses are applied to the spine. The measurement of signals of this sort (on extensively instrumented dummies) indicates the extreme importance of head supports in avoiding serioud injury in rear-end collisions.

(3) Finally, Fig. 28 shows an experimental arrangement for determination of the dynamic characteristics of a human being in a simple steering or piloting task. On the face of an oscilloscope, a target spot R is moved back and forth horizontally. The position of a second spot is controlled by the man who moves a joy or control stick. As the target spot moves, the man is asked to follow by appropriate right and left motion of his spot. From experiments such as this

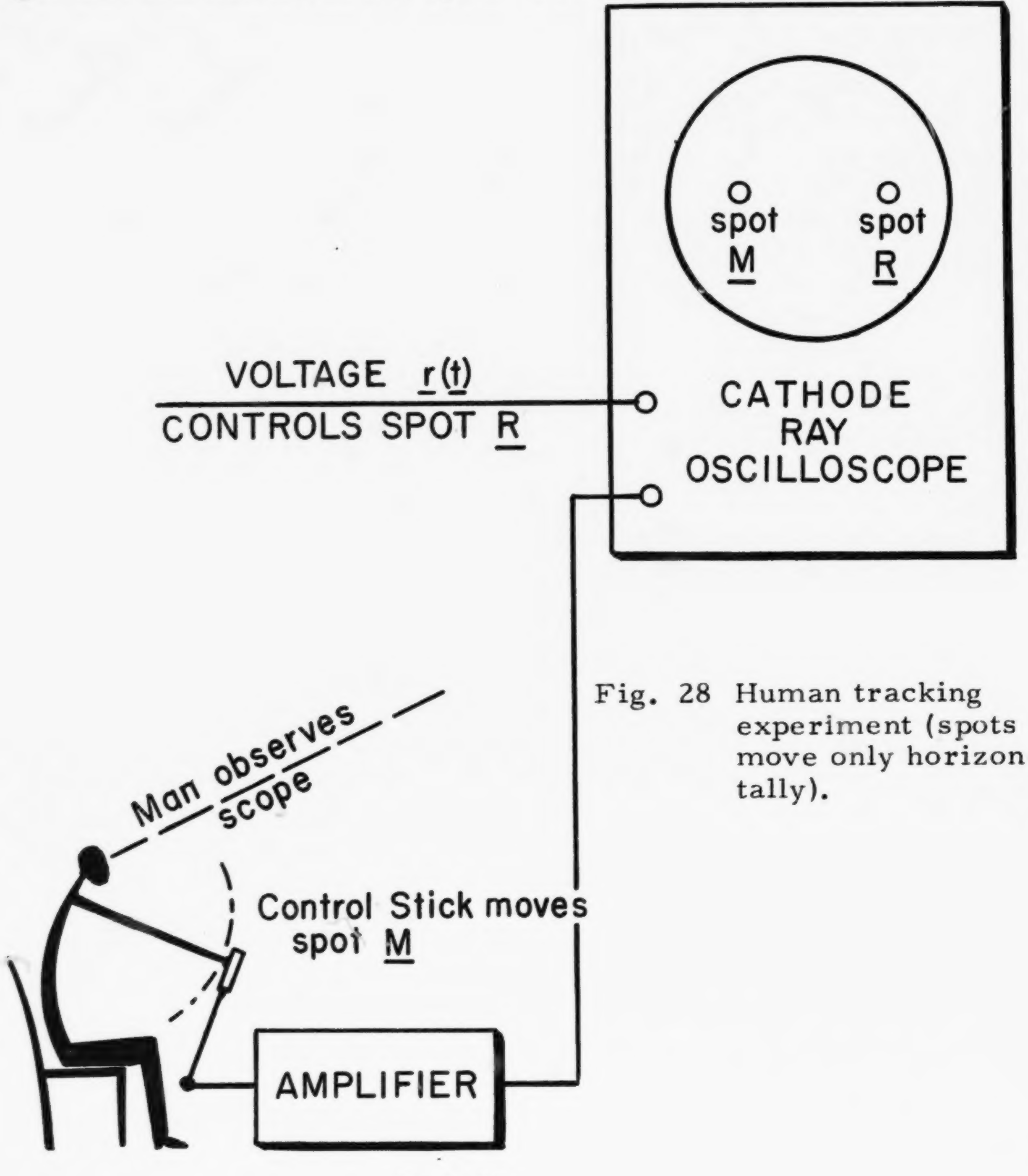

Fig. 28 Human tracking experiment (spots move only horizontally).

(initiated by A. Tustin in England in the 1940's), it was found that the human being can be characterized by:

> A time delay (0.2-0.4 second): the man observes an error, but his response occurs a fraction of a second later.
>
> Proportionality: the man positions his spot proportional to the location of the target spot.
>
> Differentiation: the man measures the velocity of the target spot and uses this to predict the future location.
>
> Integration: the man integrates the past error and corrects accordingly.

Thus, the human controller is a complex, dynamic system; his behavior in such a tracking task obeys complex laws or rules which depend upon many factors -- his degree of fatigue, his motivation to succeed, etc.

In the experimental measurement of human beings in a control task such as described above, the target spot must be moved at random. If the spot is moved sinusoidally, the man quickly learns the regular nature of the motion and he is able to anticipate accurately the future motion. The experiment then degenerates into a measure of the man's ability to understand the exact character of the motion and to adjust his joy stick accordingly.

These three examples are described merely to emphasize that there are important situations in which the signals are very far from sinusoidal. In spite of these examples, however, the sine signal is by far the most important in applied science.

Other sinusoids

(1) Why is electrical energy distributed to homes by a sinusoidal voltage?

(2) How do radio navigation systems work?

(3) How can sine signals be used to measure altitude in an airplane?

We conclude the chapter with a brief consideration of each of these three questions indicating some uses of sine signals.

(1) Why is a sinusoidal voltage variation used for distribution of electric energy, rather than some other signal shape? The reason is very simple: the sine signal is unusual in that addition of two sinusoids of the same frequency results in another sinusoid at the identical frequency. As a result of this property, when the character of the electric energy network is changed, the signal shape is not altered. If the electric range is turned on in the kitchen, the voltage at the television set remains sinusoidal in shape. If the voltage signal were triangular in shape, for example, this property would no longer be valid: electrical equipment would have to be designed to operate on a wide range of different voltage signals.

(2) How can radio signals be used for navigation--i.e., for locating oneself at sea or in the air? The Loran* system currently in use is one example of such an application. Precisely timed pulses of sine signals are transmitted from a widely separated set of radio stations around the world. On the ship, which is attempting to determine its own position, these various signals are received. The relative arrival times of the pulses from the different transmitters determine the relative distances of the ship from the transmitting station. For example, if the pulse arrival times from stations A and B are such that we know $(d_B - d_A)$, the ship is located along a curve such as shown in Fig. 29. If signals can be received from three stations, we can locate the ship's position. In actuality, the equipment is designed so that the operator need only adjust a few dials to read his location automatically.**

---

*An acronym from long range navigation.

**More recently, signals from satellites at known positions have been used. Such a location in orbit permits the transmitter to cover a larger portion of the earth, and the radio signals are less susceptible to radio interference and poor reception.

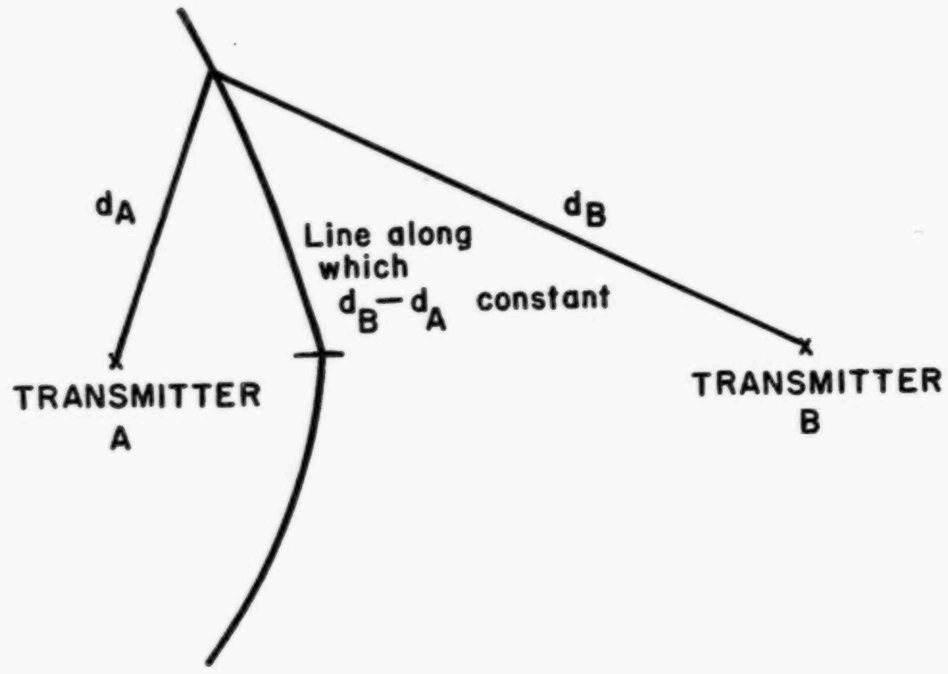

Fig. 29 Principle of Loran operation.

(3) How can sine signals be used in an aircraft altimeter? Several different schemes are possible. We might simply transmit radar pulses downward from the aircraft and measure the time until the echo returns from the surface of the Earth. An alternative scheme involves transmission of a continuous, frequency modulated signal downward. The frequency is steadily varied. The difference between the frequencies of the transmitted and echo signals then measures the time for the round trip by the radio signal (Fig. 30). For example, if the transmitted frequency is increasing at the rate of 100 cycles per/sec every microsecond, the increase is 1075 cycles/sec in 10.75 microseconds (the time required for a radio wave to travel two mile. Hence, an altitude of one mile would correspond to a frequency difference of 1075 cycles/second. In an actual altimeter, the frequency obviously can not increase forever, and the actual frequency variation takes the form shown in Fig. 31. Obviously, erroneous readings are obtained around the flyback time (the short time in Fig. 31 when the frequency is decreasing back to the start of the gradual, slow rise); but this error can be avoided automatically by proper design or considered negligible if the flyback time is very short and the periods of correct echos are relatively very long.***

***One might wonder why this altimeter is used instead of a simple radar system. It turns out that electronically frequency can be measured very accurately. Hence, when an unknown quantity such as altitude can be converted directly to frequency, it can be measured with a high degree of precision.

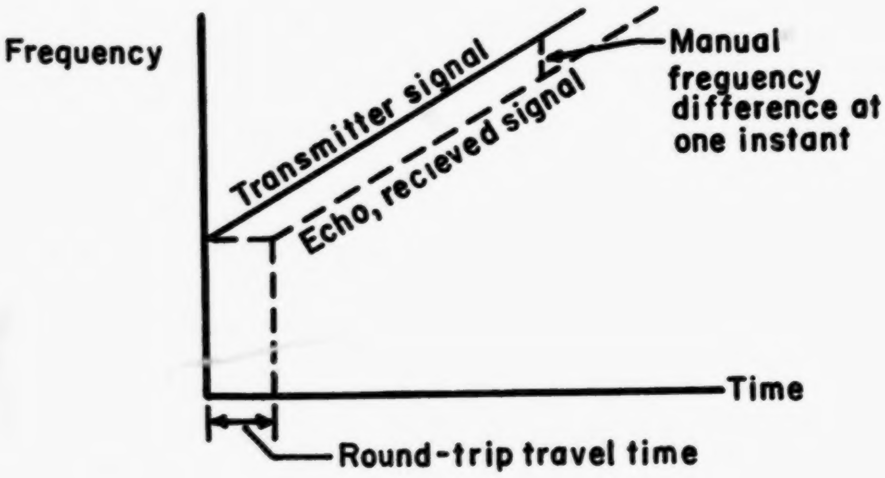

Fig. 30 Operation of a common radio altimeter.

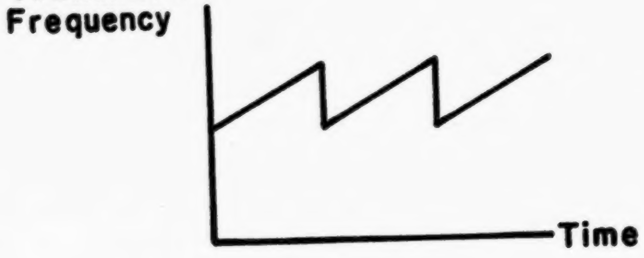

Fig. 31 Actual frequency signal in altimeter.

Questions for Chapter B-5

1. Use the data from Figure 1 of Section 2 to predict the solid waste production in the year 2000. Both the graphical and algebraic techniques for prediction should be used. How valid is this prediction?

2. Use Figure 5 from Section 2 to answer the following questions.

   (A) Why is there no profit for over three years?

   (B) What is the significance of t_A?

   (C) Why is it important to minimize t_B?

3. Why is the problem of a car bouncing on a bumpy road being studied in this chapter?

4. The approximate elongations of a spring produced by each of a series of forces is record in the following table.

   (A) Make a graph of force versus elongation.

   (B) What is the value of the spring constant?

   (C) From the graph, predict the elongation produced a force of 1.3nt and 4nt.

Force (newtons)	Elongation (meters)
0.0	0.00
0.5	0.02
1.0	0.04
1.5	0.06
2.0	0.08
2.5	0.10
3.0	0.12

5. (A) What is the relationship between the period and frequency of a sine wave?

   (B) If a sine wave had a period of 1/20 of a second, what is its frequency?

6. Two sine waves of the same frequency, had amplitudes which differed by a factor of two. If one was out of phase with the other by 180°, what would be the result of combining them.

7. In a radar system, we can not transmit a second pulse until all important echoes from the first pulse have been received. If the maximum target range of any significant echoes in 100 miles, what is the minimum allowable spacing between successive transmitted pulses? This antenna rotates through a full 360°; during this rotation, we wish to send at least 7200 pulses (20 every degree). What is the maximum allowable speed of rotation of the antenna?

8. Radar echoes have been observed from both the Moon and the planet Venus. What length of time is required in each case for the radar echo to return?

9. A police radar is used to measure the speed of cars on a thruway. If the transmitted frequency is 3000 megacycles/second (3×10^9 cycles/second), the police radar antenna is stationary, and the car is moving at 80 miles/hour, what is the Doppler shift? If the equipment must measure car speed to an accuracy of ± 2%, what accuracy is required in the measurement of the difference of the transmitted

and received frequencies? How would these answers be changed if the transmitted frequency were shifted to 3×10^{10} cycles/second? to 300 megacycles/second?

10. (A) If an insect is 18 feet from a bat, what is the time delay between transmitted and echo pulses (the velocity of sound in air is approximately 1100 ft/sec)?

    (B) What is the maximum range of the bat cruising sonar system if an eacho must be received before the next pulse is transmitted?

    (C) If the bat approaches the insect at the relative speed of 10 ft/sec and the transmitted frequency is 30,000 cycles/second, what is the received frequency?

11. Why is the transmitted frequency lowered when the bat is homing in on an insect? (The bat by this time is moving directly toward the target and angular position measurement is less important).

APPENDIX

Editorial note: The following material is reprinted from a government report publised in June of 1967*. These pages emphasize the importance of change and the problems which are generated by change. The appendix is not an essential part of the course, it is not necessary background for any of the ideas in subsequent chapters, but it is an excellent description of the dynamics of technology as they influence the lives of all citizens.

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*Appendix A is Chapter 1 of "A Strategy for a Livable Environment," the report by the Task Force on Environmental Health and Related Problems to the Secretary of Health, Education, and Welfare, June, 1967.

THE ENVIRONMENT

At the two-thirds point of the 20th Century, man has discovered that he cannot act toward his surroundings with the abandon of a caveman. For countless thousands of years, man has treated this planet as a dumping ground, boundless in its ability to absorb insults.

But now, the factory smokestack and the automobile tailpipe, symbols of industrialization and exploding population, combine to foul air, land, and water, and to change in both obvious and subtle ways the quality of life.

For generations we assumed Nature had the ability to absorb an increasing number and variety of environmental insults. And for a while it did. Now nature has rebelled. The lashback today threatens metropolis, town, and village, and to a growing degree open countryside as well. Sadly deficient in precise knowledge of the growing and changing array of hazards in our environment, we know enough to realize that we must mend our ways.

We cannot keep adding more wastes in the air.

We cannot turn more rivers and streams into open sewers, and lakes into cesspools.

We cannot befoul the land with the discards of abundance.

In short, we cannot engage in biological and chemical warfare against ourselves. Our health and well-being--and those of future generations--are at stake.

Man lives in delicate equilibrium with the biosphere--on the precious Earth-crust, using and re-using the waters, drawing breath from the shallow sea of air. While these can cleanse themselves, they can do so only to a finite point. That point is being reached and passed in many places in the United States. It is not only necessary that we take preventive action, it is also urgent that we take steps to restore the quality of our environment.

The public will have to understand the limitations of Nature. Understanding begins if we think of the Earth as somewhat like a submarine or a space capsule. Air and water supplies in all three places, are limited. Some of us understand what is required to survive and work effectively in hostile cosmic regions or deep in the sea. All of us must understand what is required to live in the finite capsule of air, water, and land that is our own environment.

Present Situation

This Chapter outlines the background against which the Task Force recommendations were made. The Task Force has examined the history of environmental health problems and has scanned the broadest possible variety of these hazards to Americans.

It has reviewed the current condition of man's living space. When viewed in its fullest context, the rapid deterioration of finite amount of water, air, and soil makes it clear that present trends cannot be permitted to contunue. In fact, they must be reversed.

Not only should the rate and direction of environmental pollution be changed, the citizenry also needs protection from a conglomerate total of environmental health threats--not only air and water pollutants, but also combinations of these, plus noise and crowding, safety hazards, and other factors.

An individually-acceptable amount of water pollution, added to a tolerable amount of air pollution, added to a bearable amount of noise and congestion can produce a totally unacceptable health environment.

It is entirely possible that the biological effects of these environmental hazards, some of which reach man slowly and silently over decades or generations, will first begin to reveal themselves only after their impact has become irreversible.

Thus, one paramount conclusion resulted from the many diverse lines of inquiry which the Task Force pursued: An effectively coordinated environmental health protection system is mandatory, one predicated on the basic premise that the environment affects man's mental as well as his physical health and welfare. An approach toward environmental health protection which is limited to concern for less than the total range of hazards that do or may exist in man's environment must be viewed by the Department as inadequate.

The problems arising from our productivity and growth are truly unprecedented. No other nation has produced so many things for so many people.

The history of our exploitation of the environment since early last century reveals many examples of misuse and abuse and unsuccessful attempts to control the environment. From an agrarian economy concentrated along the East Coast, the country grew into the breadbasket of the world; from a handful of mills along tumbling New England rivers sprang an industrial giant that has no equal in all the world. To serve this industrial revolution came the skilled and unskilled from far and near to cluster into the growing cities. Through the wonders of research and the marvel of mass production came new products and new processes --adding up to a higher standard of living than the world has ever known.

Today's world is profoundly different from that of even a few decades ago. But progress has been purchased at a price: a million traffic fatalities since World War II; cities bathed in a sea of pollution; lives stressed by noise, squalor, and crowding.

Our mode of living requires that we travel, so we accept the risk of being maimed or killed on the highway or on city streets. Some 50,000 die each year from traffic accidents. However, the driver or the pedestrian has some control over his safety. He can readily appreciate the acute nature of the danger to his life and limb. But many environmental hazards are more subtle and are beyond an individual's perception and control.

Neither the growth of metropolitan populations nor their accelerating dependence on fossil fuels for energy and motor vehicles for transportation serve to explain fully why many cities have reached or are approaching a crisis in air pollution.

To understand the situation we need to look at the dynamics of free air.

It might at first seem that there is enough air to absorb whatever insults man might hurt at it. However, when pollutants are released to the atmosphere, the degree of mixing is confined to the lowest levels of the air mantle. Luckily this air, especially in the United States, is usually in motion. Consider what happens when pollutants are released.

Under normal conditions, the atmosphere has a large capacity to cleanse itself. Nuclear testing has given us knowledge of the global transport of air masses. Radioactivity from atomic bomb tests in the Lob Nor desert of Sinkiang Province in China can be detected in the United States because the air mass circulates around the planet in a matter of weeks.

When the winds stop blowing, local pollution hazards can increase. Technically, one talks of an atmospheric stasis which means a calm condition or dead air. Given a prolonged stasis, the pollutants concentrate in the lowest levels of air and then trouble begins. This is most common during the fall and winter when there is less sunlight. Then the ceiling tends to dip closer to the ground so that under conditions of a temperature inversion the mixing air is confined to a layer approximately 2500 to 1500 feet above city streets and factories.

Future in Doubt

It is difficult to predict future levels of sulfurous air pollution from the burning of fossil fuels, because such a predicition must be predicated on assumptions about trends in fuel use and advances in pollution control technology. Authorities agree that premium fossil fuels will be in short supply in the foreseeable future and that reliance will have to be placed on coal. According to one authoritative estimate, continuation of present control practices would see a 100 percent rise in sulfur levels by 1980 as a result of increased consumption of sulfur-bearing coal and oil, that is from a present level of 24 million tons per year to 48 million tons annually. If even the most rigorous control technology were developed and applied, sulfur emissions by 1980 would reach an estimated 32 million tons per year. Electric power plants, which account for about half of the sulfur discharged to the air as a result of fuel burning, will continue to be a major source of sulfurous pollution long after nuclear energy becomes a mainstay source of electric power, simply because so many coal and oil burning power plants will be built in the next 25 years to supply rising demands for electricity.

Without nuclear power and without controls, the year 2000 would look very black from a pollution standpoint. Then the sulfur dioxide pollution would amount to 75 million tons from power-plants alone (annually). Long before that time, without benefit of controls and nuclear power, Americans would have to restrict their use of electricity or pay very much more for the kilowatt because of the scarcity of cheap low-sulfur coal and oil.

Efforts to bring an end to environmental hazards often have proceeded without adequate attention to their effect. For example, the development of efficient braking systems for motor vehicles--surely a life-saving technological achievement--has led to increased exposure of the public to asbestos particles produced by the gradual wearing of brake linings. There is scientific basis for concern that these particles may promote lung cancer over long periods of time.

Similarly, the change from hard to soft detergents, a move aimed at reducing a serious water pollution problem, led to the introduction into the environment of a new compound which is believed to be killing large numbers of fish by attacking their eggs.

In essence, then, the changes that have occurred in this country as part of its transition from a small agrarian nation to an urbanized, industrialized world power have given rise to environmental problems which we understand but little. This limited understanding has caused a failure of our society to recognize the full impact of environmental hazards on human health and welfare, and it has led to sporadic, fragmentary efforts to meet some of the most flagrant of environmental problems.

We stand at a point in history when our capacity to enhance or degrade the environment is literally beyond reckoning, but we do not now fully understand how to use this capacity for the benefit, rather than the harm, of our and future generations.

Not only have we overwhelmed many of nature's processes for environmental stability, we have misued, without knowing it, biological processes upon which the preservation of life depends. By allowing tiny amount of pesticides to enter our waters, we have set in motion processes that can lead to the destruction of birds that feed on fish, that feed on plants, that draw the pesticide from the water. Our ignorance of the consequences of our deeds may be innocent, but it is ignorance we can no longer tolerate.

Nor are the effects of environmental change manifested solely in threats to man's physical well-being. The pressures of our industrial culture must certainly produce threats to social and psychological welfare. Less difficult to measure, perhaps, these psycho-social effects of environmental hazards are nonetheless cause for concern in a Nation where mental and social ills are recognized as major problems. Are they not to a significant degree major environmental problems? It seems certain that they are.

Additional Evidence

Man's affluence has its source in the extraction and exploitation of natural resources. But the use of these resources has resulted in an environment abused. Strip mining has left ulcers on the land. Our forests have been empited of their timber. Dust storms of the thirties recall the price of land neglect. Chemical agriculture has laid down a barrage of deadly insecticides, fungicides, and herbicides to kill off plant pests and diseases; but the residues infiltrate the food chain. Banks of rivers are littered with the accumulated debris of fish kills due to these water-borne residues.

In a recent report dealing with the increasing pollution of the air, water, and land, the National Academy of Sciences National Research Council stated: "Pollution is an undesirable change in the physical, chemical, or biological characteristics of our air, land, and water that may or will harmfully affect human life or that of other desirable species, our industrial processes, living conditions and cultural assets; or that may or will waste or deteriorate our raw material resources.... Pollution increases not only because as people multiply, the space available to each person becomes smaller, but also because the demands per person are continually increasing, so that each throws away more year by year. As the earth becomes more crowded, there is no longer an 'away.' One person's trash basket is another's living space."

The trends of population expansion and urbanization will continue to funnel increasing loads of pollutants into the environment and will place increasing strains on both health and environment-protection resources.

In 1900, the population of the United States was seventy-six million, and--though our cities were steadily growing at that time--urbanization was still a trend of the future. Our population today approaches 200 million and may reach 235 million, by 1980. Already, nearly three quarters of this Nation's inhabitants are densely packed into 200 urban centers. Demographers estimate that before the next turn of the century, "super-cities" will stretch from Boston to Washington, Buffalo to Milwaukee, San Francisco to San Diego. The problems of these huge urbanized land masses will be vastly greater than those of the present cities.

The impact of population growth, technology, and urbanization on man's environment is accelerating. There is no sign of any stability or plateauing in man's collision with his environment. As people crowd more densely together, the environment changes with increasing effect--often unpredictably, and what is most serious of all, possibly irreversibly.

Inadequate Effort

The United States needs to take stock of its environmental condition and to recognize the urgency of the situation.

Americans make more things than other people, and they make far more than half of the world's trash. This year's rubbish would fill 36 lines of box cars stretching from coast to coast.

Ours is a mobile society. Think, for example, about the advances made in locomotion. Before the beginning of the 20th Century, an ambling railroad train provided man's fastest regular means of transportation. The Wright brothers' flight in 1903 was shorter than the distance from wing tip to wing tip of today's jet transports. This year Americans will fly a total of 70 billion passenger-miles in commercial airliners at speeds up to 650 miles per hour. We are now planning to fly the Supersonic Transport (SST) nearly three times that fast.

This year, the 90,000,000 motor vehicles in use will burn an estimated 60,000,000,000 gallons of gasoline, or about 700 gallons for the typical automobile. This means that each automobile in the country will discharge in a single year over 1,600 pounds of carbon monoxide, 230 pounds of hydrocarbons, and 77 pounds of oxides of nitrogen.

Even through the potential deleterious effects of radiation have been known for several decades, recent incidents concerning uranium miners indicate the general gaps in our understanding of the need to control radiation hazards in the environment. At the present time many uranium miners in the United States are being exposed to excessive amount of radioactive gases. These gases decay into radioactive daughter products and by attaching to particulate matter in the air, enter the miners' lungs. Such exposure has produced a marked increase in the incidence of lung cancer among uranium miners. With this situation existing for many years, the Federal Radiation Council has nevertheless failed to come forth with standards for the occupational exposure of uranium miners. Approximately 10,000 miners have been employed for some period in underground uranium mines prior to January 1, 1967. Dr. Leo Gehrig, Deputy Surgeon General of the Public Health Service, estimates that 529 of these miners will die of lung cancer. Now we realize, too late, that over a thousand such miners in the United States have been exposed to cancer-producing radiations which may be expected to reduce their life expectancy by several years.

Infectious hepatitis appears to be directly related to contaminated drinking water, but very little is known about how the disease-causing agent gets in the water or how it can be taken out.

Traces of cobalt were used in beer manufacturing for foam control, and when the side effects of cobalt were fully examined, the practice was rapidly stopped. But no one knows what the total effect of that foamy-cobalt interlude on the public will be

The Food and Drug Administration has estimated that the American people are being exposed to some 500,000 different substances, many of them over very long periods of time. Yet fewer than ten percent of these substances have been catalogued in a manner that might provide the basis for determining their effects on man and his environment. Again, our ignorance of potential hazards is perilously great.

Too Little Known

Too often, the undramatic nature of an evolving health hazard has kept it off page one and out of news broadcasts.

Public information usually focuses on dramatic disclosures or on pollution episodes where an acute health hazard results. For this reasons, most people have some knowledge of smog, fallout, fish-kills, and drug abuse.

Modern technology brings man into contact with a vast array of substances and processes new to the human race which have the potential of causing new health problems.

Man is not defenseless against the onslaught of modern technology. Knowledge gained through research--and applied--can enable him to deal with the great majority of environmental hazards. But he is still a long way from adequate understanding of the intricate web of life which links plants, animals, and man.

Health experts have repeatedly pointed out that grave, delayed physical manifestations can result from repeated exposure to concentrations of environmental pollutants so small that they do not make one ill enough to send him to the doctor. Environmental pollutants can have cumulative effects, especially because they accumulate in certain tissues and organs.

These effects can take delayed forms such as cancers, emphysema, and reduced life span, and they can even extend to following generations. In other words, the most serious effects of pollution may be those whose effects are delayed and subtle--those which we do not fully appreciate or take steps to prevent.

We have learned how to enclose a hundred men in a metal capsule and keep them healthy for prolonged periods of time below the surface of the ocean. In these nuclear submarines men live only a few feet from a nuclear power plant. They live in a closed system which is carefully organized and monitored to provide a compatible environment. Even more stringent life-support systems are required for manned spacecraft.

People on Earth must begin to think of their planetary home as a closed system--as a kind of huge spacecraft, which, in fact, it is.

The thrust of the Task Force's Report is that we must begin to manage all aspects of the environment so as to ensure the physical and mental well-being of the American people.

Overview Essential

Since the environment and man's relationship to it are so complex no simple solution or simple approach can be sketched out which will allow the Federal Government to correct overnight centuries of misuse.

The tools available to the Nation to do the job are insufficient. Jurisdictional disagreements among those responsible for environmental protection create problems and too often inaction. Nowhere is there the capability of making the enlightened assements of policy affecting the economy.

Yet one is no less important then the other. A weak economy means human distress. A diseased environment also means human distress.

The Task Force recommends that the Secretary of Health, Education, and Welfare, as a major step toward meeting the challenge of environmental protection, urge the President to seek Congressional authorization to establish a Council of Ecological Advisors to provide an overview, to assess activities in both the public and private sectors affecting environmental change, and to act in an analyzing capacity; to be in a commanding position to advise on critical environmental risk-benefit decisions; and finally, to be instrumental in the shaping of national policy on environmental management.

Even the abbreviated effort which the Task Force has made to examine the nature and extent of the Nation's environmental problem leaves no doubt that there must be a radical increase in the national commitment to protection of man's health and welfare from threats--present and future--in the world about him. The recommendation above is but one of many presented in this report. To all of them the Task Force assigns the highest order of importance.

The practice of medicine is becoming more and more imbued with the concept of treating the whole man, not merely a collection of his symptons. This same concept is urgently needed in our efforts to deal with problems of environmental health.

Our orientation must be to the total man in his total environment, to the cumulative effects of a growing number of environmental hazards on a receptor--man--who can respond to them in an incredibly complex manner. It is not sufficient to narrow our interest to the effect of air pollution on the lungs, the effect of noise on the ear, or the effect of crowding on the psyche. We must identify the interrelationships of these and all other forms of environmental insult on the whole man, on his physical and mental health, his productivity, and his ability to enjoy the fruits of our culture.

Chapter B-6

CHANGE IN DRIVEN SYSTEMS

B-6.1 INTRODUCTION

One day we read the following news item in the newspapers: "The U.S. Air Force has launched a revolutionary Spy-in-the-Sky which can sweep over military bases in any foreign country, take pictures and return packages of film on command from ground controllers. It may even operate a television system which could be monitored from the ground! "

On another day we read: "Senator Warren G. Magnuson called today for an international agreement to protect the world's fish supply. He said he will propose a conference for such an agreement when he talks to Russian officials in Moscow later this week."

In a Memorandum of Decision of the Superior Court of the State of California, we read the following: "This is an action for wrongful death. The action arises out of an automobile accident which occurred on May 16, 1960 on a highway known as the Carmel-Pacific Grove Cutoff in Monterey County, California. On the date in question, Don Wells Lyford, a young man 16 years of age, was driving a 1960 Corvair automobile on this highway and proceeding in the direction of Carmel. The highway involved was a two-lane highway through a wooded section with a number of curves. At the exit of a right hand curve and the beginning of a curve to the left, the Corvair automobile went across the center line and into the opposing lane of traffic and collided with a Plymouth automobile which was proceeding toward Pacific Grove. Don was not thrown from the Corvair but, as a result of the collision, he received certain injuries from which he died before reaching the hospital. Don, the only occupant of the Corvair automobile, made no statement before his death relative to the cause of the Corvair going out of control."*

What do these three stories have in common? They are all news; they are all interesting; they are all unfinished stories; they may have some immediate or long range effect on our lives. But what is important in these stories is that they all involve modeling and they also involve dynamics. We do not attempt to predict the action of a satellite, or the future of the world's fish supply, or the action of a 1960 Corvair, without building a model first.

It is a matter of common experience that an automobile does not give an equally comfortable ride under all circumstances. Moving along a rough country road at low speeds, the passengers may be jostled uncomfortably. On the other hand, it is also possible to encounter a bumpy ride on a smooth superhighway at certain speeds of travel. Sometimes, a reduced speed gives a smoother ride; at other times, a higher speed gives a smoother ride. An old car may give a poor ride at all speeds.

It is also a fact that skyscrapers bend in the wind by a discernible amount. The top of the Empire State Building can deflect as much as several inches when a modest wind is blowing. Under some conditions, such motions can have surprisingly large effects on activities in the building.

Power or telephone transmission lines strung between poles can interact with the wind and "sing". When the wind blows over a thin wire, little whirlpools of air called eddies are formed on the downwind side of the wire. The same phenomenon can be observed in flow around a stationary object in water; as for example, in the flow of a river around a bridge abutment. In the case of wires, the sound of the singing transmission line may be annoying. Then again, in a steady wind, wires can vibrate with enough intensity to break.

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*Memorandum of Decision #771-098, July 29, 1966

All three of the above phenomena are examples of behavior (up-and-down motion of the car, to-and-fro motion of the skyscraper, "singing" of the transmission line) that can be described by the same model. This model applies to a great variety of physical situations from the design of measuring instruments to the development of radio receivers. It is increasingly useful in the comprehension and the management of complex situations such as those found in economic and biological systems.

B-6.2 MODELING THE AUTOMOBILE RIDE

To describe the frequently-encountered phenomenon of back-and-forth (or oscillatory) motion, we will start with the problem of finding a suitable model of the automobile suspension system. The model must be simple, so that we can analyze it, understand it, and deal with it quantitatively. The model should include the essential features of the problem, so that no effects of crucial importance are omitted.

An acceptable model should describe the motion of the passengers in the car under various conditions of operation. This motion requires dual treatment: firstly, we must consider its direction and speed and secondly, we should consider the forces that are responsible for the effects that are produced.

Let us consider for the moment the second aspect of this motion — the relation between the forces which act and the motion which is produced. We have already studied this general relationship between force, mass and motion in section B-3.8. Although we have not as yet examined the nature of the forces which produce the up-and-down motion of the car, we are aware of the fact that these forces are related in some manner to the "bumpiness" or irregularity of the road. Our model of the system should therefore include road irregularity as an important element. The other element of importance in our model must be related to the mass of the car and the passengers.

The automobile can display various types of motion. For example, we can be jostled up and down, or from side to side. In an actual road situation, all of these must, of course, be given consideration. But because we are attempting to discover how such motions come about,in the simplest possible fashion, let us single out only one of these motions for further study; namely, the up-and-down motion.

We shall discuss the origin of the forces which give the mass of the car the motion that can make the passenger feel uncomfortable. These forces must somehow enter the model. We will consider a wheel of the car going over a bump as in Fig. B-6.1. On a level road at any instant, there is a force, F_1, from the road acting on the tire (this force is necessary to balance the gravitation force acting on the car). The axle of the wheel is then at a distance, d_1, above the road. When the car passes over a bump, the tire is flattened to some extent. As a result, the distance d_2 between the axle and the road is smaller than d_1, and the elevation of the axle is higher.

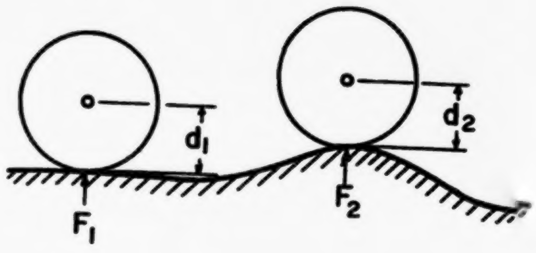


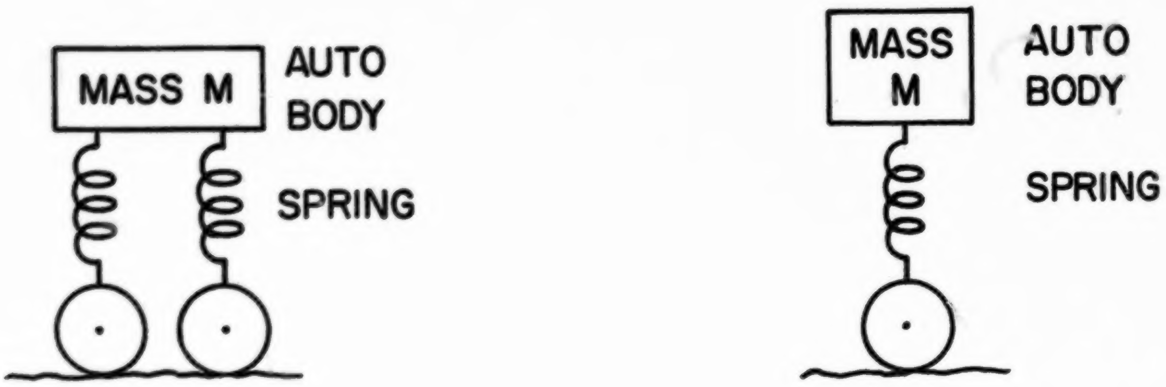
Fig. B-6.1. Forces on a tire going over a bump.

Not only is the tire compressed, but, with reasonably sized road bumps, the spring which connects the car body to the axle of the wheels may also become slightly compressed. Since this compression of both tire and spring is responsible for the effect which we are anxious to study and to model, let us examine it in greater detail.

With the car motionless on the road, we observe that the car body is in equilibrium under the action of two sets of forces: the force due to gravity acting downwards and the force due to the springs and tires acting upwards. If several people enter the car, the car body again achieves a state of equilibrium but under changed conditions: the greater downward force of gravity, which occurs because of the increased weight of the passengers, produces a compression of the springs and the tires, until the upward force arising from the increased compression of these components becomes equal to the new downward force which arises from this added weight of the passengers. We note that an increase in the compression of the spring and tires is thus equivalent to an increase in the upward force on the car body. Fundamentally, therefore, the bump in the road which momentarily compresses the tires and the springs produces an additional and sudden application of an upward force to the body of the car.

Let us develop a very simple model of the up-and-down (vertical) motion of a car. In Fig. B-6.2a the car has been reduced to an equivalent bicycle. Its mass is lumped into a rectangle, and each of the two rear wheels and springs as well as the two front wheels and springs have been replaced by a single wheel and spring. This model can retain the general vibratory motions of the car in the vertical direction but in addition rotational motion about an axis perpendicular to the page is also possible (e.g., the back up and the front down). In other words, this model can pitch. The model now differs from the actual car since roll from side to side is no longer possible.

The model of Fig. B-6.2a is more complex than it need be since it permits pitching motion. To simplify the model further, we replace it with the model illustrated in Fig. B-6.2b where the car now has the appearance of a unicycle. It will neither pitch nor roll, but will only vibrate in a vertical direction.



(a) Equivalent bicycle

(b) Equivalent unicycle

Fig. B-6.2. Simplified models of a car suspension system.

We combine the entire mass of the car into the box above the spring and also combine the four tires into the one wheel of the unicycle. But how can we combine the four springs into a single spring? If we test springs made of various materials we find that in all cases, the magnitude of a force Δf required to extend or compress a spring from a given length is proportional to the extension or compression (Δd) of the spring. The proportionality factor k is called the spring constant. Thus,

$$\Delta f = k\,\Delta d$$

This fact seems to have been first observed by Robert Hooke in 1678 and is called Hooke's Law. Note that the additional force Δf exerted by a spring which is extended (or compressed) by Δd is $-k\,\Delta d$. It is equal and opposite to the force exerted on the spring. If the spring is stretched (or compressed) through a distance Δd from its unstretched state, it will therefore exert a force

$$\Delta f = -k\,\Delta d$$

The negative sign simply tells us that the spring always exerts a force in the opposite direction from that in which it is deformed. A stretched spring tends to shrink and a compressed one tends to extend to its undeformed length.

Let us return to our model of the up-and-down motion of the car. The car actually has four springs, each with the same spring constant k_a. If the car moves up or down each spring deforms and exerts a force $-k_a d$. Thus, if we are to have a unicycle with a mass equal to that of the entire car and with a single spring which can replace the four springs on the car, then the unicycle must have a spring which is four times as stiff as each of the car's springs. In other words, the spring must exert four times the force of each of the car springs for a given deflection. Hence, in our model the spring constant must be $4\,k_a$ (For simplicity, we shall now call this "total" spring constant k).

We can simplify our model even further! The wheels and tires are not particularly important in the study of the up-and-down motion of the car. As long as we include their "springiness" in the spring of the model, we can eliminate them. Thus our model is no longer a unicycle. It is simply a mass M on a spring k as shown in Fig. B-6.3., where $k = 4k_a$.

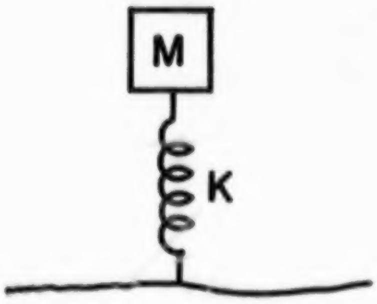


Fig. B-6.3. Simplified model of the up-and-down motion of a car.

What does Fig. B-6.3 have to do with an automobile? It certainly does not have the appearance of an automobile. Its appearance is not however important. It is simply a model of a special feature of the automobile that we wish to study. It is an example of a dynamic model -- a system which we have isolated from a complex situation for special study. It is by no means the complete system with which an automotive engineer must deal, and its worth will depend on whether or not it helps in understanding and designing an automobile to give a comfortable ride. The simple model of Fig. B-4.3 may appear to be over-simplified. On the other hand, such bold idealizations lead to progress in apparently unmanageable or complicated situations.

We have idealized the combination of the actual springs in the car, the "springiness" of the tires, and the fact that there are four tires and springs; all of these effects have been modeled by a single spring which is characterized by a single number k, the so-called spring constant. If we measure force versus distance for an actual car, we find that Hooke's Law is not followed precisely. For purposes of design, the assumption of Hooke's Law is important. With it, we have produced a model that we can analyze. Thus, we can obtain quantitative relationships between the various factors entering into the design.

B-6.3 A STUDY OF THE MASS-SPRING MODEL

To begin our quantitative study of the model, let us draw the model with reference to a coordinate system, as shown in Fig. B-6.4. We have assumed that the shape of the road has been given in the form of distance above the horizontal (y_r) as a function of the distance x along the road.

The distance d_0 is the rest, or the static position of the mass m (auto body) above the horizontal when the car is stationary. The displacement of the body from that rest position is labeled y in Fig. B-6.4, and it is the fluctuation in y that the passengers feel when they are riding in the car. Thus, the automobile designer is interested in the behavior of y over a variety of road shapes.

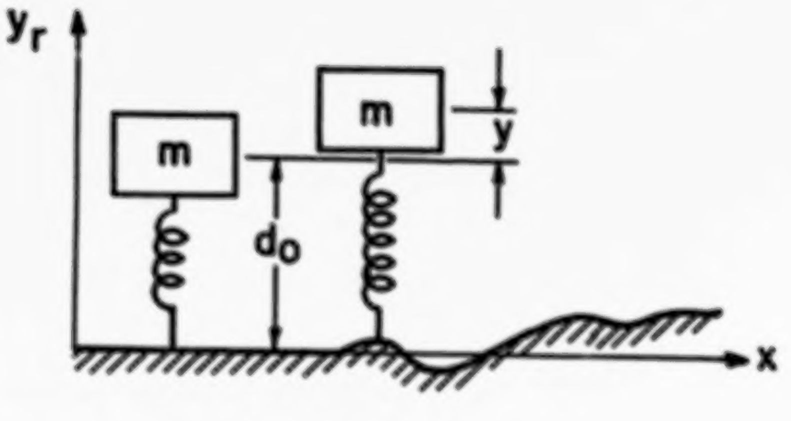


Fig. B-6.4. Model of suspension system showing spring and coordinate system.

Let us examine what happens in a particularly simple situation; namely, a flat smooth road with only one bump, as shown in Fig. B-6.5. We assume that the model car moves to the right at a uniform speed and passes

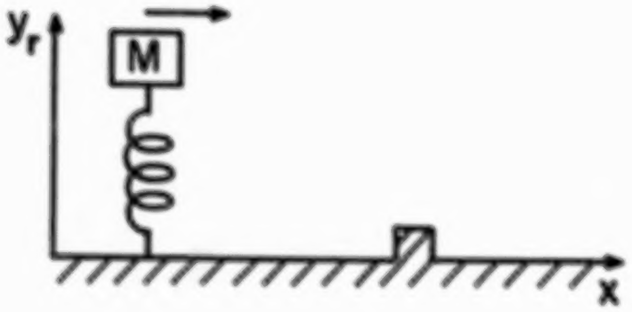


Fig. B-6.5. A simplified road surface.

over the bump. The resultant displacement of the mass is indicated in Fig. B-6.6 (for the moment, we merely state this answer; we are not interested in the proof that it is the correct answer). The mass, or car body, is set into oscillation by the bump, and this oscillation persists for many complete up-and-down cycles. In other words, the oscillatory motion is periodic, it repeats

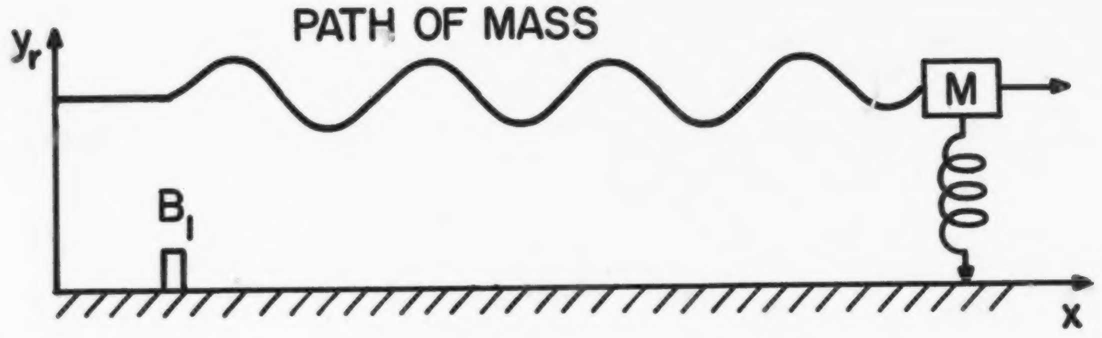


Fig. B-6.6. Vertical displacement caused by motion over a narrow bump B_1.

itself again and again. The time taken to complete one cycle is called the period. This motion is such that each cycle is exactly the same as any other.

The mass-spring model has also another important aspect. From Fig. B-6.6 we can see that bump B_1 sets the mass and spring into an up-and-down oscillation whose form is traced out in that figure. This smooth, snaky curve has a particular significance; namely, it occurs in many situations. For instance, in Fig. B-6.7 such a curve is traced out by the end of a swinging pendulum as paper from a roll is pulled beneath it at a uniform speed.

Basically, this curve results whenever a particular situation exists. This motion occurs when a mass is subjected to a force which opposes its displacement. The crucial condition for generating this particular motion is that as the mass departs from its rest position, there must be a force tending to restore the mass to its rest position. Furthermore, this restoring force must be proportional to the displacement from the rest position. In the case of the mass-spring system of Fig. B-6.5, this force is supplied by a Hooke's Law spring. In the case of the pendulum, the restoring force is supplied by gravity and satisfies the proportional condition provided the amplitude of the oscillation is small.

Because we encounter motion of this sort so often, it has been given a special name, simple harmonic motion, or, alternatively, sinusoidal motion. The snaky curve itself is called a sine wave.

Sine waves have several properties which are particularly important for us. Their size is usually stated as half the total excursion as indicated in Fig. B-6.8, and is known as the amplitude. Sine waves of various amplitudes are shown in Fig. B-6.9. The frequency of a sine wave is the number of "up-and-downs" completed in one second. To make this clear, we note that after the

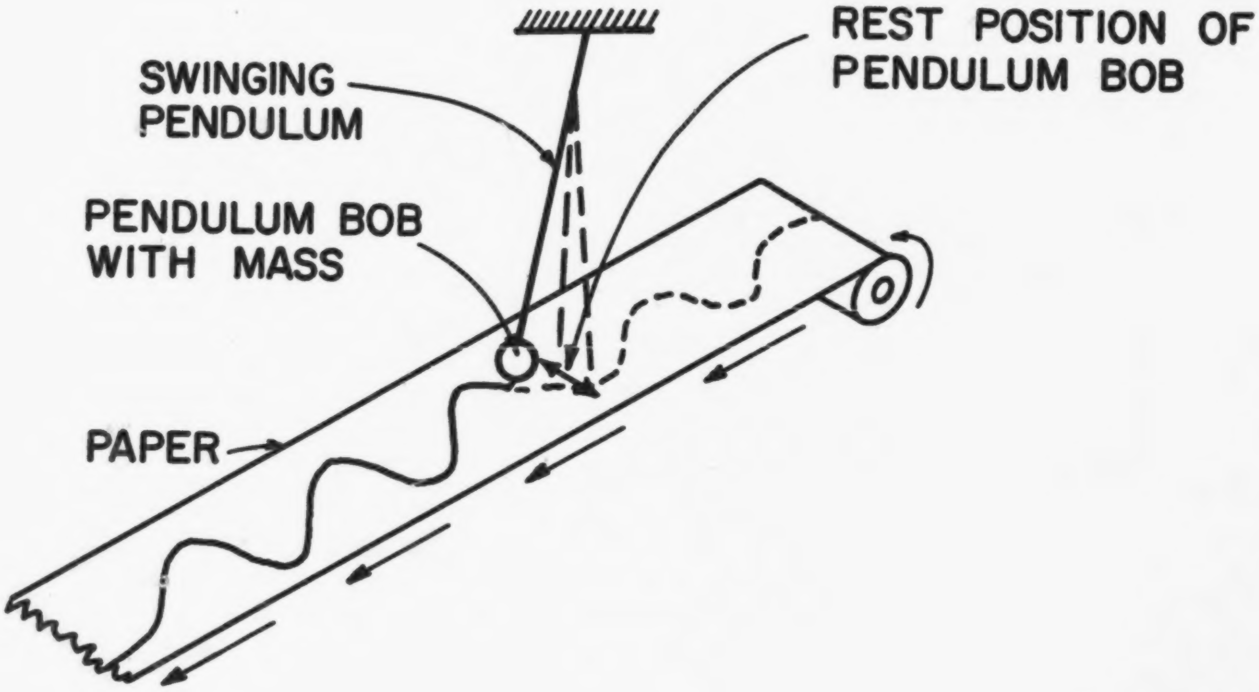

SWINGING PENDULUM
REST POSITION OF PENDULUM BOB
PENDULUM BOB WITH MASS
PAPER

Fig. B-6.7 Tracing out the oscillatory motion of a swinging pendulum.

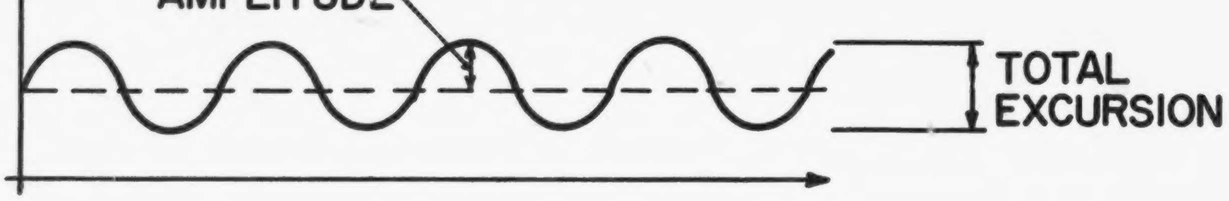

AMPLITUDE
TOTAL EXCURSION

Fig. B-6.8 Definition of the excursion and amplitude of a sine wave.

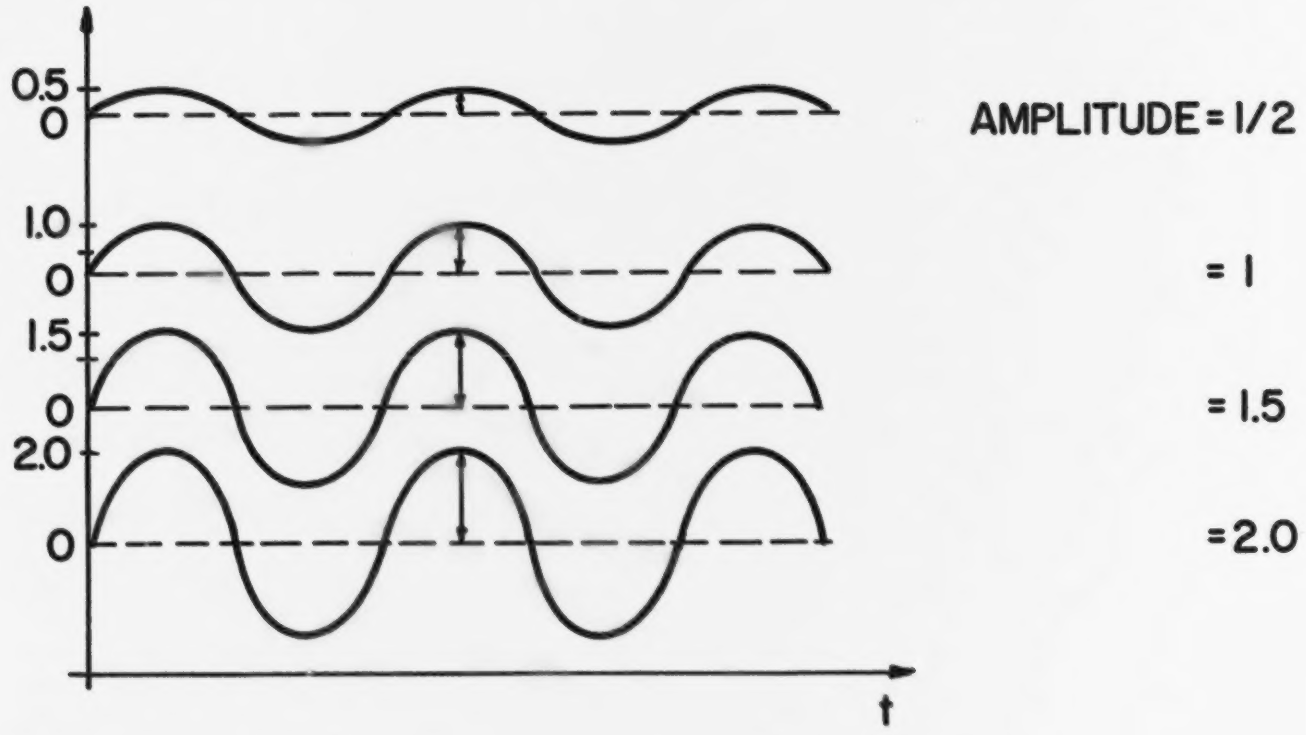


Fig. B-6.9. Sine waves of various amplitudes.

mass in Fig. B-6.6 encounters B_1, it bounces up and down, and we can count the number of oscillations that take place in each second. This number is the frequency of the oscillation. Sine waves of various frequencies are shown in Fig. B-6.10. We note that the frequency (denoted by the letter f) of each can be found by counting the number of cycles or periods in one second (easiest way to do this is to count the number of crests or troughs)

Sine waves can be combined in a rather simple fashion. In the case of the mass-spring combination of Fig. B-6.6, consider what would happen if there were two bumps on the road instead of one, as in Fig. B-6.12. If bump B_1 were absent and another bump was present further along the road (Fig. B-4.11) then the car motion would be identical to that shown in Fig. B-6.6, except that it would be shifted so that its beginning would coincide with the position of the second bump B_2 as shown in Fig. B-6.11. (If B_1 were larger than B_2 by some factor, say 1.5, then the amplitude of the motion resulting from B_1 would be 1.5 times larger.) However, bump B_2 is not alone (Fig. B-4.12). The total motion must be some combination of the motions caused by B_1 and B_2 separately. Indeed the total motion resulting from B_1 and B_2 together is the sum of the individual motions. This sum is shown in Fig. B-6.12.

In this example (Fig. B-6.12) it happens that the amplitude of the excursions is larger than that produced by either B_1 or B_2 alone. This is not true for all situations; if B_2 were moved slightly, we can find a location for B_2 such that the motion ceases altogether. The motion curve would then appear as in Fig. B-6.13.

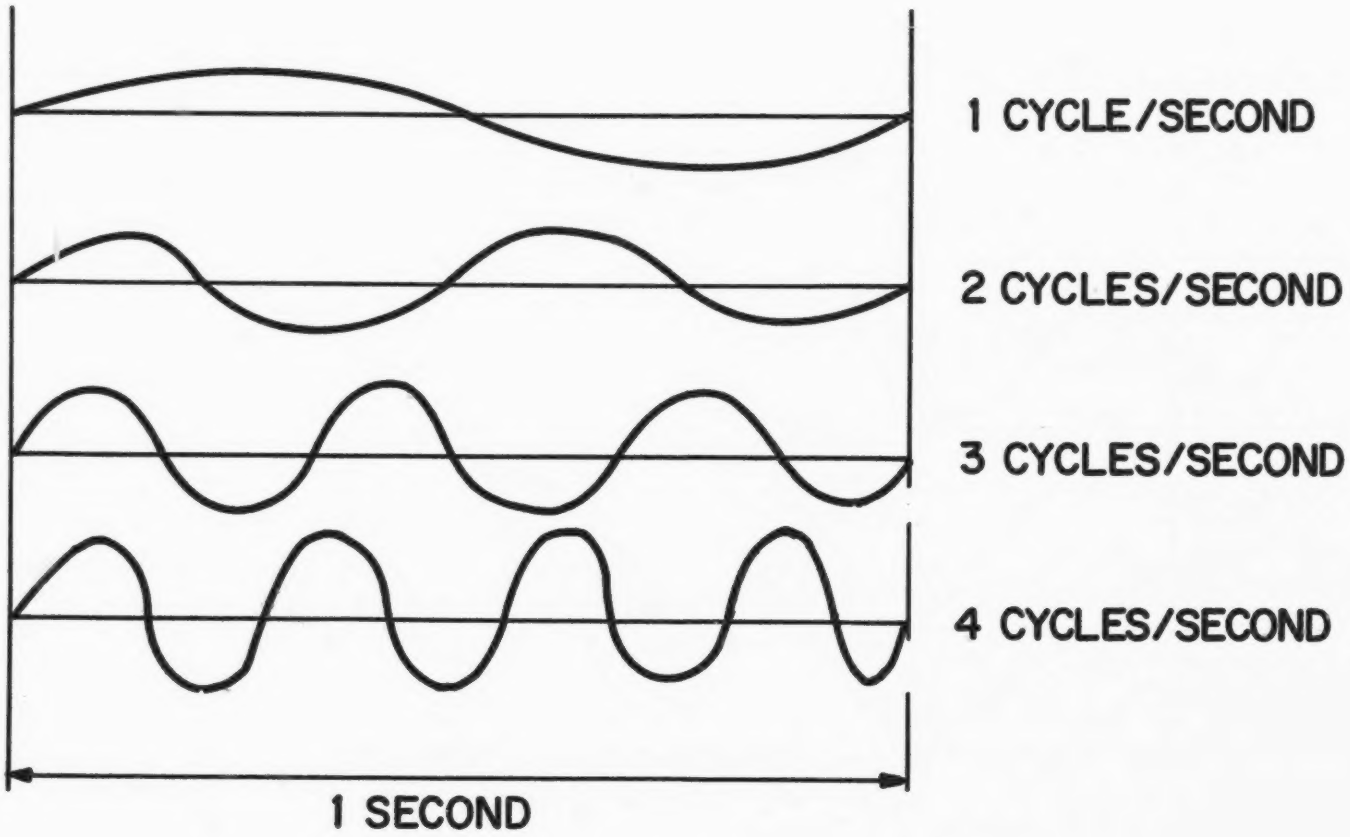

1 CYCLE/SECOND
2 CYCLES/SECOND
3 CYCLES/SECOND
4 CYCLES/SECOND
1 SECOND

Fig. B-6.10. Sine waves of various frequencies.

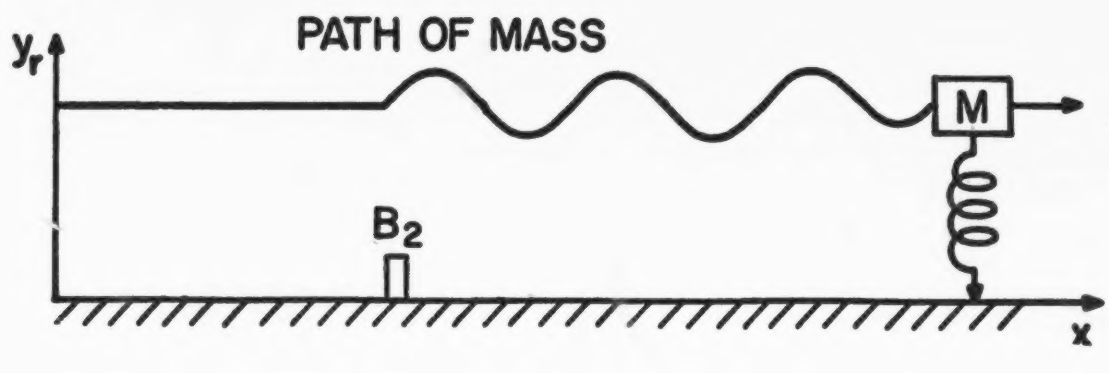

y_r
PATH OF MASS
M
B_2
x

Fig. B-6.11. Motion resulting from bump B_2 alone.

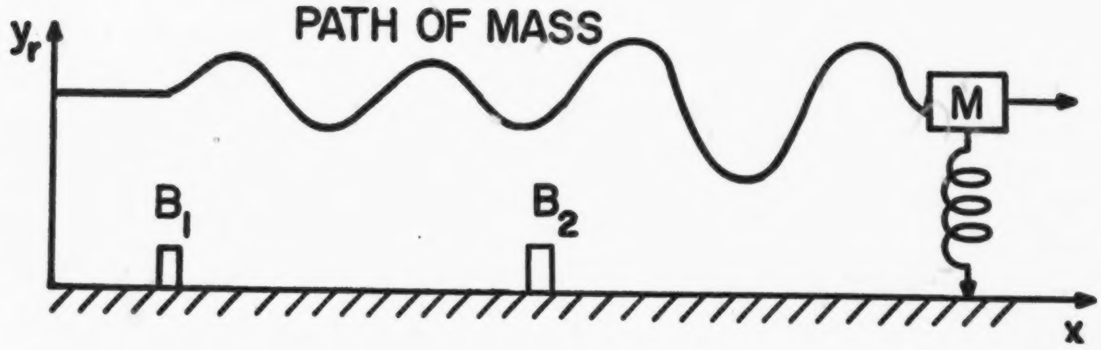


Fig. B-6.12. Total motion caused by both B_1 and B_2.

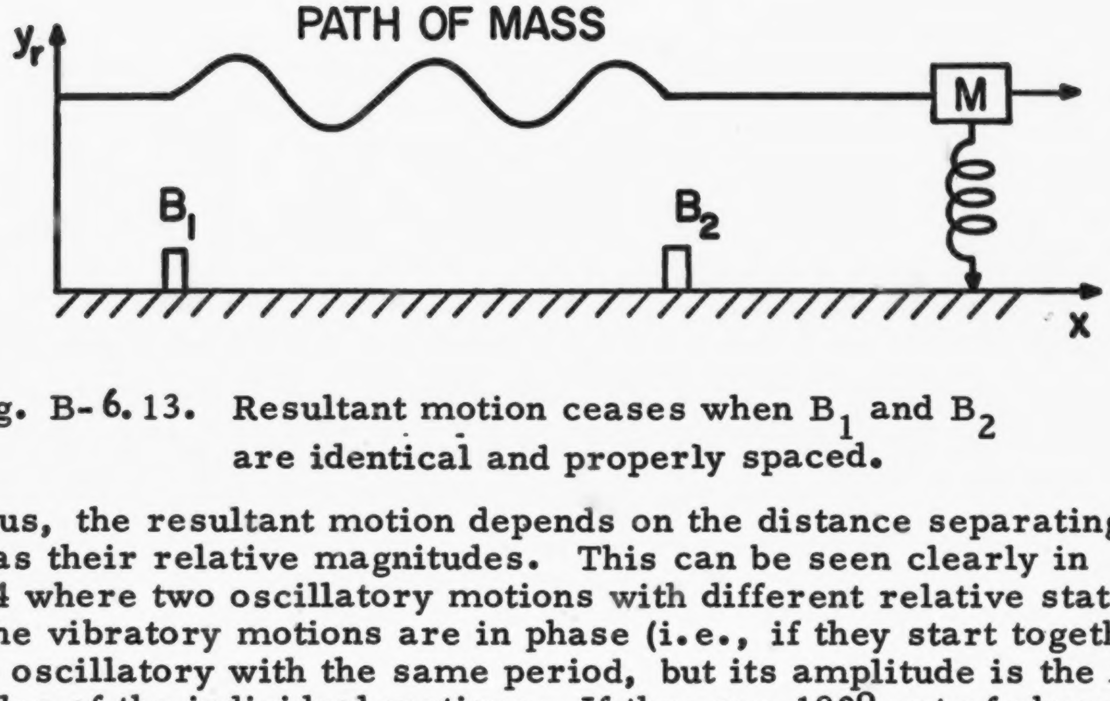


Fig. B-6.13. Resultant motion ceases when B_1 and B_2 are identical and properly spaced.

Thus, the resultant motion depends on the distance separating B_1 and B_2 as well as their relative magnitudes. This can be seen clearly in Fig. B-6.14 where two oscillatory motions with different relative states are added. If the vibratory motions are in phase (i.e., if they start together) their sum is also oscillatory with the same period, but its amplitude is the sum of the amplitudes of the individual motions. If they are 180° out of phase (i.e., if they begin one-half cycle apart), their sum is zero since they are the negative of each other. If they are 90° out of phase (i.e., if they start a one-quarter cycle apart), their sum is oscillatory with the same frequency or period, and the amplitude is greater than the amplitude of any one but less than the sum of the amplitudes. The sum of two harmonic oscillations, with equal periods but different starting times (or phases) and different amplitudes, is always harmonic and of the same period. In other words, the sum of sine waves of a given frequency is another sine wave of the same frequency but of different amplitude.

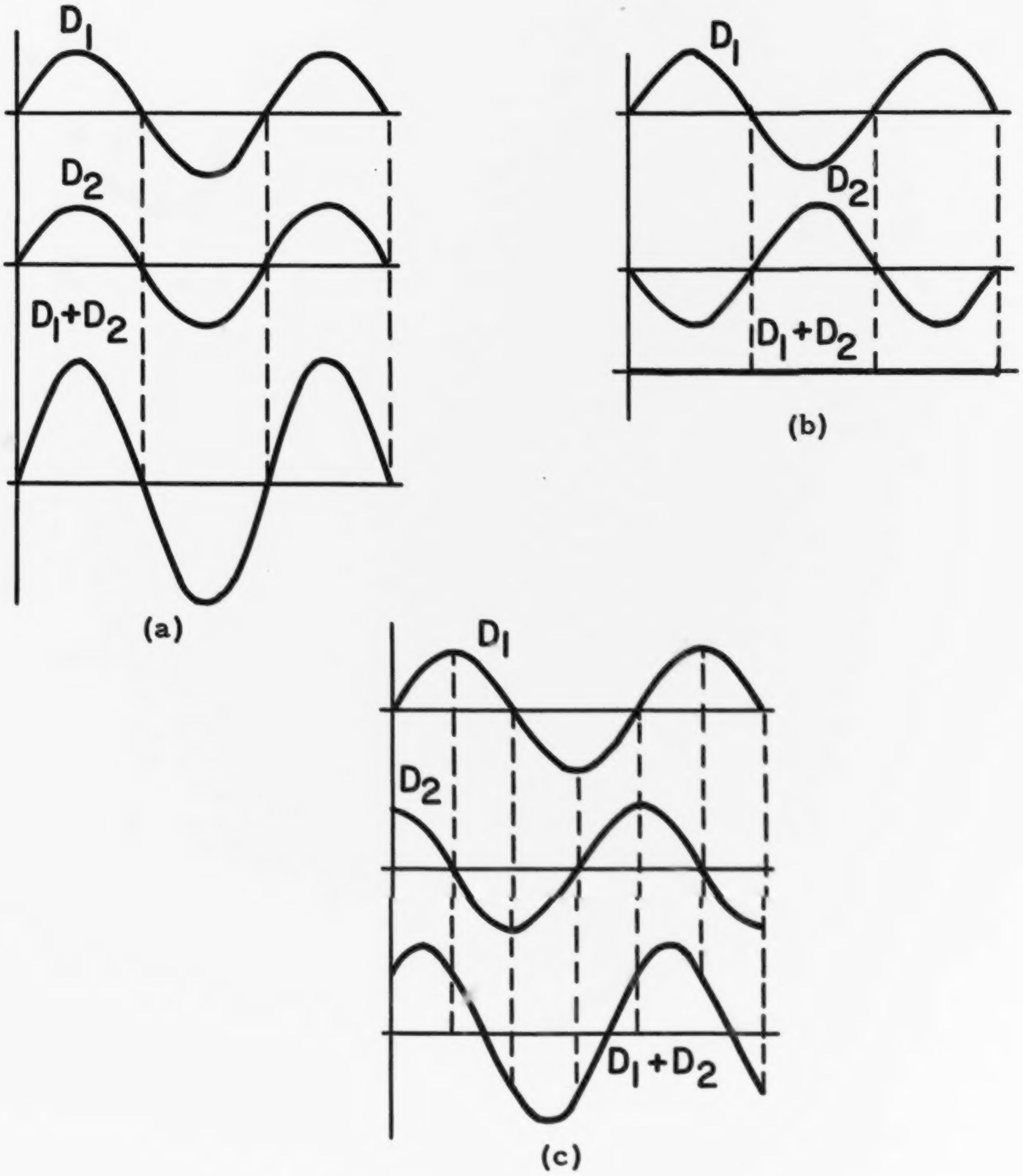

D1
D2
D1+D2
(a)
D1
D2
D1+D2
(b)
D1
D2
D1+D2
(c)

Fig. B-6.14. The sum of two oscillatory motions: (a) of equal magnitudes and in phase, (b) of equal amplitudes and 180° out of phase (step) (c) of equal magnitudes and 90° out of phase (step).

This additive method is valid for finding the total motion no matter how many bumps there are or what may be their relative positions. We can consider any road as if it were made of a series of "elemental" bumps all exactly alike except for height, as indicated in Fig. B-6.15. Thus, the additive property permits us to compute by simple addition the motion for any road surface, once we know the motion for an elemental bump. This particular idealized model based on Hooke's law is a very useful one. (This method does not work however, for a spring which does not follow Hooke's Law.)

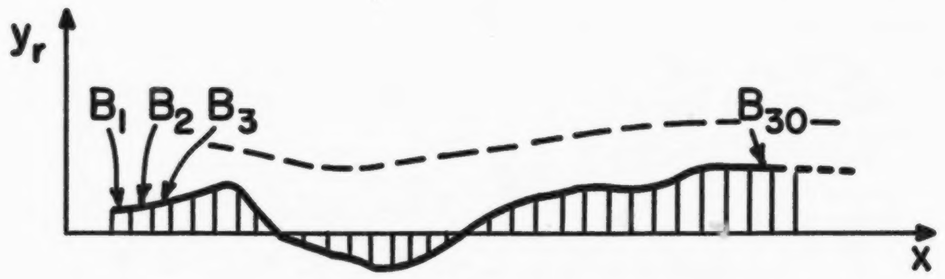


Fig. B-6.15. An illustration of describing any road surface as a series of separate bumps.

B-6.4 VERTICAL MOTION OUTPUT AS RELATED TO ROAD WAVINESS INPUT

We have already observed that an abrupt disturbance produced by a bump in the road will set our model of an auto body into sinusoidal motion. If we reexamine Fig. B-6.6 and visualize the behaviour of the car, we may describe the event as follows: the input to the mass-spring system is B_1 which produces a sinusoidal motion in the mass-spring system. Figure B-6.4 represents a more general case. Here the input to the system at any instant is expressed by the ordinate y_r at that instant, and the output of the system is represented by the ordinate y. The block diagram of Fig. B-6.16 illustrates this relationship.

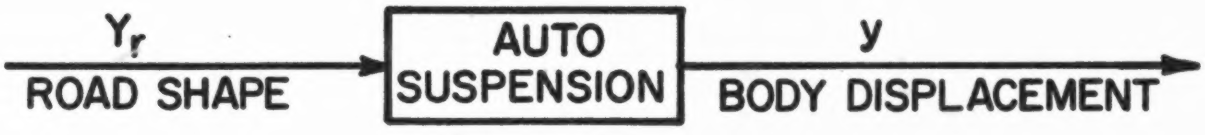


Fig. B-6.16. The relationship between y and y_r.

We are interested in the relationship that exists between y and y_r. This relationship must be such that the car passengers are not subjected to values of y (body displacements) which produce an uncomfortable ride. This condition must hold for all reasonable values of y_r.

One method of investigation of the effect of the variations in y_r involves the use of the principle of superposition - the additive method which was outlined in the previous section. This would be laborious, and there would be no concise way of summarizing the results from the many different situations selected for testing. It would be useful however, if we could find an input which resulted in an output [illegible] identical shape or form. Then we could merely compare the magnitude [illegible] of these motions. For instance, we would be able to conclude that "a road disturbance is reduced in size by a factor of five". This would be a very concise conclusion.

Happily, there is an input shape which permits such a convenience. It is the sine wave. If the road is sinusoidal in shape, the motion of the car body is sinusoidal. The output sine wave has the same frequency but the amplitude may be different (larger or smaller) and the output sine wave may be delayed in time. These effects are illustrated in Fig. B-6.17, where we have labeled the input and output amplitudes as Y_r and Y. By comparing these values, we can state concisely the effect of the suspension in smoothing out road roughness.

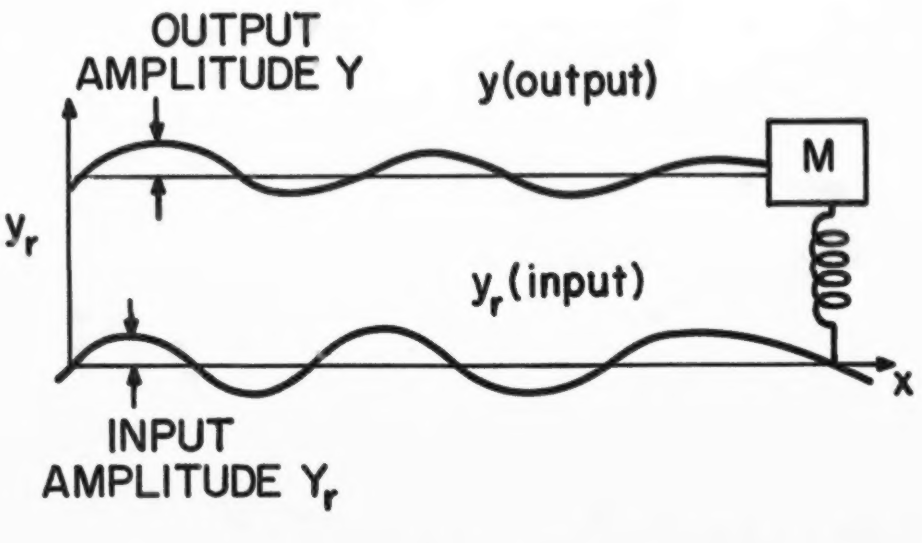


Fig. B-6.17. The effects of sinusoidal road shape on auto motion.

Let us consider, then, a car moving along a sinusoidal road, as illustrated in Fig. B-6.18, and measure the ratio of output amplitude to input amplitude, y/y_r. Let us measure this ratio for a variety of sinusoidal frequencies of the road and plot the results as indicated in Fig. B-6.19. The most prominent feature of these measurements is that, at one frequency, marked f_o, the output amplitude may become much larger than the input amplitude. This frequency is known as the natural frequency, or the resonant

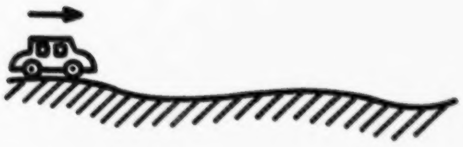


Fig. B-6.18. Car traveling on sinusoidal road.

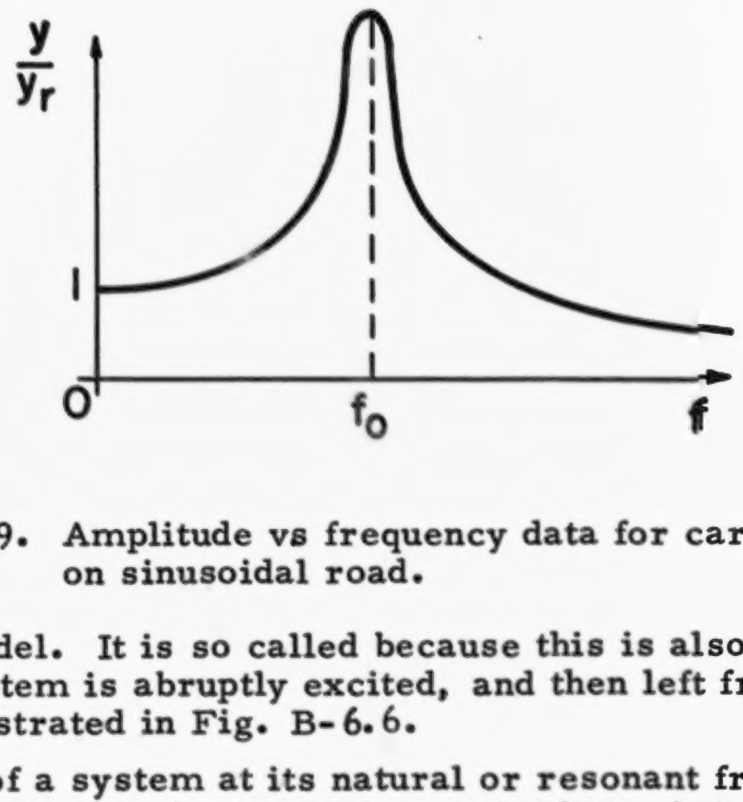


Fig. B-6.19. Amplitude vs frequency data for car moving on sinusoidal road.

frequency of the model. It is so called because this is also the frequency which results when the system is abruptly excited, and then left free to vibrate. This situation is also illustrated in Fig. B-6.6.

Excitation of a system at its natural or resonant frequency results in a very large output because each successive excitation reinforces the output from the preceding ones. Figure B-6.12 shows two successive disturbances reinforcing each other. If B_2 were followed by B_3, B_4, and so on, each at an appropriate point to add to the size of the output, it could grow very large indeed. In the case of sine-wave excitation, the bumps are not as abrupt, but each successive crest of the sine wave reinforces the preceding crests.

Let us redraw Fig. B-6.19, and plot on the horizontal axis the ratio of the excitation (road) frequency to the natural (car) frequency (Fig. B-6.20). The ordinate axis (vertical axis) is the ratio of the output to input amplitudes (car's amplitude to road amplitude) as before. We see that the curve has three general regions of different types, labeled 1, 2, and 3. If the excitation

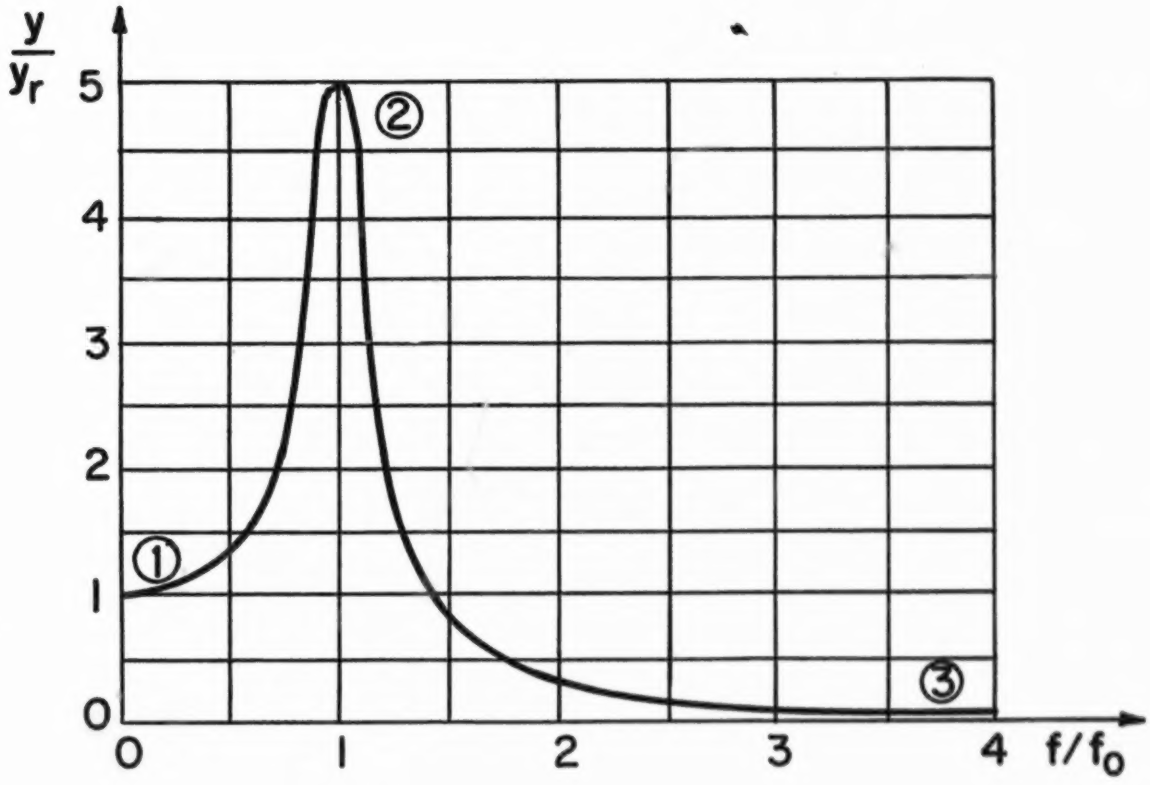


Fig. B-6.20. Amplitude response-excitation curve for undamped mass-spring system.

frequency is small in comparison to the natural frequency (region 1), the output-input ratio is nearly unity; if the excitation frequency is almost the same as the natural frequency (region 2), the output is many times as large as the input; finally, if the excitation frequency is very large compared to the natural frequency, (region 3), the output amplitude approaches zero, since the ratio approaches zero.

It is easy to interpret Fig. B-6.20 in terms of our experience with automobiles. In the up and down direction, the natural frequency f_o of an automobile is typically of the order of 1 cycle per second. On a gently rolling road with peaks one mile apart, and in a car moving at 60 miles per hour, the observer in the car sees a "bump" or peak once every minute, or at the frequency of 1/60 cycles per second. In this case, f/f_o is 1/60 divided by 1 or 1/60. This is region 1 of Fig. B-6.20, which says that the amplitudes of the car and road should be the same. And this is indeed our experience; the tires and the car body ride in almost perfect unison, each going through the same sinusoidal motion.

On the other hand, we may move at a speed of 30 miles per hour (44 feet per second) on a concrete highway made of slabs whose joints are 44 feet apart. In this case, it takes one second to go between the joints; the observer in the car thus sees an excitation of 1 cycle per second, so that f/f_o is unity. What happens? Suppose the car is going up and down at its natural frequency and is at the bottom-most part of its swing when it hits a joint between slabs. The tires momentarily drop down, causing the spring to stretch. This applies a net downward force to the car mass, and makes it go down further, thus increasing the amplitude of its swing. But the next time it is at the bottom-most part of the swing, the car again hits a joint, which again increases the swing. If the car maintains its speed, the process is repeated over and over again, and the car will begin to oscillate violently.

This is the phenomenon of resonance, which we mentioned before. It occurs when the excitation frequency is equal to the natural frequency. Fig. B-4.20 shows a particular situation in which ($f/f_o = 1$) the output amplitude becomes five times greater than the input amplitude at resonance. In actual systems this magnification factor is sometimes so great that either something breaks or additional damping forces come into play. These forces limit the oscillation to a finite value because they oppose the motion itself. In lightly damped systems, the response can be quite violent when the excitation is at the resonant frequency. We will examine damping forces later in this chapter.

Finally, let us suppose that we are traveling over a brick road, where the joints between bricks are perhaps 1/2 foot apart, at 44 feet per second (30 miles per hour). Then we meet a joint every 1/88 of a second. The excitation occurs at a frequency of 88 times per second. The ratio, f/f_o, is now 88, and from Fig. B-6.21 the output response is negligible. We can understand this if we remember that a stationary mass does not move until a force is applied. The force in our model must come from the extension or compression of the spring. If the spring is weak, not much force is transmitted to the mass by the spring, and the mass does not undergo much acceleration and motion.

"Weak" is a relative term. What does a "weak" spring mean? Our model has shown that a weak spring is one that produces a low natural frequency compared to the excitation frequency. If such a spring is used, the car's inertia causes it to remain steady. Thus on a brick or cobblestone road, we can get a smooth ride even though the wheels are jiggling up and down quite energetically. On a superhighway, we often find that we get a more comfortable ride by speeding up. We then cause the car to go over the joints at a faster rate, and cause the car to pass beyond the resonance region of Fig. B-6.20, (region 2) into the high-frequency region (region 3).

B-6.5 THE EFFECTS OF DAMPING

In the automobile suspension system, damping forces are provided by the shock absorbers. A simplified drawing of a shock absorber is shown in Fig. B-6.21. In order to move the plunger, a force must be applied to compel the oil to flow from one side of the plunger disk to the other. Thus, the shock absorber tends to oppose relative motion between whatever is attached to its two ends. The force necessary to move the two ends A and B relative to one another depends on the relative velocity of the ends.

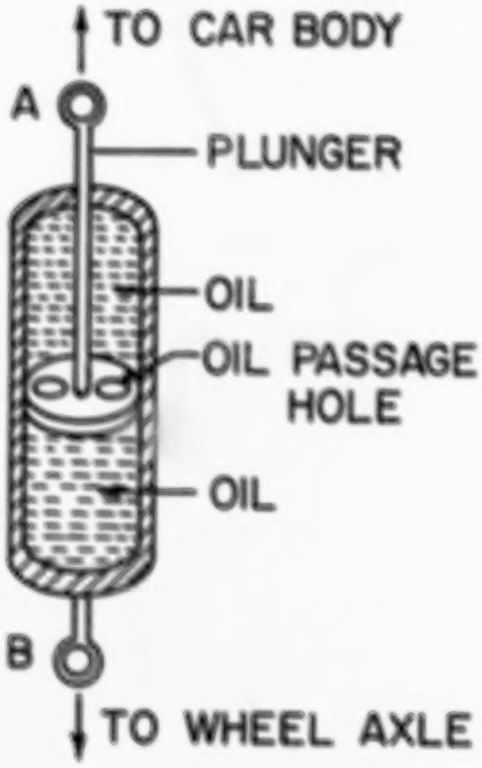

TO CAR BODY
A
PLUNGER
OIL
OIL PASSAGE HOLE
OIL
B
TO WHEEL AXLE

Fig. B-6.21. Automobile shock absorber.

Thus, shock absorbers are desirable additions to a car suspension in the configuration indicated in Fig. B-6.22. We can appreciate the effect of the shock absorber by comparing Figs. B-6.23 (a) and (b).

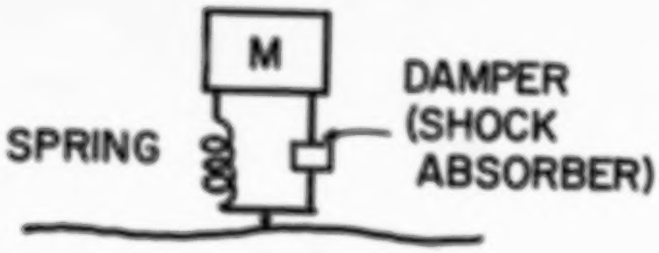
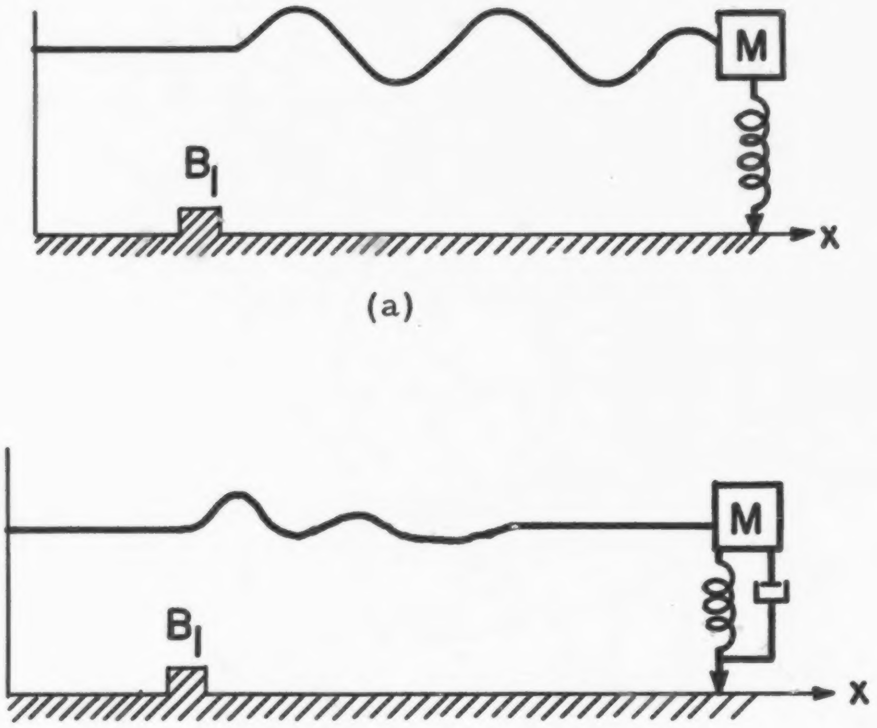

M
SPRING
DAMPER
(SHOCK
ABSORBER)

Fig. B-6.22. Model of suspension system including shock absorber.



(a)

(b)

Fig. B-6.23. The effect of the shock absorber on up-and-down motion of the car.

Part (a) is the same as Fig. B-6.6 which shows the car's undamped response. With the shock absorber, part (b), the oscillating motion dies out quickly (sometimes we say "damped" out). This action makes for a much less "bouncy" ride.

Let us consider the effect of the shock-absorber damping from the sinusoidal point-of-view. That is, how is the car body amplitude on a sinusoidal road affected by the shock-absorber damping? Its effect is indicated in Fig. B-6.24 in a plot similar to Fig. B-6.20. Again, we note that the output over input amplitude is plotted on the ordinate axis. The first thing that we notice is that, in a damped system, the possibility of an infinite response disappears, although for a lightly damped system, the response can become quite large. We find that the model gives a natural, quantitative definition of "light" and "heavy" damping. Light damping exists when the amplitude ratio y/y_r is large for frequencies close to the resonant frequency.

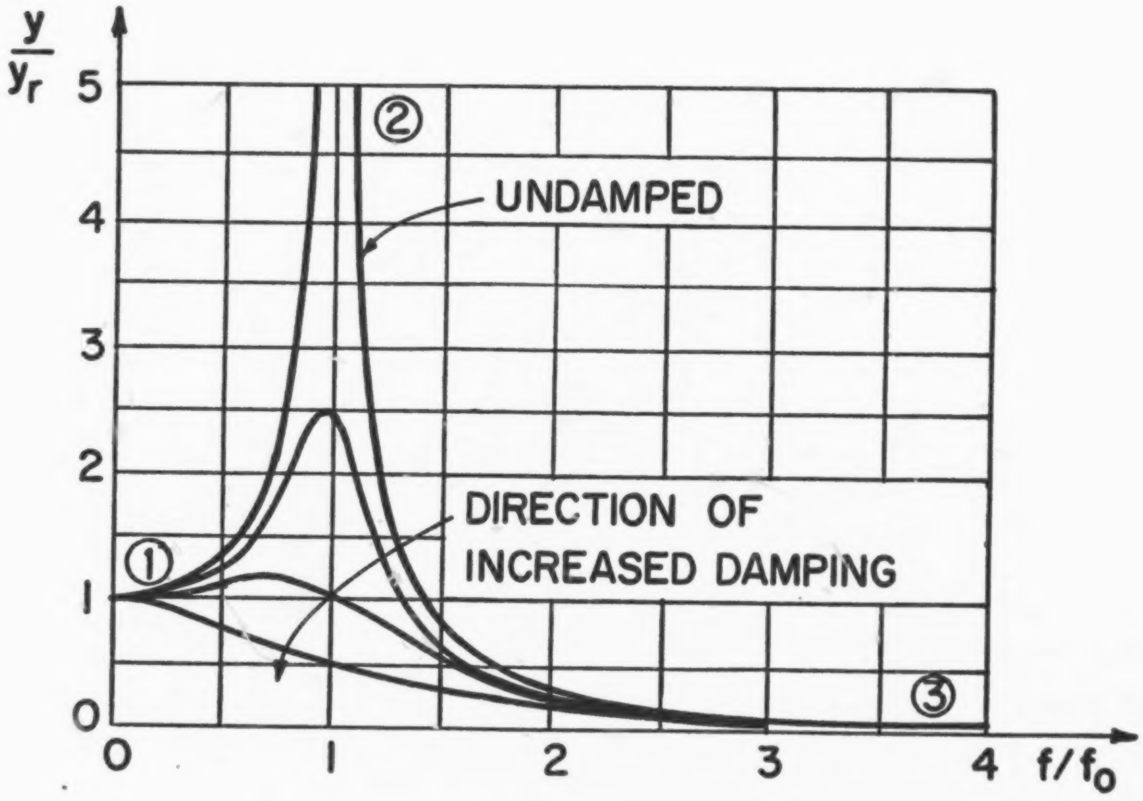


Fig. B-6.24. Amplitude response-excitation curve for a damped mass-spring system.

We observe from the model that the general features of the response excitation curve are preserved when we add damping to the model. We still have three general regions, marked 1, 2, and 3 in Fig. B-6.24. Again in region 1, where the ratio f/f_o is small, the ratio of the amplitudes is almost unity. Unless the damping is very high, excitation near the natural frequency leads to a large response(region 2). Finally, in region 3, an excitation of high frequency results in a negligible output. It is because of the similarities in the general features of the damped and the undamped case that it is useful to study the latter. On the other hand, if we design systems that are heavily damped, where we must reduce the effects of resonance, the undamped model is an oversimplification. It then becomes crucial to include damping in the model from the beginning of our model construction.

B-6.6 CALCULATION OF NATURAL FREQUENCY IN DYNAMICAL SYSTEMS

We have observed the importance of the natural frequency in the suspension system and we shall note its importance in many other instances. Indeed, the concept of natural frequency is central to our entire study of dynamical systems. It is not only the frequency at which the system oscillates if disturbed, it is also the frequency at which the system may be driven into violent oscillations.

For the simple, undamped case we have examined, the natural frequency is given by a very simple formula:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

where the spring constant k is expressed in newtons/meter, the mass m in kilograms, and the frequency f_o in cycles/second. This formula holds generally for mechanical systems if k and m are properly interpreted. For example, suppose you wish to determine the natural frequency of the up and down motion of your car. You could get into the car and have a friend measure how much the car deflected because of your weight. It is now possible to compute the stiffness k by dividing the weight by the deflection. With the mass of the car known, the natural frequency can be calculated. Similarly, to compute the frequency of horizontal to-and-fro motion, you would merely have to determine the stiffness of the spring in a horizontal direction. This could again be done by applying a known horizontal force and measuring the corresponding deflection. In some cases, however, it is easier to measure the natural frequencies with a stop watch.

But, in designing something, it is important to know the natural frequency before that something is built. In such cases, the above equation for natural frequency can be useful indeed. Let us illustrate its applicability and universality with a few examples.

As a start, we can verify the equation by means of the vertical oscillations of a car. If your weight is 600 newtons (about 140 lbs), and you find that the car sags 0.005 meters (about 1/4") when you sit in it, then the corresponding value of k is

$$k = \frac{600}{0.005} = 120,000 \text{ newtons/meter}$$

If the car has a mass of 1739 kg (about 4000 lbs) and your mass $\left(\frac{\text{weight in newtons}}{9.8}\right)$ is 600/9.8 = 61 kg, the combined mass is 1800 kg.

$$f_o = \frac{1}{2\pi}\sqrt{\frac{120,000}{1800}} = 1.30 \text{ cycles/second}$$

You can try this experiment using your own weight and a car whose weight you know to see how closely the calculated frequency agrees with what you measure with a stop watch.

As another example: suppose you are the chief engineer for the design of a 300-meter (about 1000 ft.) high skyscraper which will be of square cross section 30 m (about 100 ft.) on each side (Fig. B-6.25). You are informed that the tenant in the building will be a laboratory in which sensitive equipment that must not be excited at frequencies above 2 cycles per second is used. You know of a similarly sized and designed building in Europe that is known to deflect as much as a 0.005 m (about 1/4") inch at the top in gale-force winds which exert a pressure of 600 newtons/m^2 (about 14 lb/ft^2). It is also known that the building's mass is about 10^{11} kg. Can you assure the tenant that such equipment will not be damaged?

It may appear to be fantastic to use our equation for natural frequency

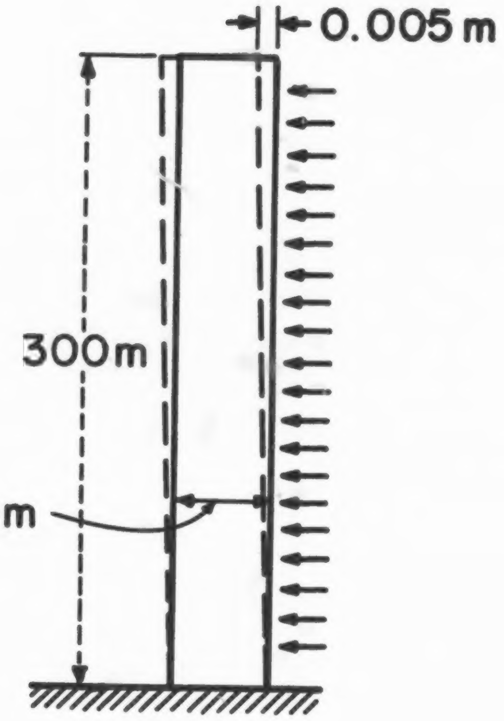


Fig. B-6.25. Skyscraper moment caused by wind.

for anything as huge as a skyscraper. After all, a skyscraper does not have the appearance of a mass and a spring. What does our equation have to do with this problem?

As we stated in the introduction to this chapter skyscrapers "wave in the wind", just as the data of the problem indicate . We can therefore model the skyscraper as a vertical reed with the entire mass of the building concentrated at the top, as in Fig. B-6.26. The springiness of the reed will oppose

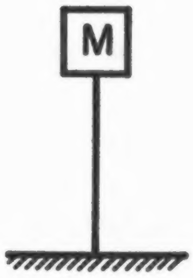


Fig. B-6.26. Preliminary model of skyscraper.

the horizontal deflection of the mass. The idealization can be carried further and we can replace the reed with a horizontal spring (Fig. B-6.27). This brings us back to the simple mass-spring system.

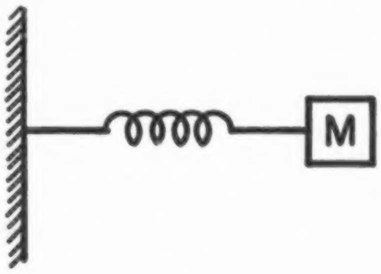


Fig. B-6.27. Final model of skyscraper.

Of course, a skyscraper is as complicated a vibratory system as an automobile. There are other ways in which the structure can oscillate, in addition to its sway in the wind. It can oscillate up and down, or it can twist around a vertical axis through its center, to name just two types of vibration. In a detailed examination of this problem, all these types of oscillation would be investigated. But you may know or guess that in this case these other oscillations are of higher frequency than the type pictured above, and that they are also very highly damped. It may then be possible to get a rough estimate of the important natural frequency by determining the stiffness from the data given for the European building.

Knowing the deflection of the top is the maximum for the building, we can assume that the average deflection is one-half of this value. But since our estimate can only be very crude, at best, let us simply assume that the spring deflects 0.005 m for a force equal to the total sidewise force corresponding to a pressure of 600 newtons/m^2 (about 14 lb/ft^2). As the area over which this pressure acts is a rectangle 300 m times 30 m, the total force can be computed by multiplying the pressure by the area over which it acts.

$$\text{force} = 600\ \text{newton/m}^2 \times 9000\ \text{m}^2 = 5.4 \times 10^6\ \text{newtons}$$

The stiffness k is therefore

$$k = \frac{5.4 \times 10^6\ \text{newtons}}{5 \times 10^{-3}\ \text{meter}} = 1.08 \times 10^9\ \text{newtons/meter}$$

From the equation we therefore calculate the resonant frequency as

$$f_o = \frac{1}{2\pi} \times \frac{1.08 \times 10^9}{10^{11}} = 0.017\ \text{cycles/second}$$

Thus, if the building is excited at 2 cycles a second as it may well be by passing trucks), the ratio of the excitation to natural frequency is about 2.0/0.017 = 125 to 1. From Fig. B-6.20, practically none of this frequency will be apparent to the tenant. If our estimate were in error by a factor of 10, so that the excitation frequency is only 12.5 times the natural frequency, we would expect a negligible amount of a 2-cycle excitation to get through. Hence we can assure the tenant that it is improbable that his equipment will be disturbed.

As a final example, imagine the following situation. You are an architect who has designed a dramatic house for a client. The house is on a cliff overlooking the Pacific Ocean (Fig. B-6.28). A bridge will be required

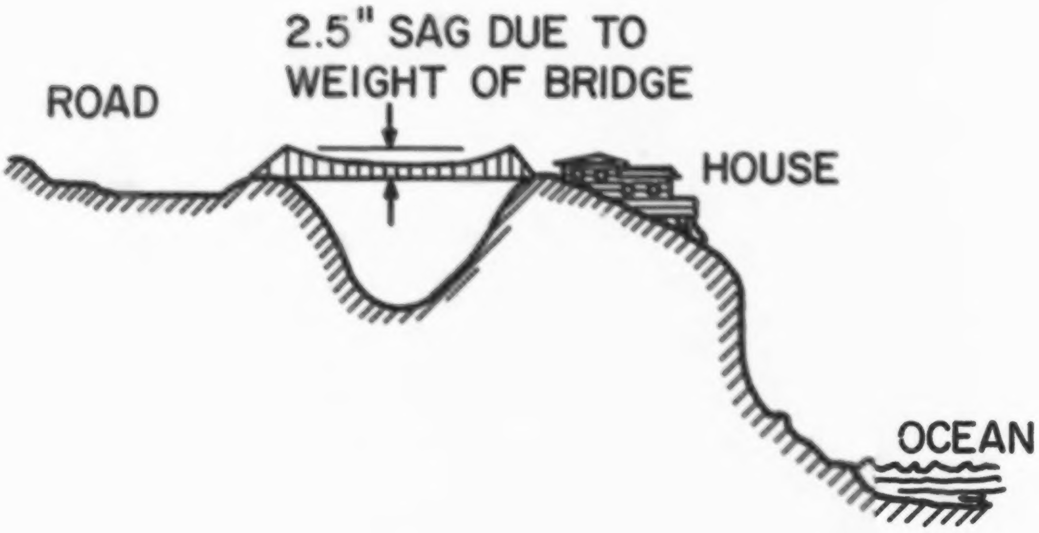


Fig. B-6.28. Bridge over gorge connecting road to house.

over a gorge to connect his house to the road. The client is worried about the effects of possible earthquakes, and you discover that earthquake waves in this area have frequencies below this two-cycles per second. You have already calculated the amount of sag in the main suspension cables when the deck mass is in place. This has been estimated at a maximum of 0.065 m (about 2.5 inches), you can calculate the natural frequency of the bridge. Is your design adequate?

Of course, all of the previous precautions hold; there are many possible ways that the bridge can oscillate. A more careful study is required, but, once again, even a rough answer can be useful. Investigators are continually amazed how often such rough calculations turn out to be accurate and informative. Let us begin with the construction of a model.

The stiffness of the bridge is mainly the result of the fact that a force must be exerted to cause the suspension cables to deflect (the deck of a suspension bridge is hung from the cables like draperies that are hung from a drapery rod). Therefore, the first step in building the model is to replace the mass of the bridge by a single, concentrated mass at the center (Fig. B-6.29). We now imagine the cables to act as gigantic rubber bands that stretch when a force is applied to the bridge. This leads naturally to the idealization of Fig. B-6.30 as the final step in the construction of our model.

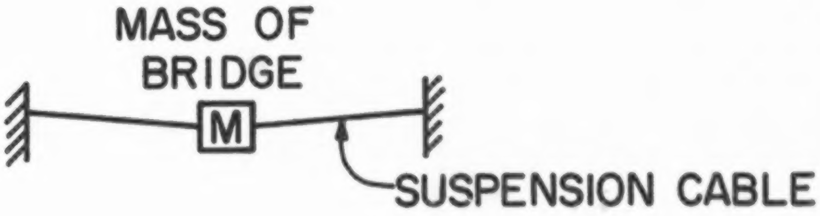


Fig. B-6.29. Preliminary model of bridge.

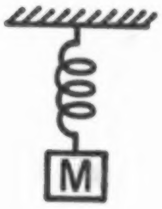


Fig. B-6.30. Final model of bridge.

This leaves the question of the computation of the natural frequency, knowing only the sag of the spring due to the weight of the mass M. Since M= W/g, and our original formula for resonant frequency is $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ we can substitute for m to obtain

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k\,g}{W}} = \frac{1}{2\pi}\sqrt{\frac{g}{(W/k)}}$$

where W is the weight of the bridge and g is the gravitational constant that converts mass to weight ($g = 9.8\ m/s^2$).

But what is W/k? It is the deflection of the spring when a force W is applied. Hence we can write our equation in still another form

$$f_o = \frac{1}{2\pi}\sqrt{\frac{g}{d}}$$

where d is the sag due to the weight of the mass in the mass spring system. If g has a value of $9.8\ m/s^2$, and if d is given in meters, f_o will be in cycles per second. Hence, we obtain:

$$f_o = \frac{1}{2\pi} \sqrt{\frac{9.8}{0.065}} = 1.96 \text{ cycles/second}$$

The bridge design seems to be in trouble because of earthquake danger. Even a mild earthquake at 2 cycles a second could excite the bridge to resonance and cause serious damage.

What can be done? You may, of course, attempt to redesign the bridge so that it is stiffer and has a much higher natural frequency; far above that produced by a typical earthquake. Or you may try to achieve a natural frequency far below the earthquake frequencies. There is a possibility that earthquakes in this region are never severe or of long duration so that the natural damping in the bridge will prevent any serious "build-up" of stresses. Perhaps more damping can be provided in the design. At any rate, this preliminary calculation shows that your client's fears may be well grounded and that you should examine your design carefully.

B-6.7 MULTIPLE RESONANCES

We have been using models with only one natural frequency or resonance. Typical systems can oscillate at many "natural" frequencies. For example, an automobile can oscillate vertically (up and down), or horizontally (to-and-fro), or it can rock back and forth like a teeter-totter in a pitching motion, to name a few of its possible motions. Each of these natural oscillations results from the interaction of different sets of masses and springs. Hence, they each involve a vibration at a different natural frequency.

Each of these different types of oscillation is called a "natural mode" of oscillation of the system. The different modes can each be driven into resonance independently. The resonant frequency in each case is the natural frequency of that mode. If we were to drive an automobile over the sinusoidally varying road that we used in our earlier illustration, and if we were to plot the magnitude of the passenger's displacement, we would discover that the curve may resemble that in Fig. B-6.31. We may find two groups of resonant

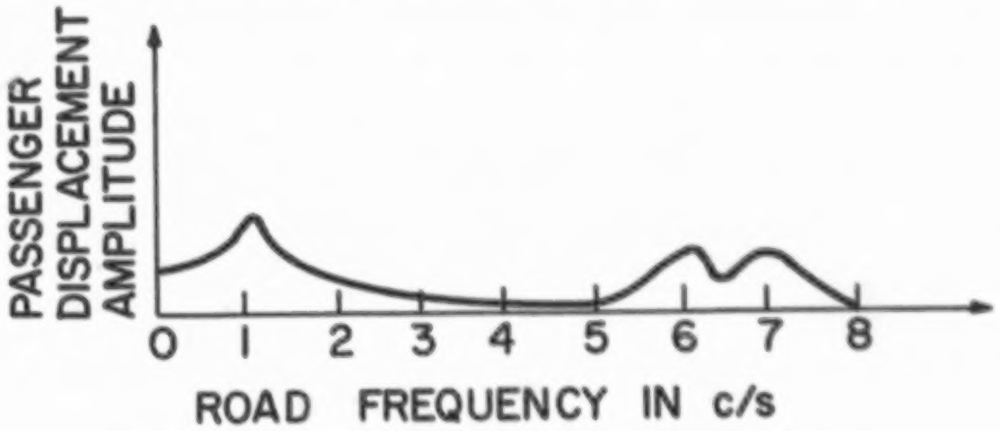


Fig. B-6.31 Passenger displacement versus road frequency for travel over a sinusoidal road.

frequencies, one in the neighborhood of one cycle per second and others in the neighborhood of 6 cycles per sec. The lower frequency corresponds to the resonance discussed in Section B-6.4. The upper frequencies correspond to those mentioned above.

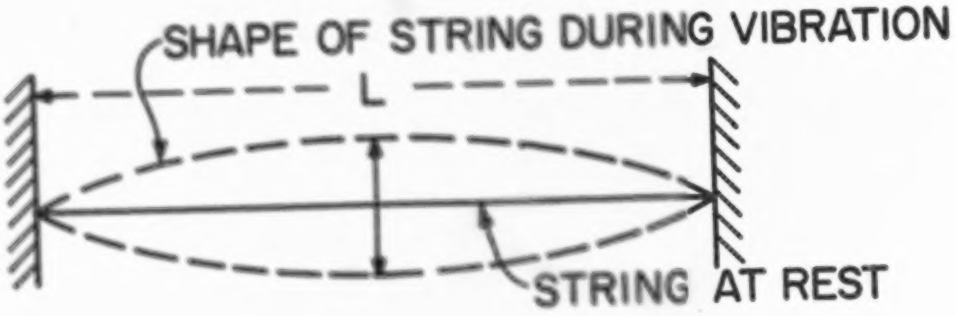


Fig. B-6.32. String vibrating in the fundamental mode.

In the case of the automobile, the resonances were quite heavily damped and therefore were not sharp. In other systems, damping is intentionally light and the resonant frequencies are prominent. Musical instruments provide many examples of lightly damped systems. For instance, a taut string or wire is a common element in musical instruments. If the string is under a tension T; has a weight per unit of length W; and has a length between stationary points L, then when it vibrates in the mode shown in Fig. B-6.32, its natural frequency is

$$f_1 = \frac{1}{2L}\sqrt{\frac{Tg}{W}}$$

where g is the acceleration of gravity ($9.8\ m/s^2$). This is the mode of lowest natural frequency.

When the string is vibrating as shown in Fig. B-6.33, the frequency is higher. In this case, the midpoint of the string remains stationary. Each

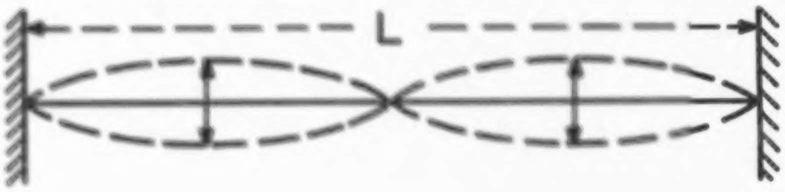


Fig. B-6.33. String vibrating in the second mode.

vibrating half-string in this case will have the same tension and, of course, its weight per unit length is the same as in the mode of lowest frequency. But now the vibrating length is half that of Fig. B-6.32. Therefore, to get the natural frequency in this case, we simply replace L in the above equation by L/2. This gives

$$f_2 = \frac{2}{2L}\sqrt{\frac{Tg}{W}} = 2f_1$$

In general, it is apparent that when the string is oscillating in k sections, the frequency will be

$$f_k = \frac{k}{2L}\sqrt{\frac{Tg}{W}} = kf_1$$

Thus, the string has an infinite number of natural frequencies corresponding to an infinite number of modes. The lowest of the natural frequencies is called the fundamental frequency. The higher frequencies are called overtones or harmonics.

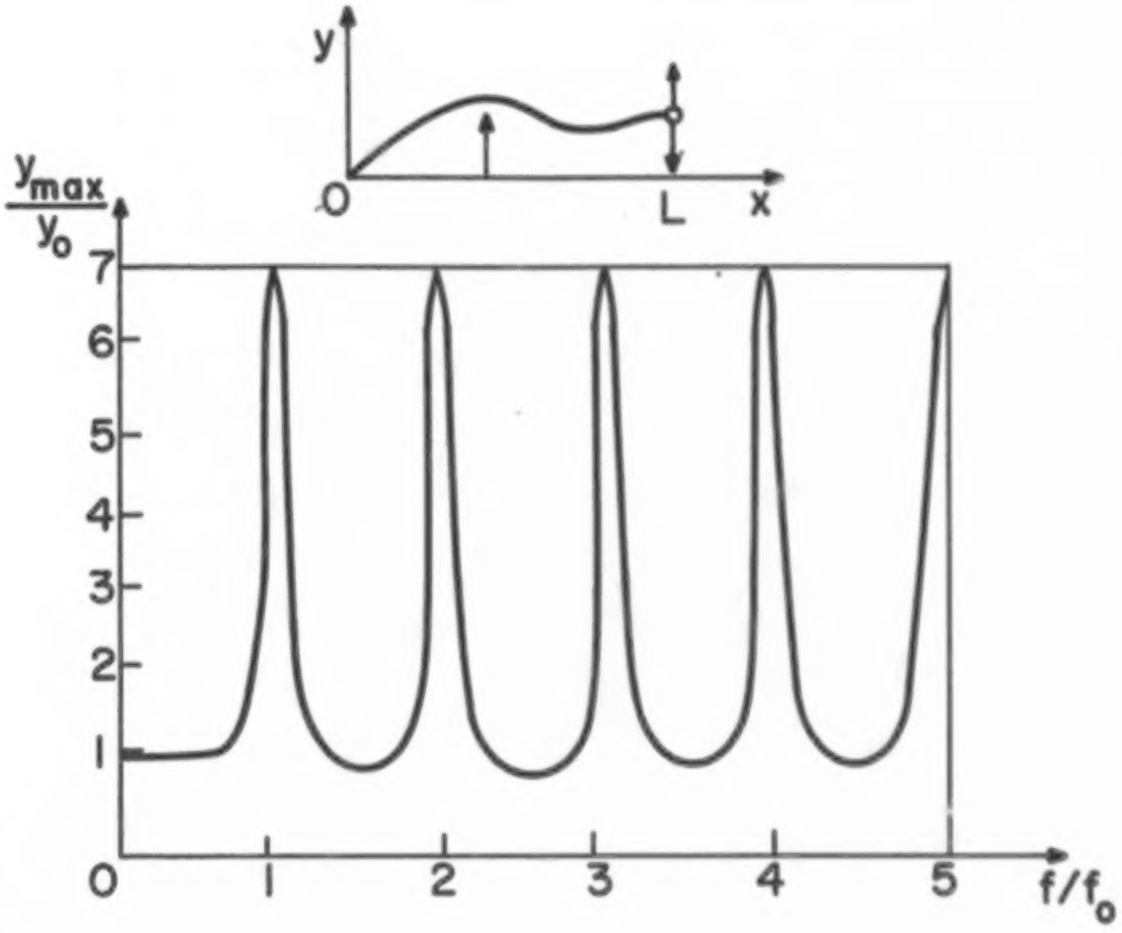


Fig. B-6.34. Amplitude-response curve for a string excited at one end.

Fig. B-6.34 shows the amplitude response-excitation curve for an undamped string being excited by sinusoidal up-and-down motion at one end. We see that the resonances in this case occur at integral multiples of the fundamental frequencies and are very sharp.

The bowing of a violin excites not only the fundamental mode in a violin string, but many of the overtones as well. The excitation of overtones produces a tone of richer quality than the excitation of the fundamental frequency alone. These components are reinforced by resonances in the violin body. The amplitudes and mixture of the various overtones varies with the violin and the bowing of the player. Mainly for this reason, two violinists playing the same violin produce different tone qualities. (There are, of course, other sources of difference which result from the manner in which notes are started and stopped).

The generation of sound is often a consequence of resonance. Air has mass and is also compressible or flexible; that is, it has a stiffness, or springiness. Mass and stiffness are the ingredients for a vibratory system. Indeed, air in a fixed enclosure like a room or an auditorium will have certain natural frequencies. In acoustical design the problem of multiple resonances is of great importance.

B-6.8 UNDESIRABLE EFFECTS OF RESONANCE

We have already discussed several examples in which systems may be driven into resonance to produce undesirable effects. The bumps or irregularities of a road can drive the car into resonance in any one of many different modes and thus produce an uncomfortable ride. Earthquakes may drive buildings or bridges at resonance so as to cause damage. It has been customary for a long column of soldiers marching over a bridge to break step to avoid a resonant collapse of the bridge. Soldiers march at about 120 steps per minute, or about 2 steps per second. On a bridge with a natural frequency of 2 cycles a second, especially a weak footbridge, a long column of soldiers marching in step could excite the bridge into large oscillations. With today's heavy bridges, built for automobile traffic, a catastrophe of this sort would seem problematical.

There are, however, no dearth of examples where resonance has caused serious destruction in systems. Often these effects arise in subtle and unanticipated ways. One example is the break-up of ships due to resonances excited by ocean waves. Figure B-6.35a illustrates a ship riding the crest of a wave. Many types of waves are cyclic or periodic and can therefore apply periodic excitations to the ship. As is the case with the other structures already studied, a ship has many possible modes of vibration and natural frequencies. It is not difficult to picture one of the basic modes of oscillation in terms of a simple model (Fig. B-6.35(d)). We first concentrate the entire mass of the ship into 2 equal masses, one at each end, and imagine the two ends to be joined by a flexible rod. When the ship is on the crest of a wave, (Fig. B-6.35b), the end masses cause the ship to drape itself over the wave, bowing the ship upward. When the ship is in the trough of a wave, the end masses cause the ship to bow in the opposite direction (Fig. B-6.35c). This motion is called bending or flexing.

As is clear from Fig. B-6.35(d), if we consider only half of the ship, we

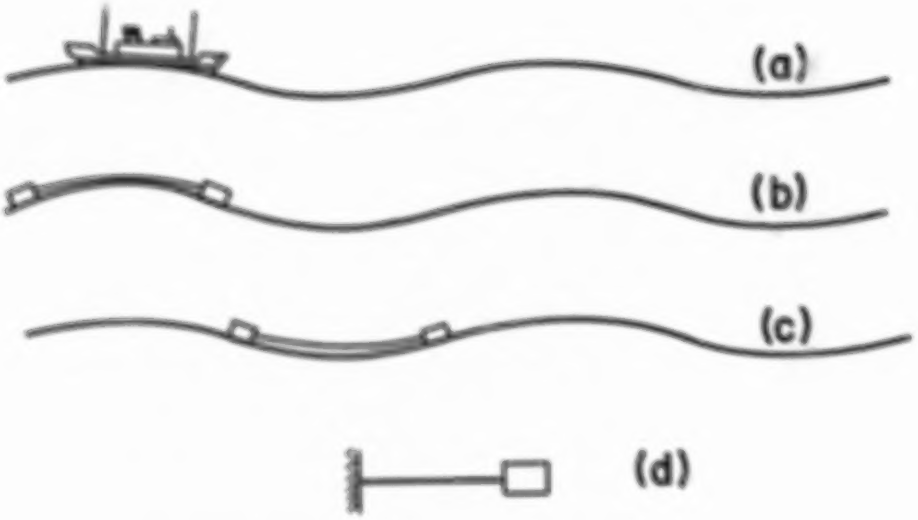


Fig. B-6.35. Model of ship bending.

have the same situation as the skyscraper of the previous section. Each half is modeled by a mass at the end of a reed, (Fig. B-6.35d). The system has a natural frequency in bending or flexing which can be excited to resonance if the speed of the ship and the period of the waves have just the right magnitudes. When the ship is executing such flexing motions that it will tend to break at the middle. Indeed, during World War II, a number of ships with welded seams collapsed at the middle in rough seas.

Another example is the "singing" of power transmission or telephone lines. When the wind blows against a thin wire, little whirlpools, called eddies,

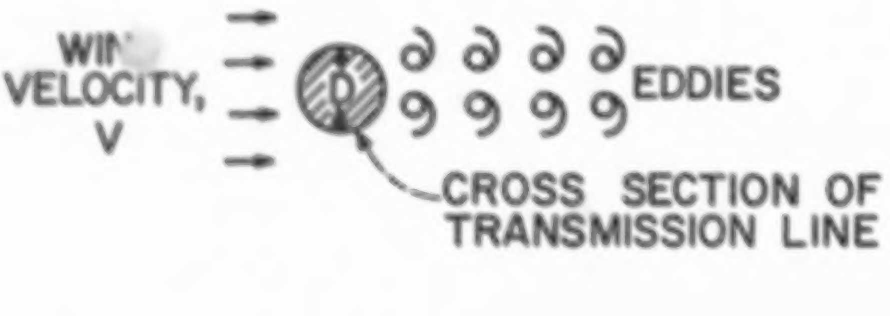


Fig. B-6.36. Excitation of "singing" transmission line by eddies.

are formed (Fig. B-6.36). The same phenomenon can be observed in flow of rainwater around an object placed in the street after a rain storm or in the flow of a river around a bridge abutment. The eddies form periodically and give rise to a periodic excitation of the wire. The frequency of this excitation is given by the following equation:

$$f = 0.22 \frac{v}{D}$$

where v is the velocity of the wind, D is the diameter of the wire, and f is the frequency of the eddy formation (the number of eddies formed per second.) The number 0.22 is a constant, the so-called Strouhal number, independent of the units as long as the length units for measuring v and D are the same. For example, v may be in feet per second, and D in feet; or v may be in meters per second, and D in meters.

It is common for the wind to excite wires, especially exposed bare wires on long spans, to resonance because of this phenomenon. The usual effect is to cause the line to "sing", sometimes with a very pleasant tone. For example, suppose the wind velocity is 30 miles per hour (44 feet per second), and the wire is 1/8" in diameter. Since 1/8" - 1/96 feet, our equation gives

$$f = 0.22 \frac{44 \text{ feet/second}}{1/96 \text{ feet}} = 845 \text{ cycles/second}$$

This is a high note in the soprano range. If the wind is steady, wires have vibrated with enough amplitude to break.

Several methods of protection against the destructive effects of this kind of resonance have been proposed. It is interesting that one of the most effective methods is also one of the simplest, and seems to have been invented independently by several people in different countries at different times. The key observation they have made is that when transmission lines are excited to resonance by wind eddies, it is usually one of the very high frequency modes of the line that is excited. As shown in the exaggerated sketch of Fig. B- .37, the peak-to-peak distance of the vibration pattern that develops is typically in the order of one foot. The solution requires the use of a helical piece of plastic about one and one half feet in length, (Fig. B-6.38). When this is wound on the line, the oscillations of the line decrease below the destructive level. The piece of plastic seems to do two things: first, it provides damping; second, the plastic helix seems to eliminate the resonant mode for an exciting frequency, for which the helix length is somewhat greater than the distance between

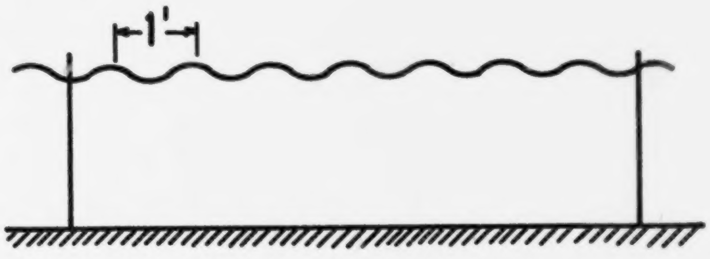


Fig. B-6.37. Mode shape of "singing" transmission line.

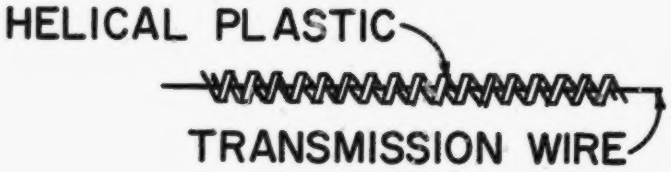


Fig. B-6.38. Damper for "singing" transmission line.

vibratory peaks.

Although this explanation seems to be plausible, the exact mechanism of the process has not been investigated in detail. In spite of only incomplete understanding, the engineering problem seems to have been solved.

B-6.9 USES OF RESONANCE

When a young child finally masters the art of pumping a swing, in a playground, he has learned that system resonance can be exploited for benefit. We have seen other examples of the exploitation of resonance in our discussion of musical instruments. There are other cases of useful applications of the resonance phenomenon. In building his world, man has used resonance as one of the keystones. For example, the effect plays a vital role in radio and communications systems of all kinds. To close this chapter, we will cite a few examples of the uses of resonance in the highly technological world which surrounds us.

One of the oldest and still one of the most useful applications of resonance is in the mechanism of the clock or watch. It is perhaps easier to describe this use of resonance in terms of a pendulum clock, although the principle is the same in all practically clocks except the electric clock which depends upon alternating current. First, we note that the pendulum executing small oscillations is equivalent to a mass-spring system. The restoring force, which is supplied by the spring in the mass-spring, is provided by gravity in the pendulum clock. The resonant frequency of the pendulum depends only on the length of the pendulum and the gravitational constant, g. In particular,

$$f_o = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Thus, the pendulum displays a very convenient property; namely, that its natural frequency is easily regulated. One simply varies the position of the bob along the rod. thus changing the value of L and thereby the period of the swing. This suggests the use of the pendulum as a timing mechanism. If it were excited at resonance, a small effort should make the pendulum oscillate at a definite frequency, which is determined only by the position of the bob.

One complication enters immediately: how can we provide the excitation? The excitation must be at the resonant frequency. To apply the excitation at the correct time one might conclude that another clock would be needed. This is not a very comforting thought; it raises the question, "how was the first clock built?" A means of overcoming this dilemna depends on the use of the clock itself as the mechanism for timing the excitation.

A sketch of a basic clock mechanism is shown in Fig. B-6.39. For clarity, only two teeth of the geared wheel are shown. In Fig. B-6.39a, the pendulum is shown in the vertical position. Imagine it to be given a small clockwise push. This will cause the fork to swing clockwise, releasing the gear tooth. The weight will then pull the gear counterclockwise so that the fork is struck by the next tooth (Fig. B-6.39b). The overall effect of this impact is to give the pendulum a small tap and also to rotate the fork so that it catches that same tooth when it reaches the position shown in Fig. B-6.39c. The gear is thus prevented from further rotation until the pendulum again reaches the left-hand portion of its motion (Fig. B-6.39b). The net result is a series of taps on the pendulum one at the end of each stroke as in Fig. B-6.40. In a real

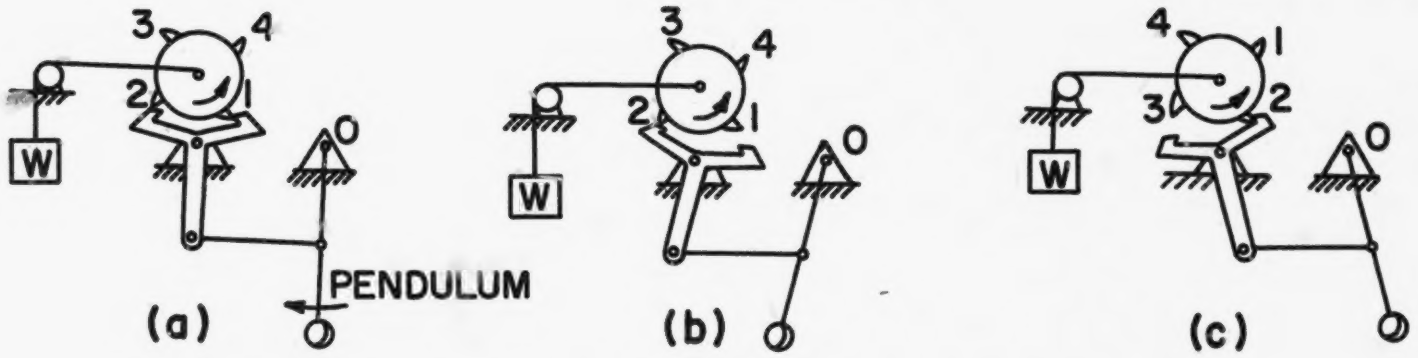


Fig. B-6.39. Basic clock mechanism.

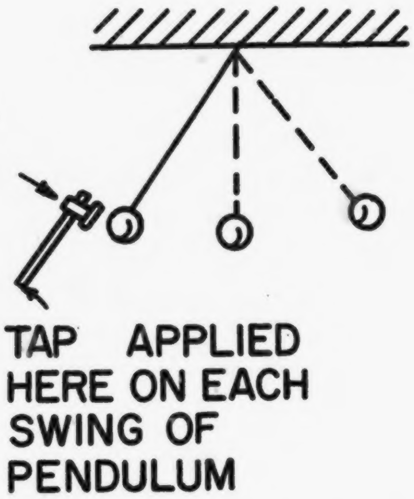


Fig. B-6.40. A pendulum motion analogous to basic clock mechanism.

clock mechanism teeth are spaced all around the geared wheel so that the cycle of operation can be continued indefinitely as long as there is an activating force, which is provided in this clock by the weight W.

In wrist watches a rotary mass-spring system is used to replace the pendulum as the resonating system. A coiled spring and balance wheel is used instead of the weight. In modern electronically-operated wrist watches which do not require the winding of a spring, a similar system is used; namely, the clock actuates a mechanism that automatically applies a periodic driving force. This idea has been used as the basis of timepieces from the invention of the pendulum clock to the present day.

A look at the response-excitation curves for a lightly damped dynamical system suggests another use of resonance--as a filter which is sensitive to sine waves of a particular frequency. Suppose we have as input to a resonant system a signal containing sinusoids of many frequencies. For example, in Fig. B-6.41, we show three sinusoids of frequencies 1, 2.2, and 4.6 cycles per second, and with amplitudes of 1/2", 1", and 3/4". Suppose we ride on a road whose form is equivalent to the sum of the three sinusoids. This sum is shown in Fig. B-6.42.

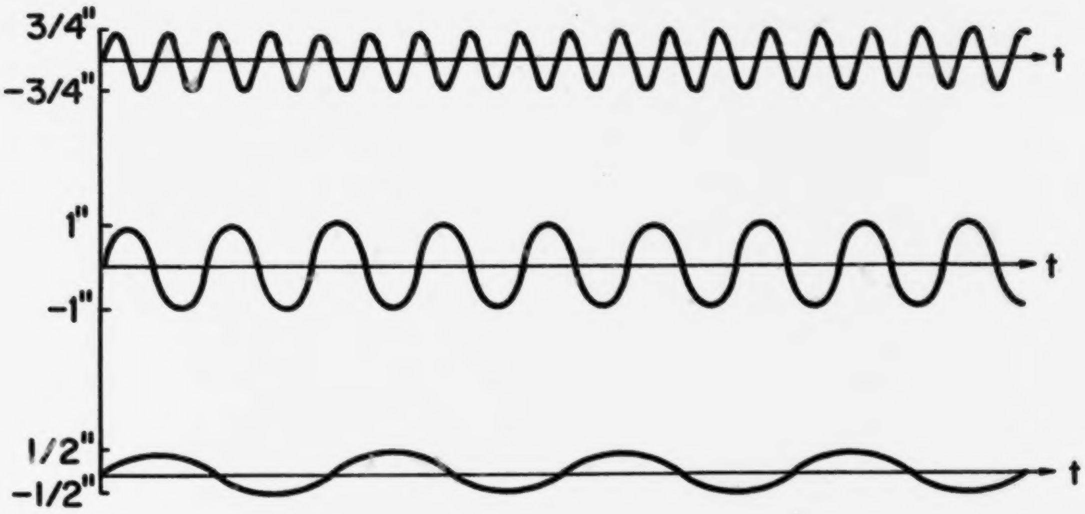


Fig. B-6.41. Sinusoidal components of input function.

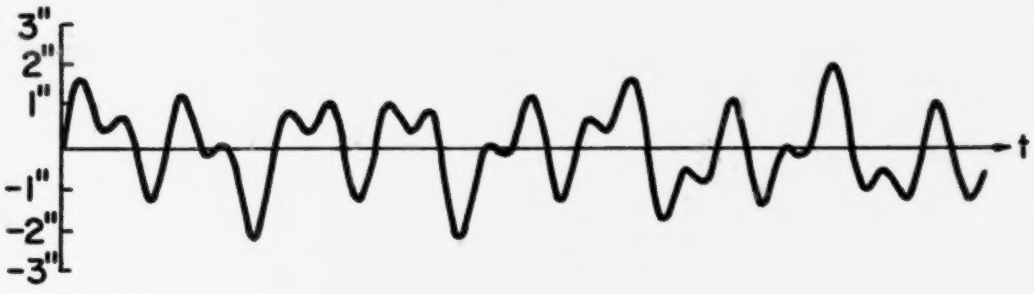


Fig. B-6.42. Composite input signal that is the sum of the three sinusoids Fig. B-6.41.

What will happen depends on the natural frequency of the suspension system. But if we use shock absorbers with very light damping to obtain a sharply resonant system, we can expect the car with a natural frequency of 1 cycle/sec to pick out the signal in the road of 1 cycle/second. That is, the higher frequencies will excite the car only slightly, and the car will respond as if only the sinusoid frequency of 1 cycle/second were present. (Remember that in the mass-spring system, the response to an excitation consisting of several components is the sum of the responses to the components taken individually.) In fact, by measuring the amplitude of the response at 1 cycle/second, we could use Fig. B-6.25 to compute the amplitude of the one cycle/second component of the road shape.

Similarly, if the suspension springs or the mass of the car were changed to give a resonant frequency of 2.2 or 4.6 cycles/second, the car could be used to measure the amplitudes of these components of the road shape. Thus, resonant dynamical systems can filter out or isolate signals of different frequencies. If a signal containing many frequencies is used as the input to a resonant system, we can obtain a signal which is almost completely that of the resonant frequency as the output. Furthermore, the amplitude of the output is proportional to the amplitude of the input signal that is filtered out.

No matter how sharply resonant the system, it will not filter out a single frequency but rather a range of frequencies. Figure B-6.44 shows the amplitude response curve for two sharply resonant systems with two values of

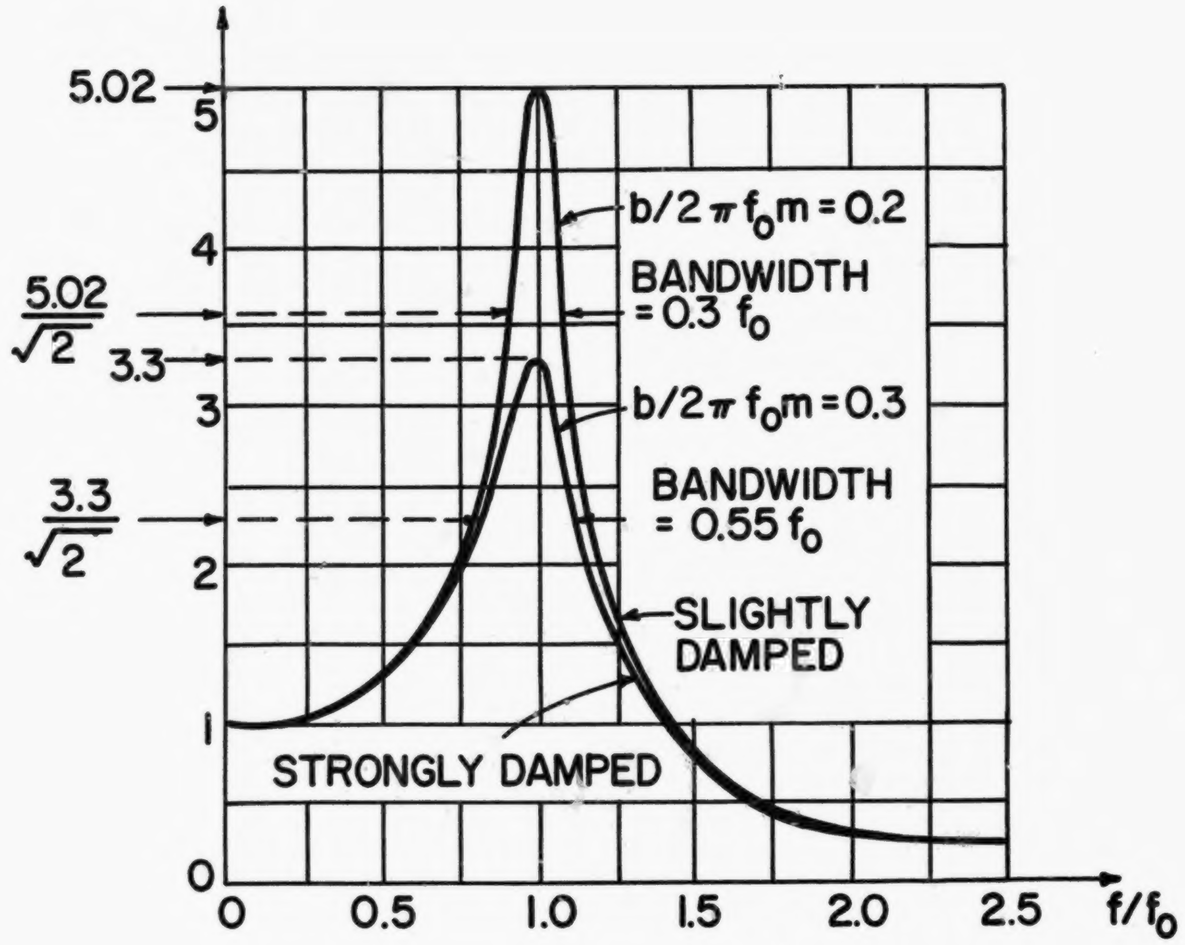


Fig. B-6.44. Bandwidth of resonator acting as a filter.

damping. The range of frequencies over which these systems are effective as filters is somewhat arbitrarily defined as those frequencies lying between the values at which the response amplitude is $1/\sqrt{2}$ of the maximum response amplitude. The size of this range of frequencies is called the bandwidth of the filter. It can be seen from Fig. B-6.44 that the bandwidth varies with damping; the greater the damping, the greater the bandwidth; i.e., the less selective the filter.

The use of the resonant frequency for filtering is very common. One example is in frequency measuring instruments. Figure B-6.45 shows a very simple frequency gauge consisting of tip masses mounted on flexible reeds.

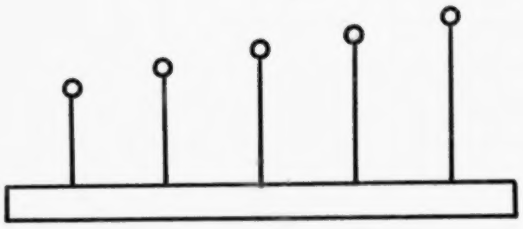


Fig. B-6.45. Reed frequency gauge.

The flexibility and length of the reeds and the magnitude of the tip masses are different so that each reed has a different natural frequency. When the base of the gauge is placed on a vibrating piece of equipment, the reed with a natural frequency close to that of the vibrating equipment is excited to large oscillations and the frequency of the vibration is determined.

The most common use of resonance as a filtering device is in radio and television. Every radio and television transmitting station emits a different frequency sine wave, which "carries" the program. The frequencies of these sine waves for A.M. broadcasting ranges from 550,000 to 1,600,000 cycles/second. Shortwave FM, and TV signals are also carried in this way. If the air is filled with all of these signals, how can the particular signal of Channel 2 be selected ? As you may guess, a resonant system is used to filter out Channel 2 from all of the other signals that are present. Of course, the resonant system used, is electrical rather than a mechanical one such as we have examined previously.

B-6.10 SUMMARY

In this chapter we examined the nature of the forces that must be applied to accelerate or set into motion material objects. Material objects have inertia; that is, the property of resisting change in motion. We also studied dynamic systems in which not only inertia but stiffness or "springiness" are present. We found that these systems have a natural tendency to oscillate when they are disturbed from their equilibrium positions. Further, we learned that there is a very simple formula for computing this natural frequency.

We also learned that the natural frequency was the resonant frequency--the frequency at which a dynamic system having both stiffness and inertia could be driven into violent oscillations by excitations of the same frequency. Damping however, could reduce the amplitude of a system driven at the resonant frequency.

The concepts of stiffness, inertia, natural frequency, and resonant frequency, and an appreciation of the role played by damping, were found to be of great importance in the design of mechanical systems in which motion occurred.

Systems were shown that had not only one resonance but many resonances, which, as in the case of musical instruments, could often be used to good advantage. We found that there were also conditions under which resonance could be destructive. One of the main tasks of structural engineers is the elimination of such resonances, which at times have led to catastrophic failures involving the loss of human lives as well as financial losses totalling millions of dollars. We also found that this phenomenon could be exploited by man for his benefit. Indeed, resonance in timepieces was one of the first phenomena exploited in developing the highly complex society in which we live. Finally, we noted that resonance is used in electrical and electronic systems as well as in mechanical ones.

APPENDIX A

In the body of Chapter B-6, the resonant frequency and the response - excitation curve for the mass-spring system were presented without being derived. In this appendix, we will derive these from the basic properties of mass and spring.

The derivation hinges upon a unique property of the sine wave when it is integrated. If, for example, an electrical sine wave is applied to the integrator section of an analog computer, the output of the integrator will also have the shape of a sine wave. No other repetitive wave shape will remain unchanged in form under this treatment.

Although the shape of the sine wave remains unchanged when it is treated by an integrator, two important changes do occur. First, the output wave is delayed one quarter of a cycle with respect to the input wave. Secondly, the amplitude of the output wave is different from the amplitude of the input wave by a factor which depends on the frequency of the input wave. This factor is equal to $\frac{1}{2\pi f}$, where f is the frequency of the sine wave. Thus if the amplitude of the sine wave entering the integrator were 8 units, then the integration will produce a wave which is delayed with respect to the incoming wave by one quarter of a cycle and with an amplitude that is $8 \times (\frac{1}{2\pi f})$ units. In general, if A_o represents the output amplitude and A_i represents the input amplitude, then

$$A_o = A_i \times (\frac{1}{2\pi f})$$

Since the output is simply a shifted sine wave, the process of integration may be repeated many times with a delay of one quarter of a cycle and multiplication of the amplitude by $\frac{1}{2\pi f}$ or $\frac{1}{\omega}$ $(2\pi f = \omega)$ for each integration. Thus a double integration would produce a sine wave which is out of step with the original wave by one half a cycle and with an amplitude which is $\frac{1}{\omega^2}$ as great as the amplitude of the original sine wave. The process of double integration of a sine wave is of particular importance in the material which follows. A general statement of the result of this operation of double integration is:

$$A_o = -\frac{1}{\omega^2} A_i$$

Here the negative sign is used to indicate the shift of one half cycle in the output wave, a condition which is equivalent to an inversion of the original wave.

This effect can be easily observed with an arrangement of equipment as shown in Fig. B-6.46.

A cathode ray oscillograph connected to the output of the signal generator displays a sine wave of frequency f. This sine wave is fed into the input of the first integrator. The output is attached to the CRO and is displayed as the dotted sine curve B. For comparison the original sine curve is shown as a solid line on the same display. The amplitude of the curve in a single integration has become $\frac{1}{2\pi f} \times A_i$.

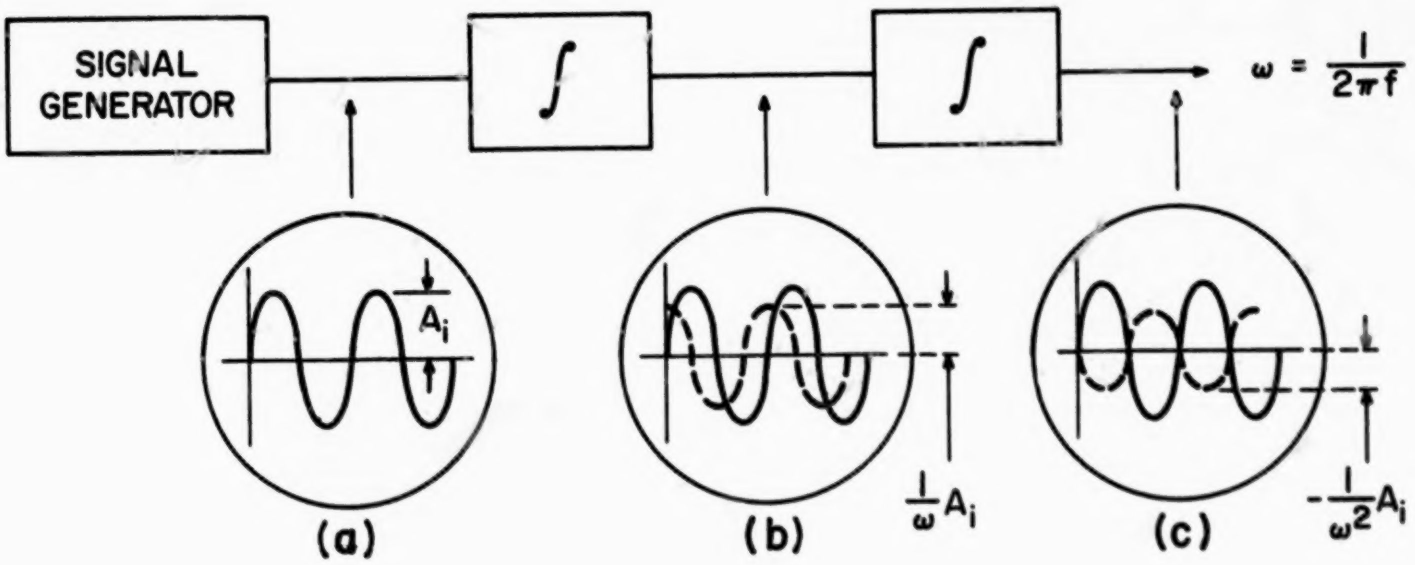


Fig. B-6.46

Display C shows the result of a second integration. Here again the dotted line represents the actual display and the solid line represents the original sine wave. The amplitude has now become $\frac{1}{\omega^2} = \frac{1}{(2\pi f)^2}$ times the original amplitude and the wave is now the inverse of the original wave. For this reason the output wave is indicated as the negative of the original, or

$$A_o = -\frac{1}{(2\pi f)^2} A_i$$

This result will be a key point in the derivation which follows:

Let us first derive the equation for the resonant or natural frequency of the spring and mass. The basic property of a spring is that the force it exerts is proportional to the compression from its original length. If this compression is represented by $\underline{d}$, then, $F_s = -kd$. The basic property of a mass is expressed in Newton's Law:

$$F_m = ma$$

In the mass spring system, the force of the spring is the force which is responsible for the acceleration of the mass, so that

$$F_s = F_m \text{ or } -kd = ma$$

In Chapter B-2 (Section B-2.5) we learned that acceleration can be converted into a displacement by a double integration; the first integration of the acceleration gives us the speed change during the given time interval, and the integration of the speed change gives us the change in the displacement of the moving vehicle during the given time interval. This relationship between acceleration during any time interval and displacement during the same time interval is quite general and can be applied to the vertical motion of the mass-spring model of the car body.

When the car body (mass-spring) is in free vertical vibration at its natural frequency, its motion is sinusoidal. We have just seen that when a sine wave is integrated twice it remains a sine wave but with a delay and a change in amplitude. For a sinusoidal motion, as displayed by the mass-spring system, the displacement, x, and the acceleration, a, have been shown to be related by the equation:

$$x = -\frac{1}{(2\pi f_o)^2}\ a$$

where f_o is the resonant frequency. We can substitute this value for x in the previous equation;

$$ma = -kx = \frac{k}{(2\pi f_o)^2}\ a$$

The a's cancel and we can then solve for f_o:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

This equation is the one quoted in the body of this chapter.

APPENDIX B

We can also derive the formula for plotting the shape of the response vs. excitation curve for the mass spring system. (Fig. B-6.20). In this derivation we are not concerned with the free sinusoidal motion of the spring mass, but rather in its motion in response to a sinusoidal input of a frequency which may differ considerably from the resonant frequency of the system. What we are concerned with is the effect of different driving fequencies on a system with a fixed mass and spring constant, which of necessity has a definite resonant or natural frequency.

Let us recall that for any sinusoidal wave the relationship between the amplitude of the displacement and the amplitude of the acceleration is:

$$y = -\left(\frac{1}{2\pi f}\right)^2 a$$

where y is the maximum displacement of the car body and a is the maximum acceleration at the same instant.

We have also observed that the acceleration of a mass is related to the force acting on the mass in accordance with Newton's Law

$$F = ma$$

Two equations are thus available in each of which the acceleration "a" appears. While Newton's Law holds for any instant, the first equation is a statement of amplitude relationship and the term amplitude indicates the maximum value achieved during a sinusoidal motion.

Thus the acceleration in the first equation represents the maximum acceleration of the car body. Since we wish to combine this equation with Newton's Law, we must make certain that the force factor F represents the maximum force acting on the car body, for only under this condition will the factor "a" represent the maximum value of acceleration.

What is the maximum value of the acting force? Obviously, it arises when the spring has developed a maximum compression or tension and is equal to the spring constant k multiplied by this maximum compression or tension. Since we represent the maximum displacement of the car body by y and the maximum change in the road level by y_r, the difference between these too maximum values must represent the change in the length of the spring. We thus can indicate the maximum value of the force in Newton's equation as $-k(y-y_r)$, and we can rewrite the equation:

$-k(y-y_r) = m\,a_{max}$ where a_{max} now represents the maximum acceleration of the car body and m represents the mass of the car body.

Therefore, $a_{max} = -\frac{k}{m}(y-y_r)$

and $a_{max} = -(2\pi f)^2 y$

Combining the above, $-\frac{k}{m}(y-y_r) = -(2\pi f)^2 y$

We have shown in Appendix A that $\frac{k}{m} = (2\pi f_o)^2$ where

f_o is the resonant frequency of the spring-mass system

therefore $$(2\pi f_o)^2 (y-y_r) = (2\pi f)^2 y$$

$$(2\pi f_o)y^2 - (2\pi f)^2 y = (2\pi f_o)^2 y_r$$

$$y(f_o^2 - f^2) = y_r f_o^2$$

$$\frac{y}{y_r} = \frac{f_o^2}{f_o^2 - f^2} = \frac{1}{1 - \left(\frac{f}{f_o}\right)^2}$$

This is the equation which describes Fig. B-6.21. At driving frequencies f which are considerably smaller than the natural frequency f_o of the spring mass system, the ratio $\left(\frac{f}{f_o}\right)$ in the above equation becomes small enough to have little effect on the denominator. The ratio of $\frac{y}{y_r}$ is then very nearly 1.

When the system is driven at $f = f_o$, the denominator becomes zero and the ratio $\frac{y}{y_r}$ then becomes infinitely great. Finally, when the driving frequency f becomes increasingly larger than the resonant frequency f_o of the system, the ratio $\frac{y}{y_r}$ becomes increasingly smaller, so that for very rapid road variations the displacement amplitude of the car body becomes negligible.

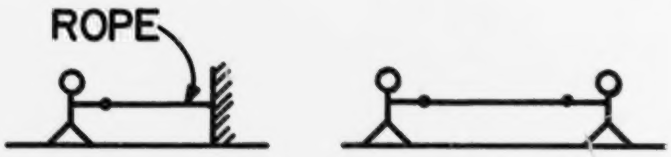
PROBLEMS

B-6.1. A rocket which has a mass of 5,000 kg is launched with a force of 190,000 newtons. What is its initial acceleration when the resisting forces are negligible? How many multiples of the acceleration of gravity is this?

B-6.2. A man is pushing a box across the floor with a force of 200 newtons. The box weighs 98 newtons and is accelerating at the rate of 18 m/s^2. What is the force of friction opposing the motion of the box?

B-6.3. A rocket sled is accelerating at the rate of 90 m/s^2. It is acted upon by a jet that exerts a force of 10,000 newtons and a retarding force of friction of 100 newtons. What is the mass of the sled?

B-6.4. You find that you can pull on a rope that is tied to a building with a force of 800 newtons. Suppose you engage in a tug of war with an equally strong opponent. What will be the force in the rope?



B-6.5. A spaceman out in space beyond the measurable pull of any planets does a space walk by using a gun that emits a gas jet.

a. If his mass is 70 kg and he applies a 7000-newton force, at what rate will he accelerate?

b. How many multiples of the gravitational acceleration on the earth will this be?

B-6.6. A large mass (history book, cement block, etc.) is hung on a length of wire from the ceiling of a room. Various people push against the mass over a period of one minute with pulses of varying frequencies but equal force. The maximum displacement of the mass is recorded for each frequency.

Frequency of pulses (Pulse/second)	Displacement (inches)
0.2	0.5
0.5	1
0.8	8
1	4
1.2	6
1.5	18
1.7	6
2	2
2.2	1.5

a. Graph the input-output characteristic of this pendulum.

b. What is the resonant frequency of this pendulum?

c. If this pendulum were allowed to swing freely, at what frequency would it swing?

B-6.7. If the mass of a car is 2000 kg and its suspension system has an effective spring constant of 72,000 N/m, find the amplitude of the road if the road frequency is 2, 4, 8, and 10 rad/s and the amplitude of the car is 0.01 meter.

B-6.8. Suppose the car is going twice as fast as in Problem B-6.7. Find the road amplitude.

B-6.9. If the road amplitude is 0.0089 meters and the car's amplitude is 0.01 meters, find

a. the ratio of k/M if the road frequency is 2 rad/s. and

b. If M is 3000 kg, what is the spring constant k?

B-6.10. If the natural frequency of the car is 2 c/s and the ratio of car amplitude to road amplitude is 2.0, what is the road frequency as seen from the car?

B-6.11. If the ratio of car to road amplitude is 3.0 and the road frequency is 2 c/s, what is the car's natural frequency?

B-6.12. Suppose a car is traveling over a road at 30 miles per hour (44 ft/s) and the peak to peak distance between bumps is 22 feet. If the natural frequency of the car is 1.0 c/s, what is the ratio of the car's amplitude to the road's amplitude?

B-6.13. You are traveling in a car at 60 mi/h (88 ft/s). The ratio of the car's amplitude to the road amplitude is 3 to 2. How far apart are the bumps of the road spaced if the car's frequency is 1 c/s?

B-6.14. A seismograph is an instrument for measuring the deflection of the earth during earthquakes. It consists basically of a heavy mass suspended from a spring. The spring is suspended from a rigid mount which rests on the earth. The mass-spring system is of very low frequency, of the order of 1/10 c/s, which is well below the frequencies of earthquakes. On the basis of Fig. B-6.20, explain how the seismograph works.

B-6.15. Do Problem B- .11 on the assumption that the damping quantity has the value $b/2\pi f_o m = 0.4$. (Use Fig. B-6.24)

B-6.16. Do Problem B-6.12 on the assumption that the damping quantity $b/2\pi f_o m = 1.0$.

B-6.17. Do Problem B-6.13 on the assumption that the damping quantity $b/2\pi f_o m = 1.0$.

B-6.18. If an automobile has a mass of 1500 kg and a spring constant of 96,000 newtons/meter, what is its natural frequency in cycles per second?

B-6.19. You observe that the natural frequency is up-and-down motion of a car is 1 c/s, you also know that the car has a mass of 1000 kg. What is the spring constant k?

B-6.20. If the natural frequency of a car is 1 c/s and the spring constant is 78,800 newtons/ meter, what is the mass of the car?

B-6.21. You find that the A string on your ukelele is about one half-tone flat, so that it is vibrating at about 830 c/s instead of 880 c/s. When you tune the string up to pitch, by what per cent do you increase the tension on the string?

B-6.22. A string which has a natural frequency of 128 c/s is made to vibrate in three parts. What is the frequency of the sound produced?

B-6.23. Two strings of the same length and having the same weight are set into vibration. If the natural frequencies have a ratio of 2 : 1, what is the ratio of the tensions on the strings?

B-6.24. At what frequency would a smoke stack 5 feet in diameter vibrate in winds of 15 mi/h, 20 mi/h, 30 mi/h, 45 mi/h, and 60 mi/h?

B-6.25. The frequency of a pendulum with a mass of 10 kg is 2 cycles per second. What will the frequency be if the mass is replaced by one of 5 kg?

B-6.26. What is the frequency of a pendulum 4 feet long?

B-6.27. The following data are given for the amount of sway of a radio tower when a wind of 10-second pulses hits the tower.

North wind speed - mi/h	Amount of sway - inches
5	1
10	1.5
15	3
20	9
25	20
30	12
35	10
40	9

a. Draw a graph of the input-output characteristics for this building.

b. What is the resonant wind speed?

Chapter C-1

FEEDBACK

C-1.1 INTRODUCTION

Have you ever thought about how many of your actions are directed toward achieving some goal or purpose? If you have, you quickly realize that almost everything you do is *goal-directed*, in some sense. These goals may be very simple and immediate as, for example, when you insert a dime into a soft-drink dispenser because you wish to quench your thirst, or they may be very complex and of long-term significance, as when you decide to go to college to prepare yourself for a profession, such as teaching, law, engineering or medicine.

This purposeful activity of "seeking goals" is of course not unique to men alone, but may be found in all living things. Plants turn their leaves toward the sun and extend their roots toward moist and fertile soil. Salmon battle their way up rivers and on through rapids to spawn their eggs in the same creek in which they themselves were hatched. Even the single-celled amoeba moves away from a disagreeable substance to seek a more pleasant environment. If you accidentally cut your finger, the blood forms a clot to stop the bleeding. If excessive blood is lost, various reactions within the body begin automatically, the blood vessels contract to bring blood pressure back to normal, the spleen, which functions as a blood reservoir also contracts thus compensating in part for the loss of blood, you become thirsty and drink large amounts of water to help to make up for the loss of blood plasma. Thus, the various organs of the human body exhibit their own wisdom and seek their own goals.

In contrast with such purposeful activity as is exhibited by living things, inanimate objects appear to behave quite differently. Their behaviour is based solely on *prior* causes. Usually we do not think of inanimate physical objects as being influenced by any awareness of what may happen to them in the future. The *principle of physical causality* asserts that only *past* events can affect the present. This is one of the basic assumptions of all physical science. If you postpone a date tonight because you wish to study for an examination that will be given tomorrow, is this not a case of a future event affecting the present? Does this therefore mean that you (and other living things) are not subject to the principle of physical causality?

Inanimate objects appear to differ from living things in another important aspect, too. With living things we recognize movement towards goals, towards maturity and death, toward satisfaction and reward, towards equilibrium, towards a conscious purpose. But we do not ordinarily think of inanimate objects as seeking "goals" or behaving in "purposeful" ways. For instance, a rock loosened by the morning's rain falls down the mountainside. We should think it a bit odd if someone suggested that the *goal* of the rock is to seek a lower elevation. And, if the rock should strike and kill a snake, only an irrational and superstitious person would assert that this was the rock's purpose. Yet, if you were to throw a rock at the snake and kill it, we would all agree that to be your purpose. Why does it make sense to speak of goals and purposes in describing the activities of living things, whereas it is nonsense to speak in the same way of inanimate physical objects such as the falling rock?

Perhaps you may conclude, as many other people have done, that goal-seeking behavior is only the unique characteristic of life. But is this always so? An elevator operator has the goal of bringing the elevator to rest at each floor so that the floor of the elevator cab is level with the floor of the building. In many older elevators, the speed of the elevator is controlled directly by the operator (with some success). At the end of the day when the elevator is heavily loaded with people on their way home, the operator must begin to bring the elevator to a stop before it reaches the desired level if he is not to overshoot his mark. And if he does overshoot, he will reverse the motion to raise the elevator until its floor is at approximately the same level as the outside floor. With practice, a skilled operator can learn to adjust his actions in accord with the number of passengers and to stop the elevator at the desired level without a number of adjustments. Modern elevators now have automatic mechanisms for accomplishing the same goal much more effectively. The mechanism built into the automatic elevator remembers at which floors it should stop to pick up passengers and to discharge passengers. It can also stop at the proper level more consistently than it could under the control of a human operator. The automatic elevator thus, can be said to exhibit a goal-seeking behavior, despite the fact that it is completely inanimate. What then is the essential difference between the elevator and the falling rock that permits us to describe one, but not the other, as a goal-seeking device? The goal seeking behaviour of the elevator is made possible by the presence of a feedback arrangement in the elevator system.

The distinction between mechanistic behavior (which is governed only by past causes) and purposeful behavior (which is guided by future or desired goals) has been a battleground of debate among philosophers, theologians, psychologists and many others. Yet it is only within the past thirty years that man has discovered the concepts needed to illuminate and understand some of the issues involved. The most important of these concepts lie at the heart of this course. They are the concepts and principles that enable man to design and build machines that have the ability to perform specified tasks while adapting themselves to changing conditions, in much the same way as people adjust their behavior. These are the concepts associated with computers, information storage, optimization and dynamic models. Of them all, the most central and pervasive concept is that of feedback, which permits us to build machines and systems which display goal-seeking characteristics.

C-1.2 SYSTEMS WITH INPUT AND OUTPUT

In this chapter, we wish to develop an introductory familiarity with the concept of feedback -- an understanding of when feedback exists in a man-made system, of what can be achieved with the use of feedback, and of the characteristic behaviour that we can expect from a system which contains feedback. But before these statements can be meaningful, we need a definition of the term feedback.

Many man-made systems have a main input and a primary output signal (Fig. 1, where we call these signals u and y respectively). For example, in driving a car along a straight horizontal road, the system determining the speed of the car has the two signals.

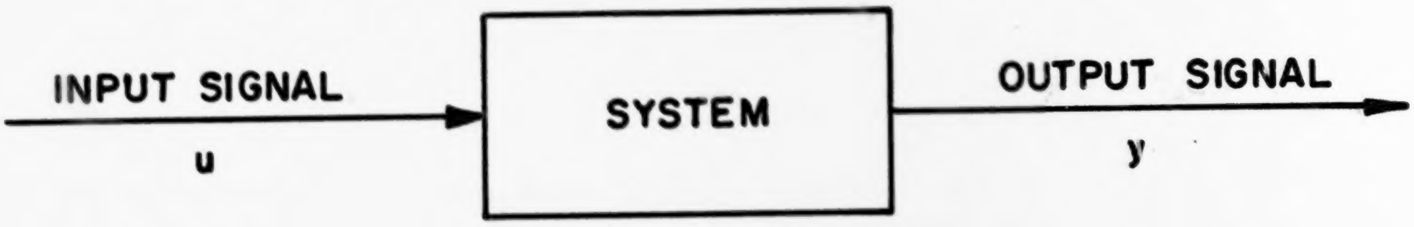


Fig. 1 Basic System

u (input signal) -- the amount the accelerator is depressed by the driver.

y (output) -- the speed of the car.

The larger the input, the more the car speeds up and the larger the value of speed y.

A better picture of the system is shown in Fig. 2. We know that in actuality the driver senses the speed y of the car. As that speed changes, the driver adjusts the accelerator position u in order to slow down or speed up the car. In other words, the driver determines u according to his estimate of y; this relationship is indicated by the block labelled Driver in Fig. 2 -- a block with the input signal y and the output u.

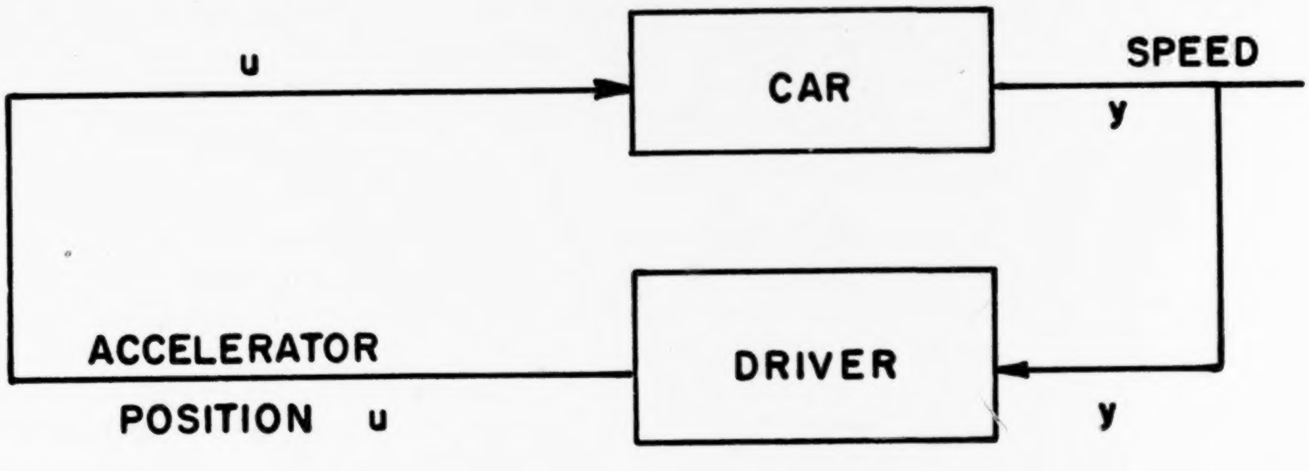


Fig. 2 Speed-control system for automobile.

Thus, Fig. 2 is a model or picture of the relative roles of the automobile and the driver in this speed control system. The automobile determines how y depends on u; the driver establishes the relation for u as a result of y (and what he would like the speed y to be).

If we compare Figs. 1 and 2, there is obviously one major difference. In Fig. 1, we have a very simple cause-and-effect relationship depicted: u causes a certain effect y (the actual quantitative value of y for a given u depends on the characteristics of the system). With a small car for example, the movement of the accelerator through a distance of one inch will not produce the same effect as the one inch movement of the accelerator of a large car. In Fig. 2, the cause-and-effect relationship is more complex. Now u not only causes a certain y but this y, in turn, results in a modification of u ; which produces a further change in u that varies y again, and so on.

Figure 2 is called a feedback system because of this characteristic: the driving input u not only determines y, but also depends on y.

Why are we particularly interested in feedback systems? What special characteristics are associated with the feedback property? In the remaining sections of this chapter, we shall attempt to develop answers to these questions. Thus, our basic objective in this chapter is to answer these questions:

(1) When does a system possess feedback?

(2) What particular characteristics are associated with the presence of feedback?

(3) When is it useful to include feedback in the design of a system?

(4) What undesirable properties may appear when feedback is used or exists?

In order to develop the answers to these questions, we will first consider in greater detail what feedback is as well as examine additional examples of feedback; we will find that an awesome range of systems involve feedback, and that feedback frequently permits systems to operate with enormously improved accuracy. Finally, in order to emphasize quantitatively these attributes of feedback, we will do a few simple, algebraic computations and experiments. The primary goal of the chapter, however, is to develop the ability to recognize feedback in systems as well as an awareness of the more important characteristics that result from its use.

We will first describe various, familiar examples of feedback in order to begin our development of an understanding of this important engineering concept. In particular, we are interested in situations in which feedback exists more or less incidentally. The next section, will consider design problems in which we use feedback intentionally to achieve certain desirable system characteristics. Thus the discussion in this section is rather descriptive and non-mathematical: our only objective is to familiarize the reader with the method of thinking in terms of feedback.

One characteristic of many real-life systems must be emphasized at the beginning: the idea that a system is commonly described in terms of a cause-effect relationship (in scientific terms, an input-output relationship).

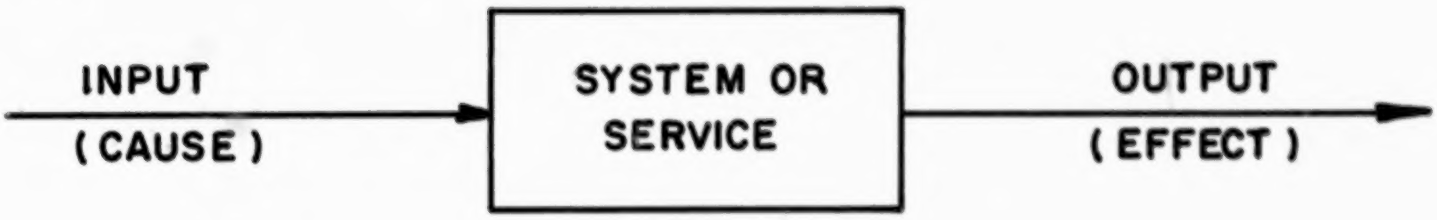


Fig. 3. Basic system.

As shown in Fig. 3, the system is driven or excited by a signal which is called the Input; the corresponding response of the system is termed the Output. The system determines the relationship between the Output and the Input -- i. e., what specific output results from a known input signal.

A picture such as Fig. 3 is the starting point for a scientific approach to understanding a system or a device. For example, if the system we try to understand is the steering mechanism of a car, the two signals are:

Input = angular position of the steering wheel

Output = heading or direction in which the car is moving

The system in this case may include the power steering device, the tires, the car, and the nature of the road surface (e. g., as the road becomes icy, the car responds very differently to a turn of the steering wheel).

Car steering is a relatively simple system, since it has only one primary input signal and one output signal. There are familiar systems with a large number of inputs and outputs -- indeed, there may be so many that we find it essentially impossible to understand what is going on inside of the system. As an extreme example of such a complex system, we can consider the transportation system on the island of Manhattan in New York City. Transportation refers to movement of both people and materials; city planning demands the design of a total transportation system (including streets and traffic control for cars, buses and trucks, pedestrian facilities, subway passageways, railroad facilities, and airplane and ship terminals). This total system must service the needs of the inhabitants and visitors as well as of industry and of government.

Just a partial listing of the various inputs to such a system emphasizes the complexity. Here an input is any signal which determines the various outputs because of the nature of the system. A few inputs are:

(1) The desired motion of people living in the city as they travel to work, shopping, etc.

(2) The desired motion of people visiting the city, both commuters and occasional visitors.

(3) The generation around the city of garbage and rubbish which must be hauled away (in the U.S. today we are generating more than four pounds of rubbish per day per person).

(4) The activities of the building industry in new construction and rehabilitation (each new building requires the transportation of extensive equipment and materials to the site).

(5) The geographical distribution and frequency of both crime and fire, with the resulting need to move police and fire equipment rapidly.

(6) Public and private health activities, demanding movement of ambulances and doctors, transporting of blood and medical supplies, etc.

(7) Food needs of the inhabitants and visitors.

(8) The weather (e.g., in a typical city in rainy weather, downtown streets are clogged during late afternoon by wives cruising around the block while waiting to pick up their husbands).

In this example, the list of inputs which affect the system can clearly be extended well beyond these few items. The intelligent design of the transportation system requires consideration of all of these factors; otherwise, we may (for example) develop a system which is satisfactory for moving people, but fails to permit rapid motion of emergency vehicles.

Thus, we are dealing here with a multi-input, multi-output system -- indeed a system which is further complicated by the fact that the system must meet certain economic, social, and political constraints. For example, in economic terms the transportation facilities are limited by cost, both in capital expenditures and in the fare charged users. Socially, new transportation means are limited to those acceptable to the public: while it might be desirable technically to construct a subway train with a maximum acceleration of twice the acceleration of gravity (twice free fall), no one would ride in such a vehichle a second time. As a final example of a constraint, the transportation system planner can not easily locate railroad tracks arbitrarily through densely populated areas or through newly constructed apartment houses.

In this course, we shall not be interested in detailed study of anything as complicated as a metropolitan transportation system. Rather, our interest is restricted to relatively simple systems, usually with a very small number of inputs and outputs (generally only one input and one output). Regardless of the complexity of the problem, however, the system is the factor which establishes the relationship between the input signals and the various outputs.

Figure 4 illustrates a final example of this cause-effect relationship. The input is the position of the accelerator, the output the speed of the car. The output for a given input is determined by the characteristics of the system: by the amount of gas fed to the engine as the accelerator pedal is depressed, the efficiency of the motor in converting this gas energy to forward thrust, the surface of the road, the incline of the road, and so forth.

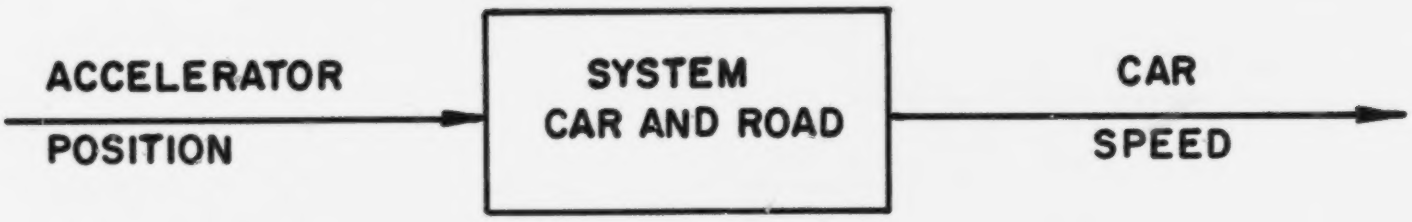

ACCELERATOR
POSITION
SYSTEM
CAR AND ROAD
CAR
SPEED

Fig. 4 Another example.

C-1.3 THE PHENOMENON OF FEEDBACK

Once we recognize that most cause-effect relationships in the real world can be represented by an input-output diagram such as we drew in the preceding section, the phenomenon of feedback is obvious. Feedback occurs when the input is determined, at least in part, by the output. In other words, if we have two systems (A and B in Fig. 5), each with its own input and output, and if we interconnect these two systems as shown in Fig. 6, the input of B is the output of A, and the input of A is the output of B, then the result of such an interconnection is a feedback system or a system with feedback. A signal entering at the input of A travels through A, then through B, then again through A, and so forth. The term feedback arises because the output of A is fed back through B to the input of A.

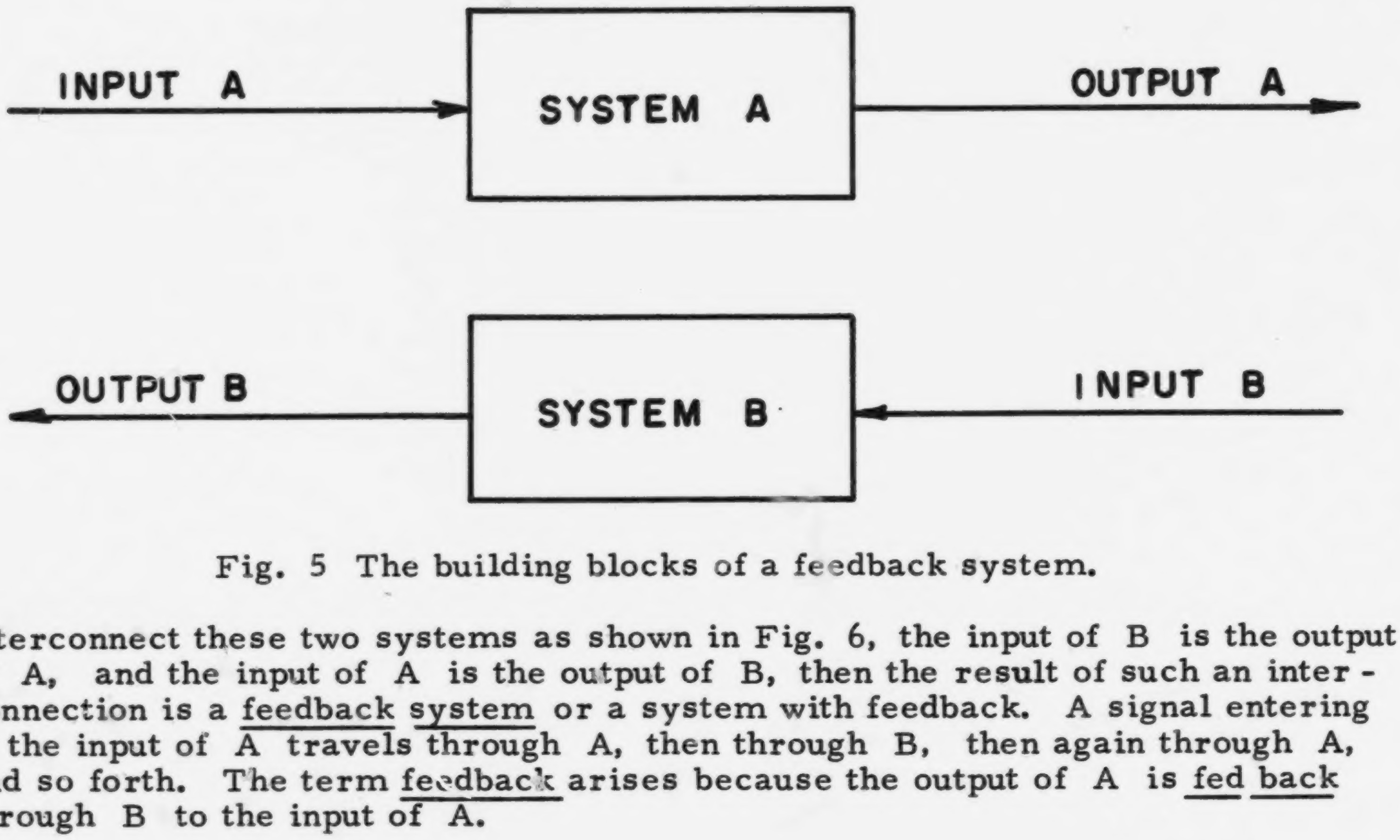

INPUT A
SYSTEM A
OUTPUT A
OUTPUT B
SYSTEM B
INPUT B

Fig. 5 The building blocks of a feedback system.

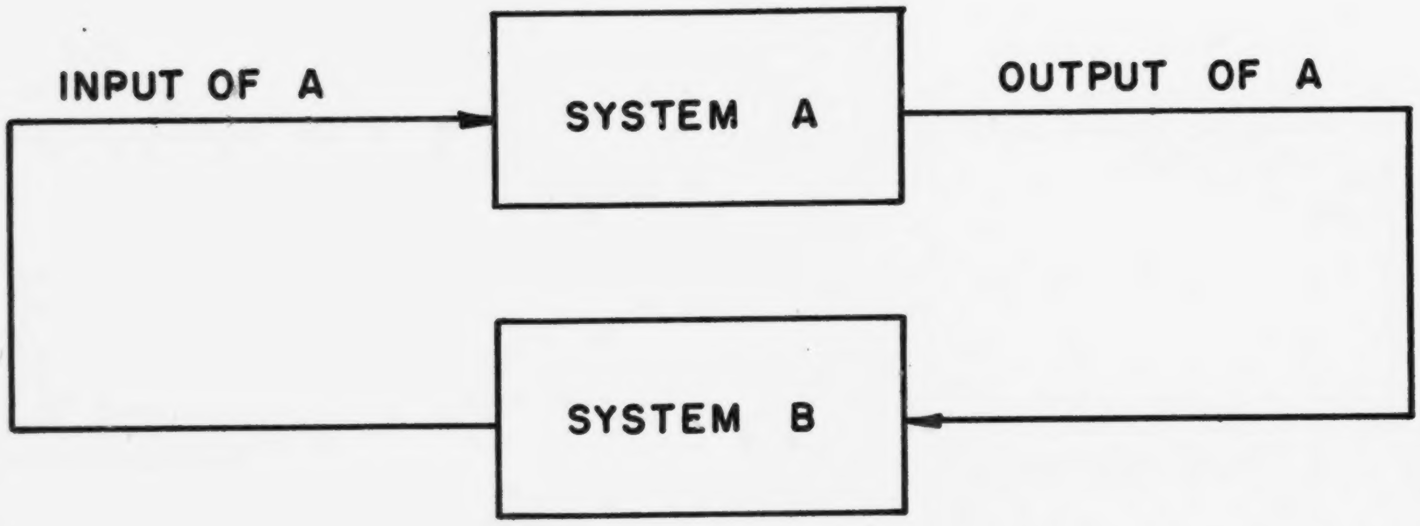


Fig. 6 A simple feedback system.

Figure 6 is not a particularly interesting system, since there is no way to start the system moving or acting. Figure 7 shows a model of a much more common arrangement; here we have added a primary input -- a path through which our system is connected to the outside world. As a result of this primary input, the system can be driven or excited, so that there is a response. Thus, Fig. 7 is a feedback system which we can expect to find in real-life situations.

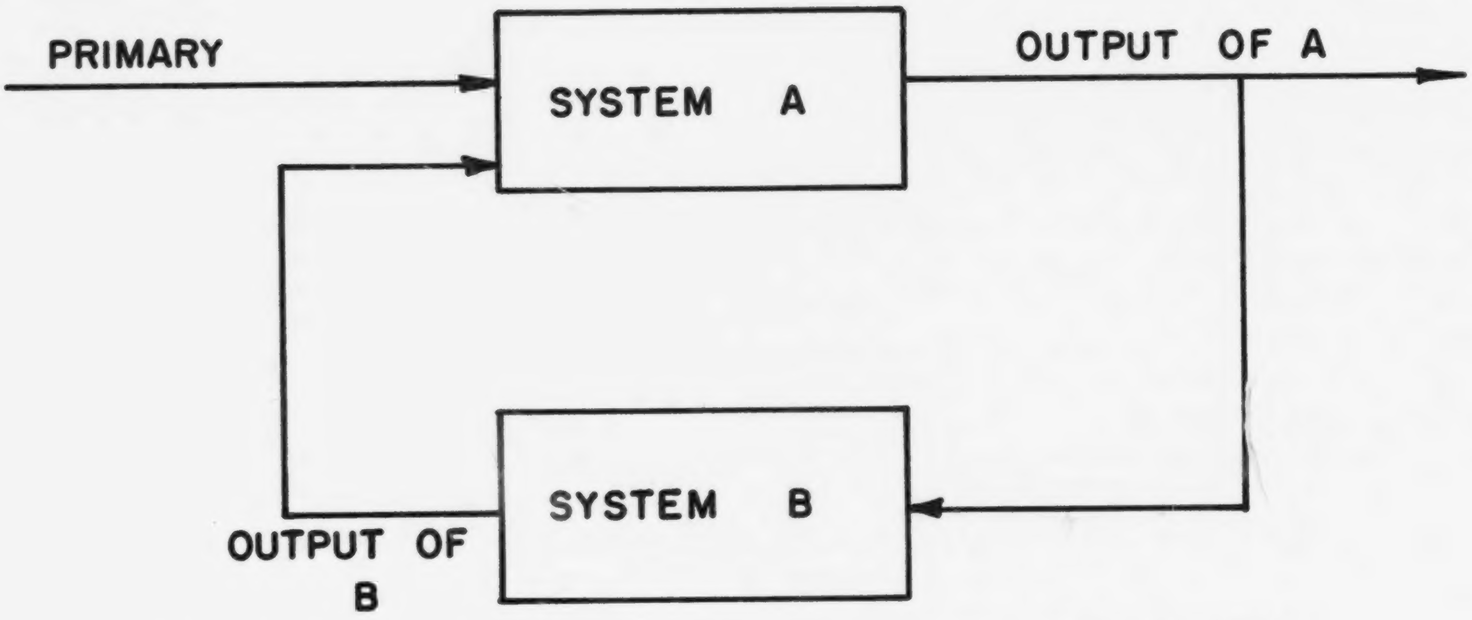


Fig. 7 Feedback system with one input from external world.

Feedback, as depicted above, exists in many familiar situations. As a first example, we consider communications between two friends. In this case, the output signals are the spoken words of the two individuals. Person A receives information from the outside world (e.g., by visual observation) which he interprets as a misdeed of person B. A then makes certain remarks (output A) to B which alienate her; she in turn makes a few sarcastic remarks to A; he replies in kind; and our feedback system quickly moves into a condition of instability. The two system outputs grow rapidly in intensity or bitterness.

The example illustrates an important characteristic of many feedback systems: a very small primary input signal (indeed a signal which may seem negligible) results in the build-up of signals within the system to extremely large values. This phenomenon (a form of instability which we shall study in greater detail in a later chapter) results directly from the closed-loop nature of the feedback system: the fact that signals can travel around and around the loop in Fig. 7. If we start with a very small signal of magnitude 10^{-6}, entering as the primary input and it is multiplied by 10 as it traverses Systems A and B in one second, it will be 10^{-5} in magnitude when it again reaches the input of A. Each trip around the loop results in a signal magnification by 10; after only 12 traversals (12 seconds), our original signal of 10^{-6} has grown to 10^6.

This tendency toward instability in feedback systems can also be illustrated by other examples:

(a) The relation between sleep and health is sometimes an illustration of a feedback system. When one becomes ill with a sore throat or nasal congestion, sleep is difficult; the lack of sleep or rest tends to permit the cold to become worse, which in turn makes sleep even more difficult.

(b) Economic system frequently exhibit feedback phenomena. When a union as large as the auto workers obtains a sizeable wage increase, costs of automobiles tend to rise. Other costs rise as the added income of the union members results in more spending nationally in other industries, hence greater demand for goods. The resulting increase in prices induces the union to ask for further wage increases. A single union, of course, does not control this inflationary spiral, and the system tendency to oscillate can be controlled by changes in government spending, taxation, and so forth. This is a system as complex as the metropolitan transportation example of the last section.

(c) The U.S.A.-U.S.S.R. armament race illustrates the same phenomenon of feedback. Here the USA learns of the development of the USSR inter-continental ballistic missiles, and launches a major program to develop more missiles than the Soviets possess. Learning of the USA missile arsenal, the USSR then undertakes development of a major anti-missile system. In order to maintain the "balance of power," the USA must then launch a major program for an effective anti-missile system. Each of these successive steps represents both a gigantic economic drain on each country and a major utilization of the technological resources of the two nations.

C-1.4 INTENTIONAL FEEDBACK

The examples of feedback in the last section were all cases in which feedback exists inherently in the system. In the boy-girl communication problem, feedback is inevitable since the boy's remarks influence what the girl says, hence what he hears as input. If feedback only occurred in such personal or social situations, and instability was the only consequence of feedback, there would be little point to our discussion of the phenomenon in detail here.

Fortunately, the concept of feedback is considerably more profound. We use feedback in a system intentionally in order to achieve remarkable improvements in system performance. One of the most useful of these accomplishments of feedback is the reduction of the influence of disturbances or unwanted signals. In this section, we first consider an example qualitatively, we then consider what additional advantages a mathematical analysis can offer in such cases.

A system normally has one or more input signals and one or more responses. The system is designed so that the input signal yields the desired output. Very often, there are also other, secondary inputs which we can not control and which may not even be predictable. As a Navy pilot attempts to land his plane on a carrier flight deck, for example, the primary input signal is his location relative to the deck; on the basis of observation of this signal, he adjusts his controls. There are also two secondary or disturbance signals; the wind gusts acting on his plane and the forces the sea exerts on the carrier. If these disturbances are too strong or violent, successful landing becomes a most difficult task.

The carrier landing problem is particularly complicated because it involves all three dimensions in space. Figure 8 shows a similar, two-dimensional problem. A ship is located 10 nautical miles (n. m.) south of a channel opening through a reef. The navigator sites the buoys or light signals marking the channel and naturally sets a heading due north. A heavy fog suddenly sets in, which tends to hide the buoys from the men on the ship.

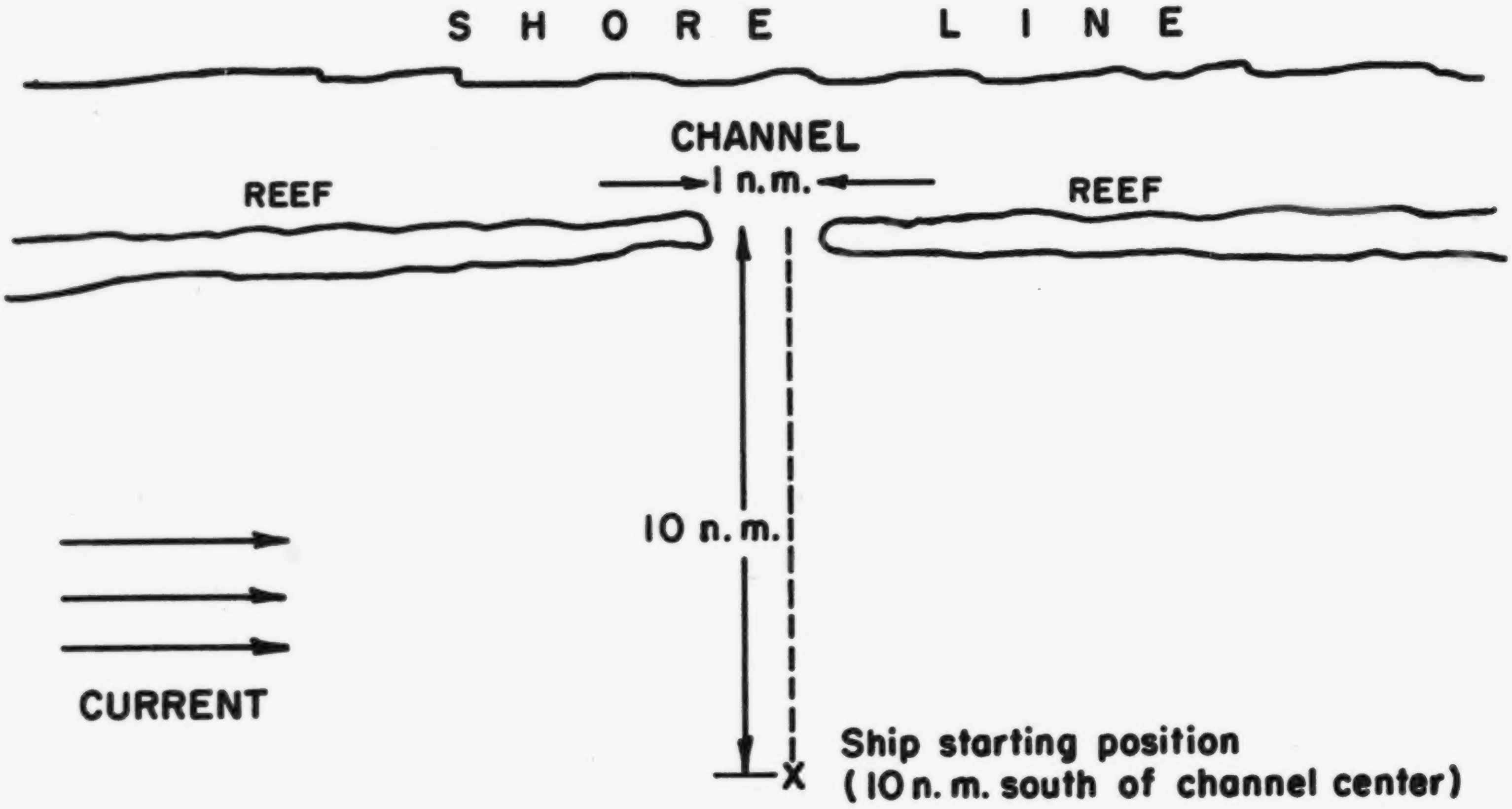


Fig. 8. Navigating through a channel

The ship steams on at 10 knots (nautical miles/hour) with a north heading. The navigator, having estimated correctly the distance to the channel as 10 n. m., assumes that the ship will pass the reef in one hour -- as indeed it will if there is no disturbance signal affecting the position of the ship.

But what happens if a current of 2 knots pushes the ship eastward? The northerly motion is unaffected, but in one hour the ship is 10 n. m. north and 2 n. m. east of its original position: i. e., 1. 5 n. m. east of the edge of the reef, and tragedy ensues.

In order to understand the system, a block diagram is useful (Fig. 9)

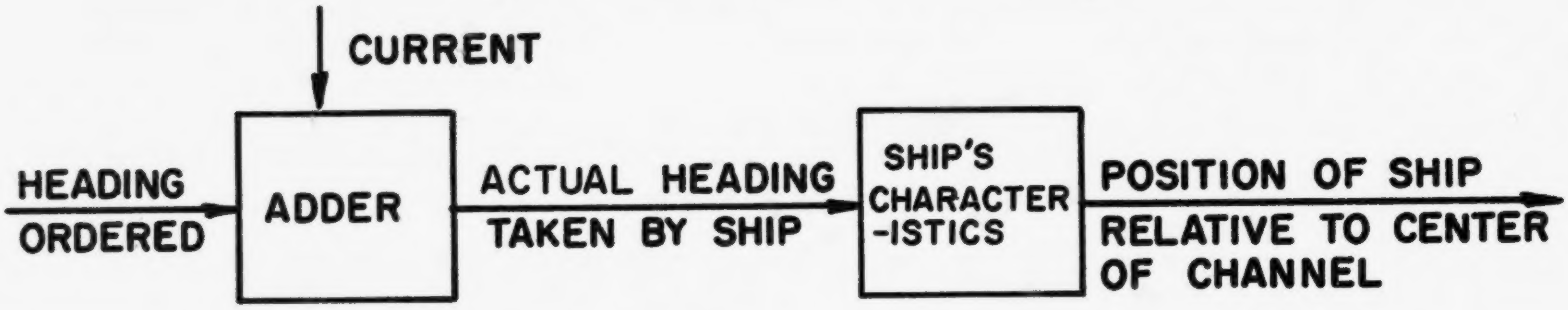

CURRENT
HEADING ORDERED
ADDER
ACTUAL HEADING TAKEN BY SHIP
SHIP'S CHARACTER-ISTICS
POSITION OF SHIP RELATIVE TO CENTER OF CHANNEL

Fig. 9 Block diagram for navigation problem.

The input signal is the heading the navigator orders for the ship. A disturbance signal (the current) adds vectorially to this heading to yield a signal which is the actual heading assumed by the ship. This actual heading determines the system output: the position of the ship relative to the center of the channel. The system operates satisfactorily if, when the north-south component of the output is zero, the east-west component is less than 0. 5 nautical miles.

The tragic ending of our story can be averted if we can use a little feedback to counteract the effect of the current. For example, perhaps after 15 minute intervals the fog lifts long enough for the navigator to complete another sighting on the channel markers. After 15 minutes, the ship is at A (7. 5 n. m. south and 0. 5 n. m. east of the channel center in Fig. 10). The navigator now orders a heading toward the channel center. Fifteen minutes later the ship is at B where another sighting is taken and a new heading adopted (we still assume the navigator is obstinate and refuses to recognize that his past errors might be the result of a constant current, for which he could compensate by aiming to the west of the channel center). If this calculation is continued, we find that the ship transverses the path shown in the figure, with

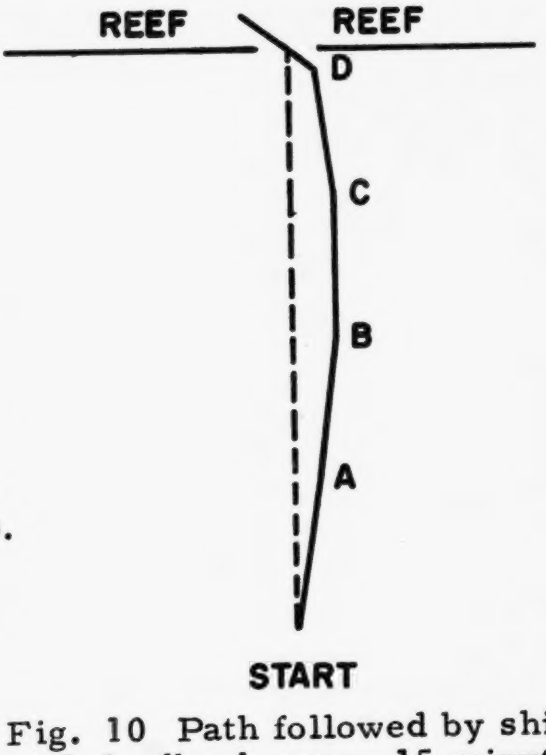


Fig. 10 Path followed by ship with feedback every 15 minutes.

sightings taken at locations A, B, C, and D.*

The block diagram of the system with feedback is shown in Fig. 11. Every 15 minutes, the switch closes and the heading is readjusted according to the difference between where the navigator would like to be and where he actually is located.

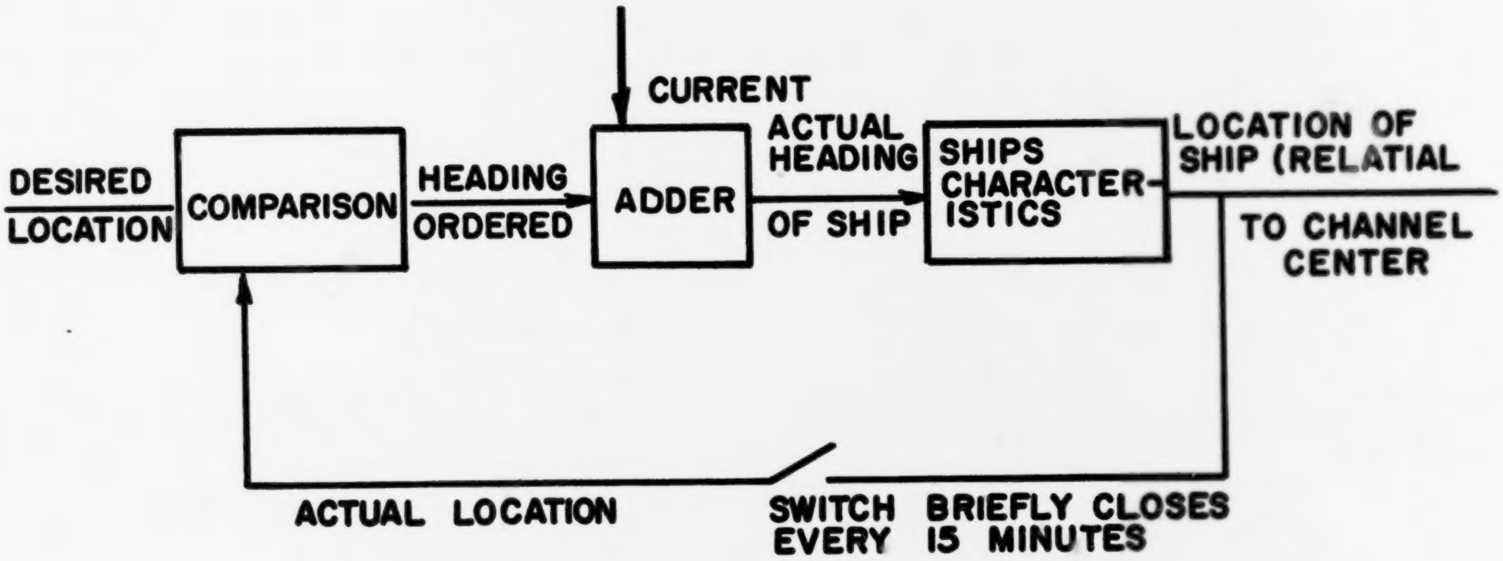

CURRENT
DESIRED LOCATION
COMPARISON
HEADING ORDERED
ADDER
ACTUAL HEADING OF SHIP
SHIPS CHARACTERISTICS
LOCATION OF SHIP (RELATIAL TO CHANNEL CENTER
ACTUAL LOCATION
SWITCH BRIEFLY CLOSES EVERY 15 MINUTES

Fig. 11 Block diagram of system with intermittent feedback

In the above example, feedback occurs every 15 minutes. If it is possible to use feedback continuously (i. e., no fog exists, so the navigator can sight the

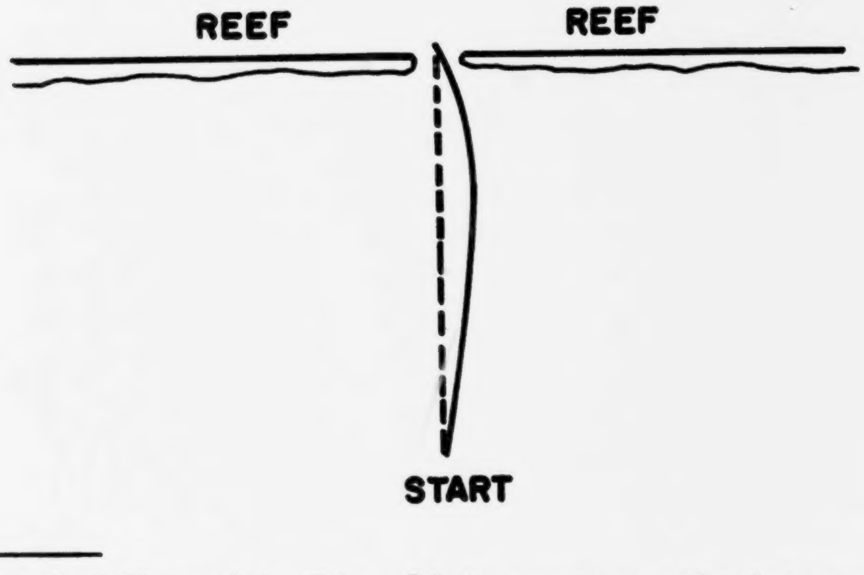

REEF
REEF
START

Fig. 12 Path of ship with continuous feedback (but navigator not estimating the current) -- essentially the same as Fig. 10 although this is a smooth curve.

---

*The path followed in Fig. 10 is most easily determined graphically if the diagram is drawn to scale. For example, once B is known, C can be found as follows. From B along a line toward channel center we mark off a distance of 2.5 n. m. to determine a point which we can call C'. This is the point the ship would reach 15 minutes after leaving B if there were no current. C is 0.5 n. m. due east of C'. Once C is known, D can be found similarly, etc.

channel markings and readjust ship heading continuously), the path followed by the ship is shown in Fig. 12. Initially the ship moves off the desired course because of the current, but the continual corrections result in final passage through the channel without difficulty. In this case, the feedback is ever-present, and the system is continuously compensating for the effect of the disturbing signal (the current).

This navigation example illustrates in rather general terms a most important use of feedback: Feedback can be used to reduce the effects of disturbance signals. Many other examples of the property of feedback can be cited. For example, in steering a car, the driver wishes to remain in his lane. On a straight road, he sets the steering wheel so the car moves along the lane. Bumps in the road, wind gusts, and unequal road surfaces under the tires are all disturbing signals which may cause the car to veer to the right or to the left. In order to compensate for these unpredictable signals, the driver uses feedback: he observes the position of the car in the lane and then turns the steering wheel to improve the position (Fig. 13). Feedback exists because the driver compares the actual position (the system output) with the desired position (normally the center of the lane). The feedback is removed if the driver keeps his eyes closed or at least away from the road.

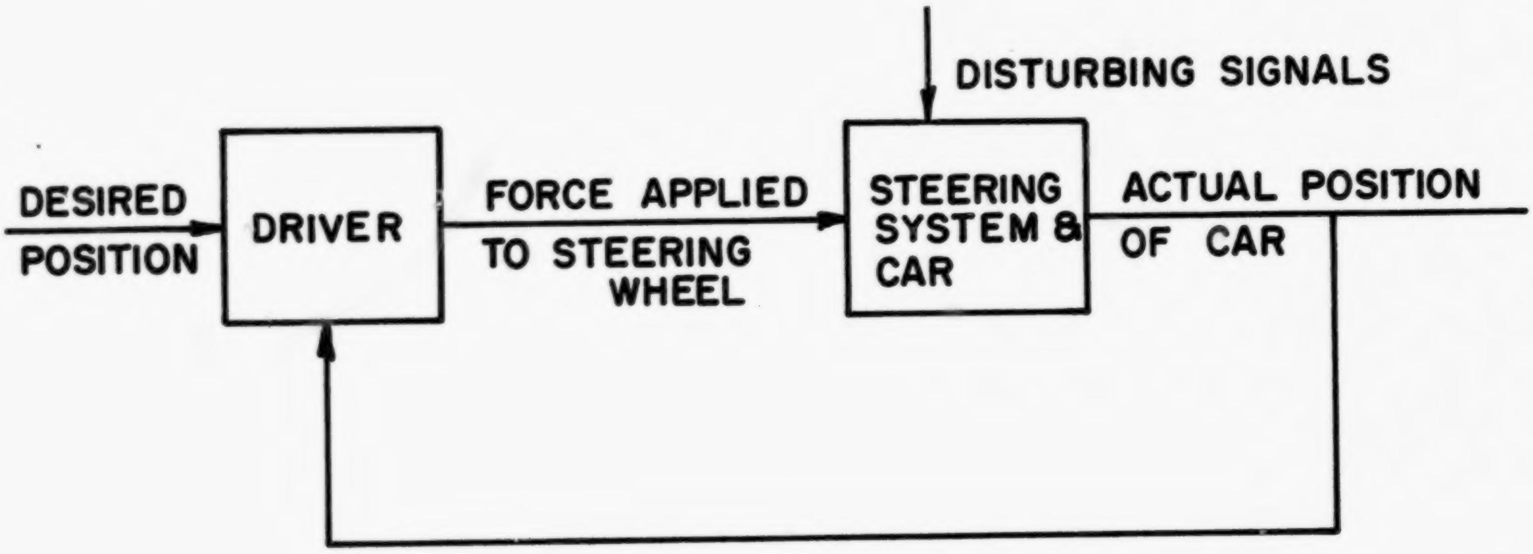

DISTURBING SIGNALS
DESIRED POSITION
DRIVER
FORCE APPLIED TO STEERING WHEEL
STEERING SYSTEM & CAR
ACTUAL POSITION OF CAR

Fig. 13 Block diagram of steering system

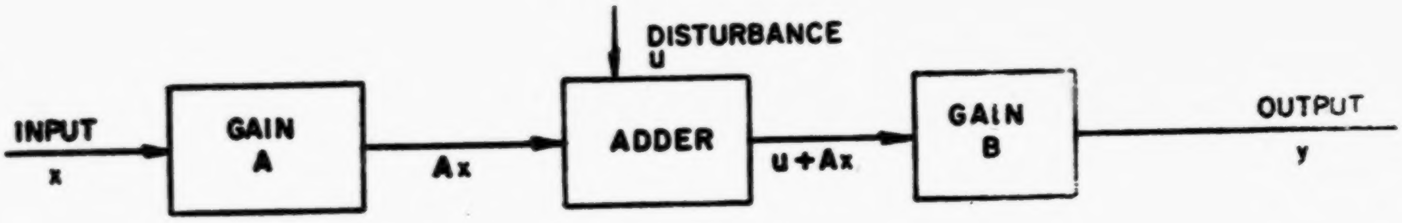

DISTURBANCE
u
INPUT
x
GAIN
A
Ax
ADDER
u+Ax
GAIN
B
OUTPUT
y

Fig. 14 System with disturbance input and no feedback

The effect of feedback or disturbance signals can be shown quantitatively if we consider first the system shown in Fig. 14. This system possesses no feedback. The input signal (which we call x) is amplified by the first part of the system to yield a signal Ax in size. We next add to this the disturbance signal u. This total signal, u + Ax, is further amplified to yield the output.

$$y = B(u + Ax) = Bu + ABx \qquad (1)$$

The output has two parts: the desired part, ABx, and the disturbance effect, Bu.

Now what can we do with the system if we add feedback? To insert feedback, we measure the output and compare it with what we would like the output to be. That is, if our system had no disturbance, the output y would be ABx; hence let us measure y and then take 1/AB of this value. Ideally, this y/AB should equal x but because the disturbance produces the factor Bu it will not. Hence the error $(x - \frac{y}{AB})$ will be used to change the output toward its desired value.

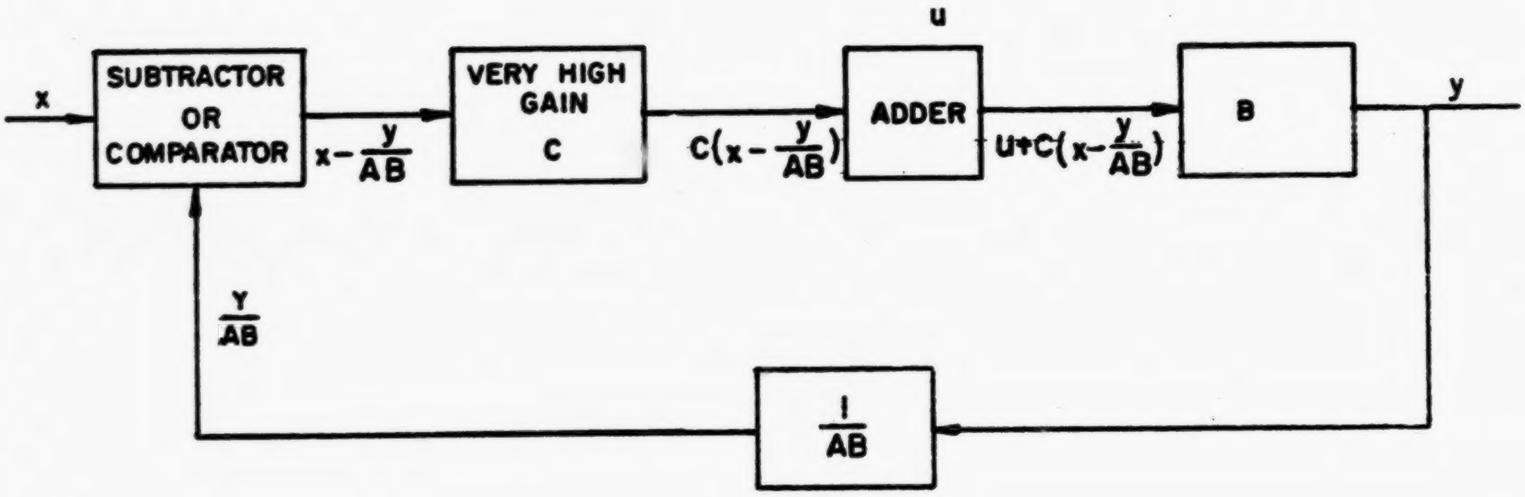


Fig. 15 System with disturbance input and with feedback

We then have the system shown in Fig. 15. How does this system operate? The various blocks shown establish the following algebraic relationship

$$y = B[u + C(x - \frac{y}{AB})] \qquad (2)$$

If we solve this equation for y by ordinary algebra:

$$y = Bu + BC(x - \frac{y}{AB})$$

$$y = Bu + BCx - \frac{C}{A}y$$

$$y(1 + \frac{C}{A}) = Bu + BCx$$

$$y = \frac{B}{1 + \frac{C}{A}}u + \frac{BC}{1 + \frac{C}{A}}x$$

$$y = \frac{BA}{A + C}u + \frac{ABC}{A + C}x \qquad (3)$$

Now we have said in the figure that C is to be very large. If C is much larger than A, the denominator term $(A + C)$ is very nearly just C. Hence, the last equation above becomes

$$y = \frac{BA}{C}u + ABx \qquad (4)$$

Equations (1) and (4) are the two expressions for the output y without feedback and with feedback. They are repeated here for comparison:

$$\begin{cases} \text{No Feedback} & y = Bu + ABx \\ \text{Feedback} & y = B\frac{A}{C}u + ABx \end{cases}$$

The wanted portion of the output (i.e., ABx) is the same in the two cases. The undesirable part is, however, quite different. If $A = 10$ and $C = 1000$, for example, feedback reduces the effects of the disturbance signal u by a factor of 100! With these same numbers and B also chosen equal to 10, the two systems are shown in Fig. 16. Comparison of these two block diagrams reveals that feedback yields the 100:1 reduction in the effects of the disturbance. The cost of this is a more complicated system, since the feedback structure involves several blocks not necessary in the simpler structure.

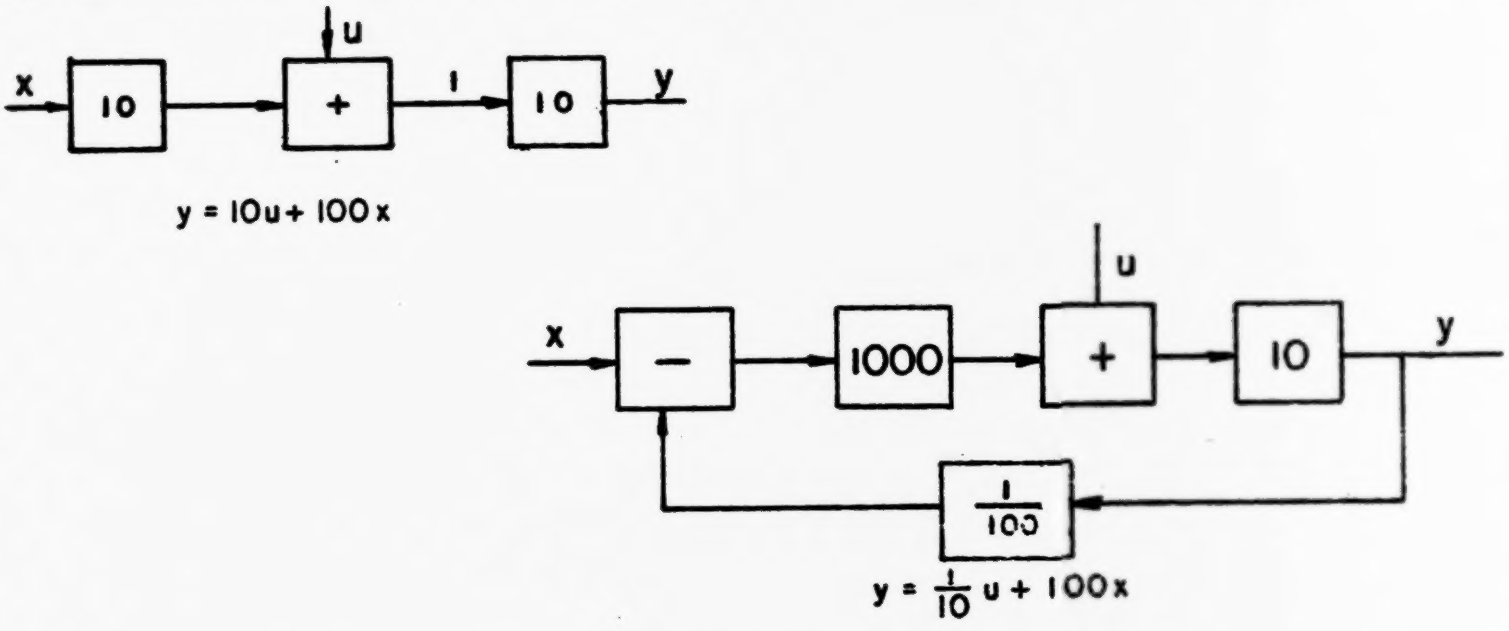


Fig. 16 Systems with and without feedback (disturbance signal is u)

In this section, two examples are considered: the former (the ship navigation problem) shows in general terms the value of feedback; the latter, represented in the above block diagrams, demonstrates quantitatively the manner by which we can often calculate the beneficial effects of feedback. The engineer or system designer must, of course, decide whether the benefits of feedback exceed the cost of greater system complexity. In many engineering problems, feedback represents the only economical way to control disturbance signals.

It is appropriate to close the section with one additional example. As you have read this section, you hopefully have participated in a systems problem. The author and publisher have combined to generate a small input to the student's mind; the system output is your level of understanding of the section.

Since, your mind may possibly have been subjected to disturbance inputs as you were reading, (interrupting telephone calls, television, and so forth), we should attempt to control the effects of such disturbances with the use of feedback. A simple form of feedback in this learning system depends on the use of a set of questions to measure the system output (your degree of understanding). If the output is not at the desired value, the questions refer you back to an earlier portion of the section so that the input signals can be repeated.

Question 1: What is a primary use of feedback? If you are at all uncertain of the answer, a complete re-reading of the section is recommended.

Question 2: When feedback is used, what signal must be measured? Once this measurement is made, feedback usually involves a comparison, Why? If you are uncertain, the navigation example should be re-read.

Question 3: In the block diagram of Fig. 17, the gain K is to be chosen so that the part of the output due to x is the same for the two systems. Determine K. Which system results in less influence of u on the output? By what factor is the disturbance effect decreased? If you have any difficulty with this problem, it would be advisable to return to the discussion starting at Fig. 14.

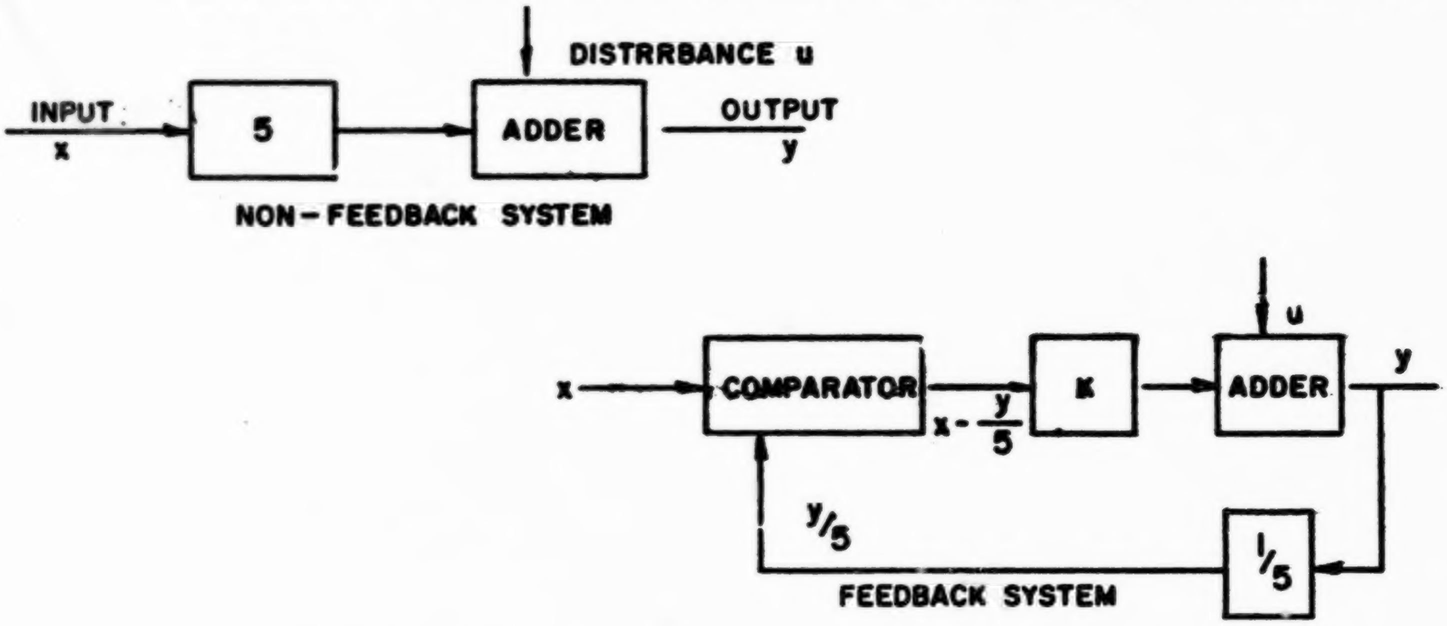


Fig. 17 Two systems with disturbance input

We have now inserted the first type of feedback. Additional feedback is inserted by your class discussions; with these various feedbacks, our theory tells us that it is certain that the student's level of uncertainty will be controlled regardless of any disturbance signals. When this system output is measured in the future by an exam, the change can not help but be marked.

C-1.5 ANOTHER ADVANTAGE OF FEEDBACK

Feedback has another very important capability: feedback can be used to compensate automatically for changes in the system. In this section, we first consider this characteristic quantitatively, then consider a few of the important and common illustrative applications.

The ability of feedback to compensate automatically for system changes is explained by consideration of Fig. 18. For the system without feedback

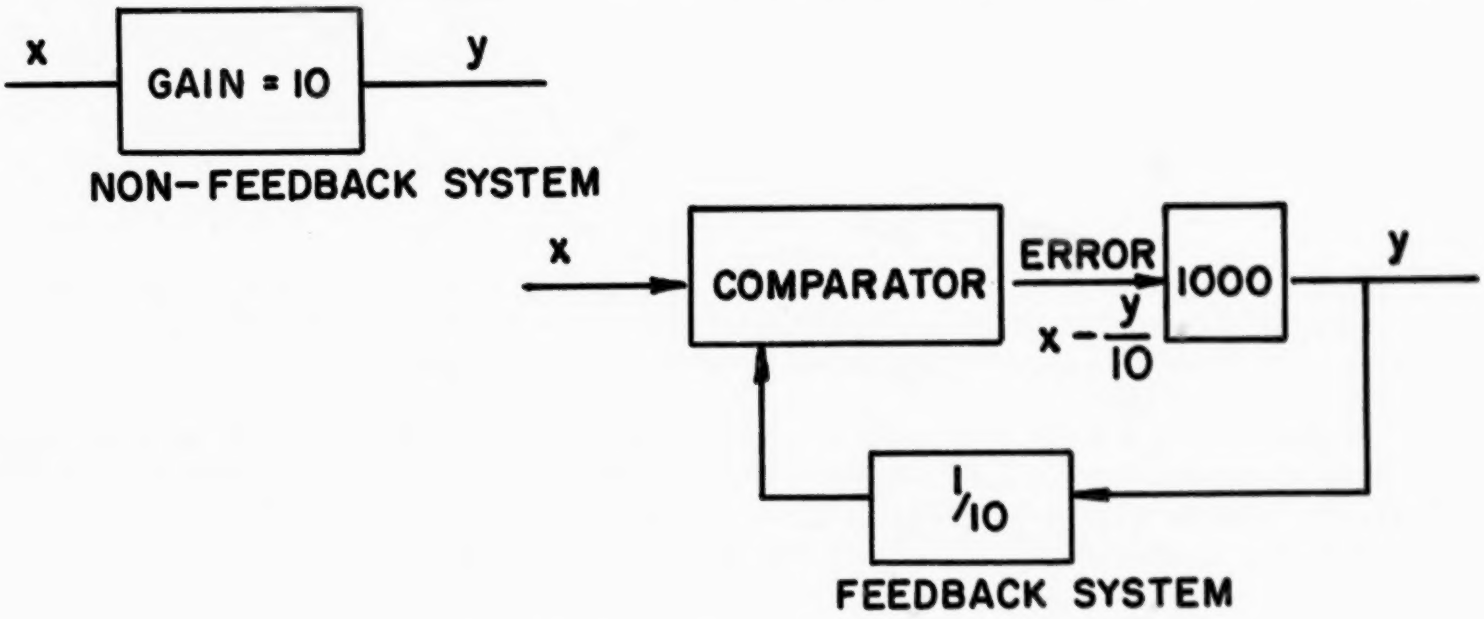


Fig. 18 Two comparable systems

we have

$$y = 10x$$

The output is 10 times the input. For the system with feedback

$$y = 1000\,(x - \frac{y}{10})$$

or

$$y = 1000x - 100y \qquad 101y = 1000x$$

or approximately

$$y = 10x$$

Thus the two systems behave in essentially the same overall fashion. Each gives an output which is ten times the input. If the input were from a microphone, for example, each system would give an electrical output signal amplified by a factor of 10.

Very often, however, amplifiers vary during operation because of changes in voltage from the electrical outlets, changes in transistor characteristics, and so forth. If the amplifier gain drops by 10%, how does each system above behave?

The system without feedback is easy to analyze. If the amplifier gain falls by 10% (from 10 to 9), the output $y = 9x$: the output also falls by 10%. The feedback system behaves quite differently. Here the amplifier gain is now 900 (10% less than 1000), and

$$y = 900\left(x - \frac{1}{10}y\right)$$

$$y = 900x - 90y$$

$$91y = 900x$$

or approximately

$$y = 10x$$

The output is unaffected by the 10% change in amplifier gain. Thus, feedback makes possible satisfactory system operation, even when the characteristics of system components change rather radically in the course of time.

The extent of the value of feedback is emphasized by the question: in the feedback system above, how much must the gain of the amplifier drop from 1000 before the output drops by 10%. Again simple algebra suffices for the calculation: if the amplifier gain is G (instead of 1000), the output is

$$y = G\left(x - \frac{1}{10}y\right)$$

If y is to be equal to $9x$ (rather than the normal $10x$)

$$9x = G\left(x - \frac{9}{10}x\right)$$

$$9x = G\frac{1}{10}x$$

$$G = 90$$

That is, the amplifier gain must drop to 90 (91% below the normal value of 1000) before the feedback-system output drops by 10%.

This rather remarkable characteristic of feedback is termed by engineers the control of system sensitivity (i.e., how sensitive the system properties are to changes in particular parameters or characteristics). Feedback can be used to keep the sensitivity small.

Certainly one of the most familiar examples of a feedback system in engineering is the regulation equipment for the central heating system in a house. In a typical oil or gas system, a single furnace supplies heat to all parts of the house (in the system with electric heaters, each room can be controlled separately).

The output of the system is the temperature, for example in the living room. The input signal is the desired temperature for comfort. The thermostat measures the output and compares this measured value with the desired value (the input which is set manually on the thermostat). The system is shown in Fig. 19.

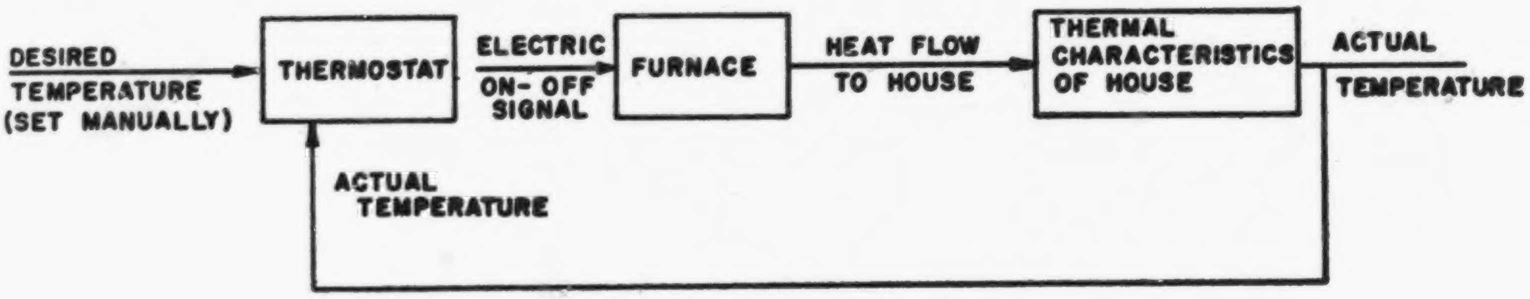


Fig. 19 Elements of household heating system

When the actual temperature drops a predetermined amount (e.g., 2°) below the desired temperature, the thermostat relay closes and the furnace is turned on. Heat is developed and flows through the house. The actual temperature rise resulting from this heat flow depends on the thermal characteristics of the house (the windows and doors that are open, the insulation provided from the outside, the degree to which air is circulating through the house, and so forth).

Regardless of these thermal characteristics, however, heat is supplied by the furnace until the actual temperature rises to about 3° above the desired temperature. At that point, the thermostat relay opens, the furnace is shut down, and the heat flow terminates. The speed with which the changes occur depends on the particular form of the system; for example, if heat is transmitted through circulating hot water, heat continues to flow into the room from the radiators after the hot water stops circulating until the water cools. In all cases, however, the basic operation of the system is the same.

If the desired temperature (the thermostat) is set at 72°, the living room temperature near the thermostat fluctuates between about 70° and 75° and never exceeds these limits as long as the outside temperature is low and the furnace system is sufficiently large to heat the house properly. The performance of the total system is completely independent of the thermal characteristics of the house and the outside temperature. The feedback permits essentially perfect system performance over an exceedingly wide range of thermal characteristics for the house. Indeed, from the viewpoint of comfort and livability, the feedback permits realization of an ideal system: we can open the front door in winter for sizeable periods of time without losing control of living-room temperature.

A second example of the astonishing ability of feedback to compensate automatically for changes in system characteristics is provided by the analog-computer kits used in this course. The integrator in this kit consists of an

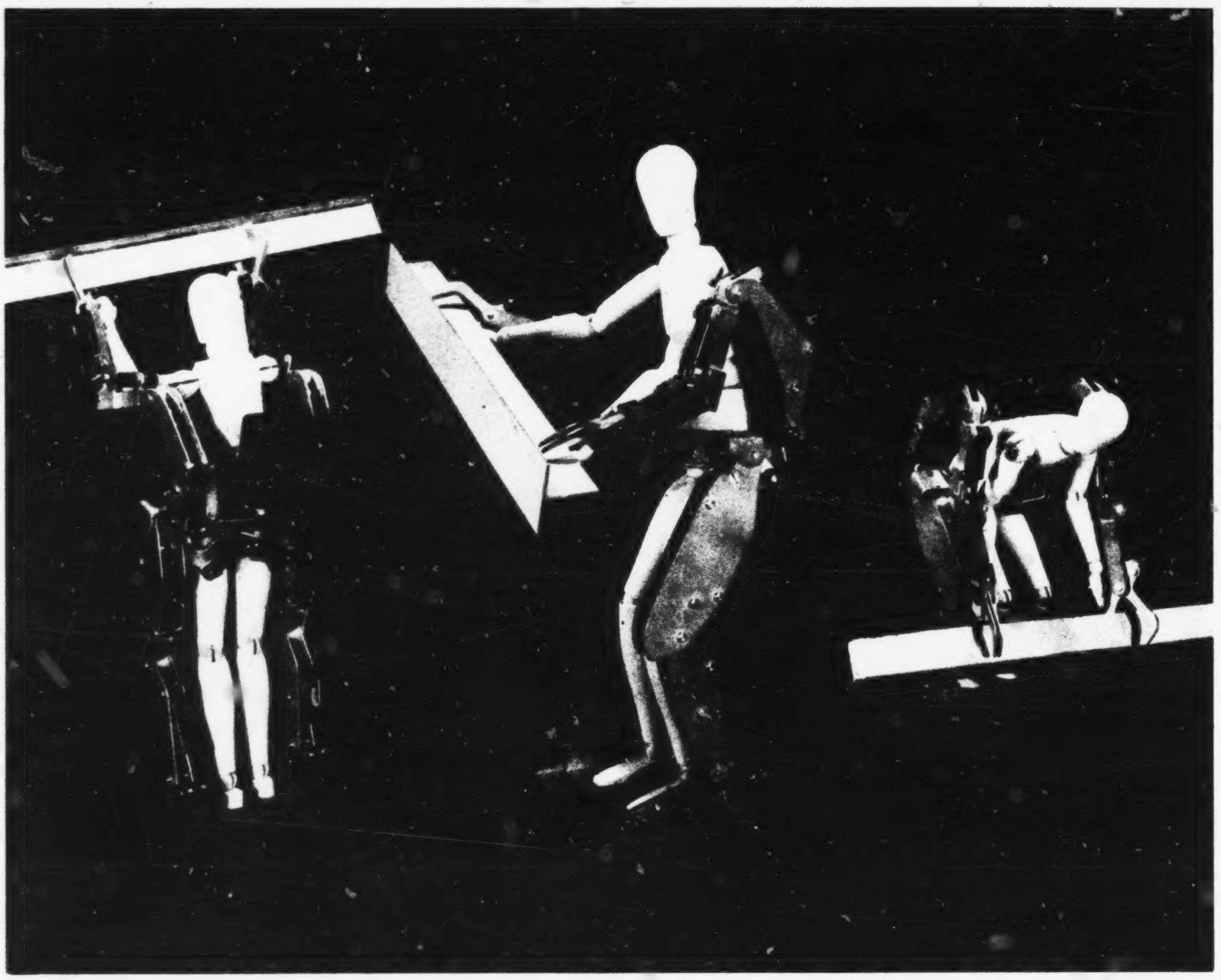


Fig. 20 Courtesy: GENERAL ELECTRIC

This is a model of a set of "mechanical muscles" that will give a human being the strength of a giant, and permit him to lift a 1500-pound load while exerting only a fraction of this force. Attached to its operator at his feet, forearms, and waist, the machine -- nicknamed HardiMan -- will mimic and amplify his movements.

amplifier with feedback through a capacitor. The overall operation of the device is essentially independent of the characteristics of the amplifier. The gain of the amplifier can change from 1000 to 500 to 5000, and the output remains an accurate measure of the integral of the input signal.

A third example is shown in Fig. 20, a picture of a mechanical amplifier to increase the effective strength of a man. The mechanical machine is worn by the operator like an external skeleton, attached to the operator at the feet, forearms and waist. As the man moves his arms, the exact motions are repeated by the machine powered by hydraulic motors (similar to the power steering and power brakes used in cars). The device can lift loads of 1500 pounds.

In order to permit the man to control the machine adequately in spite of widely varying loads, a fraction of the load forces acting on the skeleton are applied to the man's arms and legs. For example, if the machine's arm hits an object, the operator feels a fraction of the impulse force on his own arm. In this way, the machine becomes an extension of the man; the operator can use the machine in a normal way to move unusually heavy or cumbersome loads or objects that are particularly dangerous, such as bombs. The force feedback to the human being is the key to successful operation over the entire range of load conditions.

Again as in the last section, we close with one final example and attempt to add feedback to the learning system represented by the above discussion. In this learning system which involves the author, publisher, and reader, the characteristics of all three elements vary widely; we should attempt to ensure the process of understanding by adding feedback in the form of a few simple questions.

Question 1: What are the two primary uses of feedback in system design -- the applications discussed in this and the preceding section?

Question 2: For the system shown in Fig. 21, determine the percentage change in overall system gain when the amplifier gain falls by 20%.

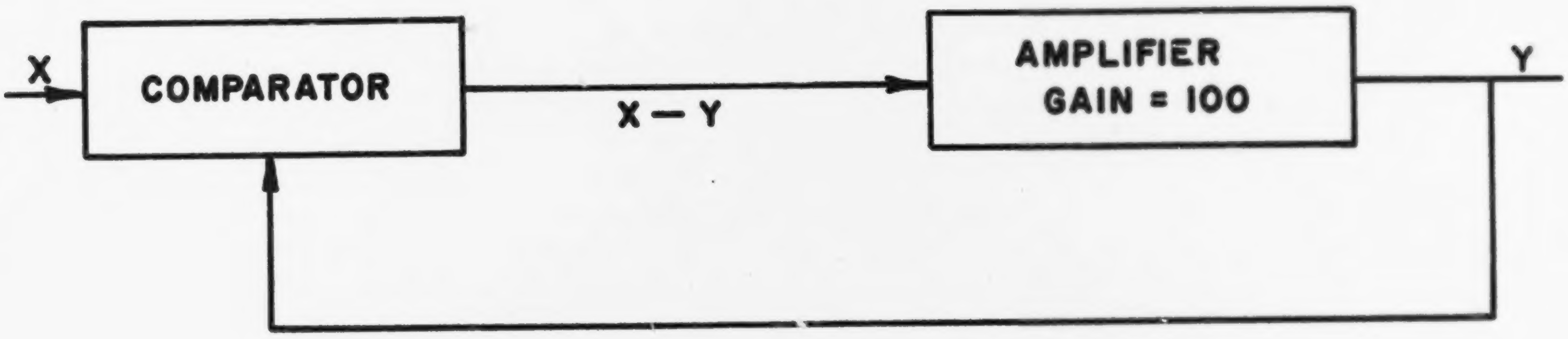


Fig. 21 Simple feedback system for Question 2.

Question 3: Figure 22 shows a more complex feedback configuration. Determine the change in overall system gain when the amplifier gain decreases by 90%. This is a rather unusual system, since it turns out that y is independent of the amplifier gain even though y is the output of the amplifier. Thus, even if the amplifier gain dropped to 0.001, the output y would be unchanged. The system begins to suggest some of the astoninshing characteristics which can be achieved when feedback is used.

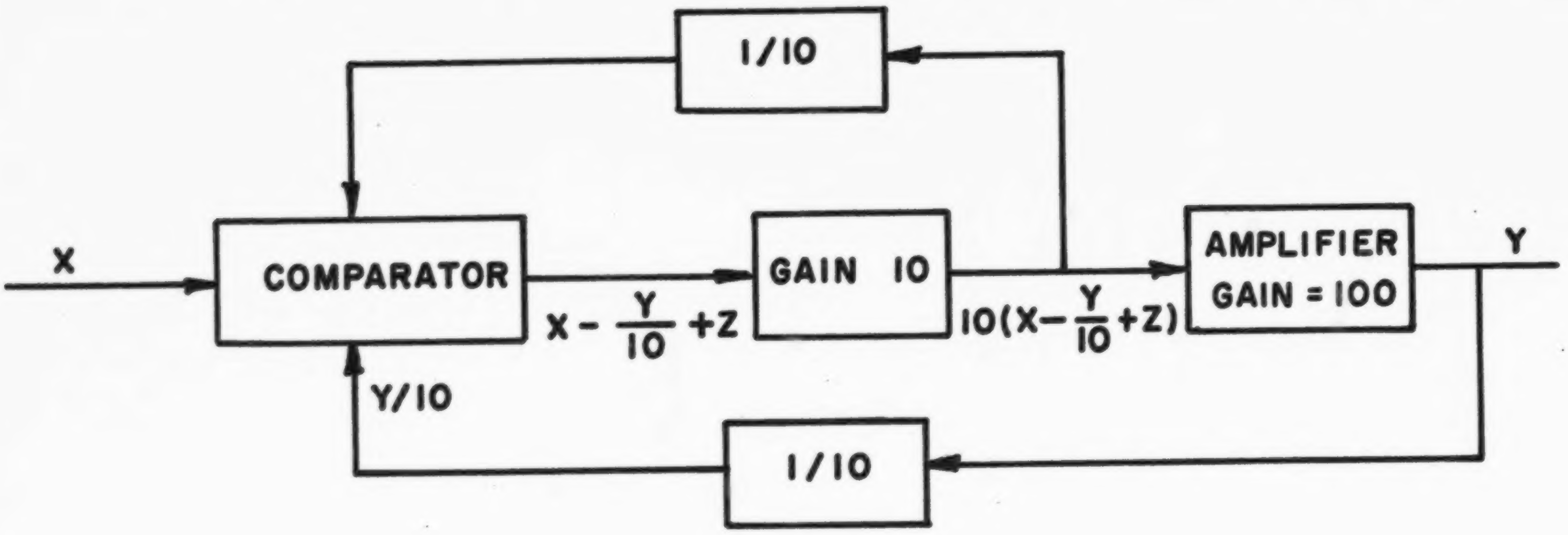


Fig. 22 A system with two separate feedback paths.

In this and the preceding section, the two basic purposes of feedback are described. We consider a familiar feedback system as a final example of these properties.

C-1.6 A FEEDBACK EXAMPLE

The ideas of the last two sections are sufficiently important so that it is desirable to emphasize them with one final example of a feedback system. For this purpose, we consider in this section the general, qualitative discussion of the system in a human being for the control of the internal body temperature. The system as described below is interesting because it illustrates an important case in which feedback is used to ensure highly accurate system performance in the presence of large disturbance signals and large changes in component characteristics.

The normal temperature of the core of the body (the internal organs and the central nervous system) is about 98.6°F (actually the British consider the normal temperature to be 98; presumably in this country we use 98.6 since this corresponds to exactly 37°Centigrade). This temperature must be controlled very accurately: cells of the central nervous system are damaged if the temperature rises as much as 7°; an even smaller drop in temperature results in greatly reduced enzyme activity within the body. In the normally healthy individual, the temperature is controlled within ± 2°F.

This control of core temperature is achieved with an ambient temperature variation of more than 100°F and the wind velocity changes from zero to very high values. We recognize the similarity between this system and the example of the last section: the control of temperature inside a house. In the human system, however, nature provides several different means by which temperature is measured as well as several different sources of heat (rather than the single thermostat and furnace common in houses).

In order to construct a model of our system, we must understand the basic elements of the system. From a thermal viewpoint, the body consists of three principal parts:

(1) The core

(2) The skeletal muscles

(3) The skin

In the control of core temperature, the human being uses several different procedures to vary the heat available in the core:

(1) The basal metabolic rate (BMR) is generated primarily in the core. The oxygen inhaled is transported by the blood to the cells where fat is stored. The oxidation or burning of this fat results in release of carbon dioxide to the blood and release of heat energy. When the BMR is measured during a physical examination, the patient is not permitted to eat for at least 12 hours in advance of the examination and reclines in a totally resting position. The net oxygen consumption of the individual is then measured to determine the BMR, the rate at which the man is converting fat to energy internally. During exercise or deep anxiety, the net oxygen consumption rises sharply (we may breathe rapidly and the heart rate increases to augment oxygen circulation). Temperature control, however, is achieved by control of the metabolic rate through the endocrine gland which receives electrical signals from the brain which dictates an increase or decrease metabolism.

(2) The muscles also provide a source of heat. If the skin detects a sharp drop in outside temperature, electrical signals are transmitted from the brain to the muscles to order shivering. Here adjacent muscles (the same ones normally used for motion or useful work) operate in an uncoordinated fashion, with the result that there is very little useful work and most of the energy is converted to heat. (Metabolism in the muscles also results in heat generation).

(3) The skin is used to effect changes in internal temperature in two ways. First, the blood flow to the surface of the skin can be controlled (this is called the vasomotor effect). When heat flow out of the body is to be decreased, less of the warm blood goes to the skin (e.g., during a cold shower). Secondly, sweating permits the loss of heat by evaporation, and is particularly important when the temperature of the surroundings is higher than the body temperature. (Dogs have very few sweat glands, they pant to increase evaporation from the tongue and the mouth).

Thus, the human being has four primary ways of exerting control over the internal body temperature: the variation of the metabolic rate, shivering, varying blood flow to the skin, and sweating. Each of these four types of control is actuated by signals from that region of the brain which determines temperature control. This part of the brain is supplied electrical information about the internal body temperature and about the skin temperature. These signals are produced by nerve sensors which respond to temperature and to temperature changes.

We now have developed a general understanding of the system operation and are ready to construct a block diagram which shows the important elements of the system and the various feedback paths. The output of our system is the internal body temperature (the temperature of what we have called the core). Another output signal of the body (Fig. 23) is the skin temperature. In addition to the four primary input signals to the body, there are also disturbance inputs which influence the core temperature -- signals which result when exericse is performed and when the temperature of the surroundings varies.

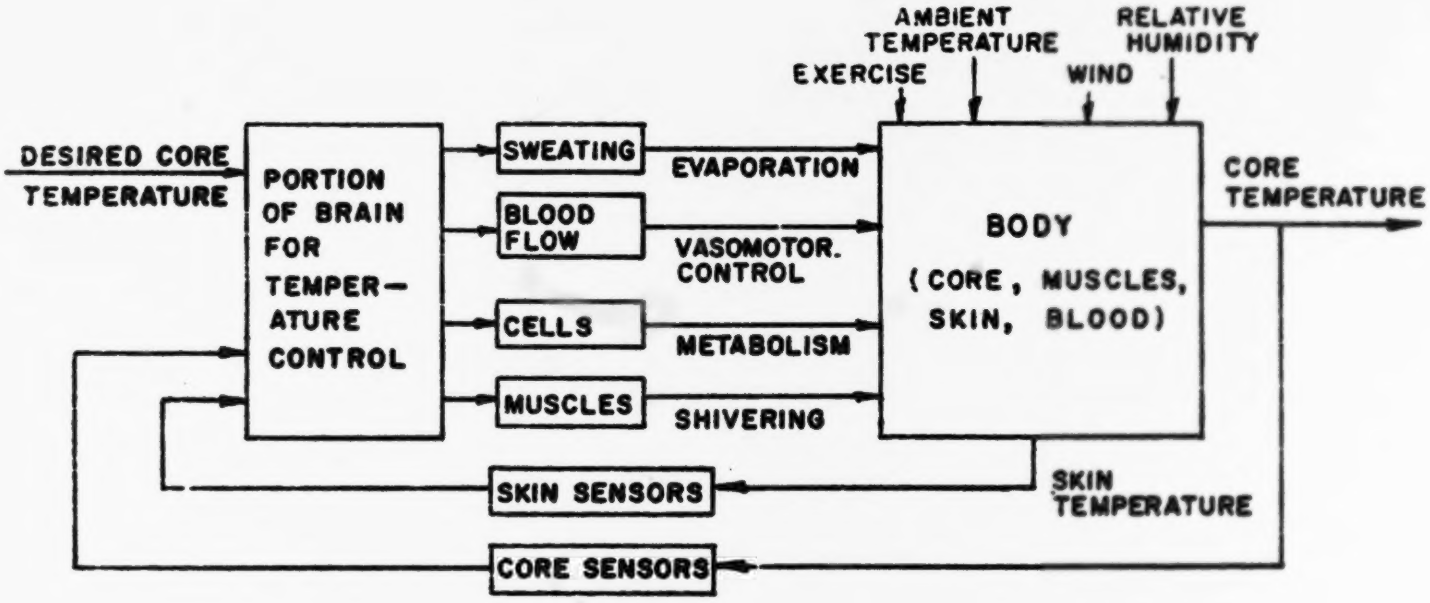

AMBIENT TEMPERATURE
RELATIVE HUMIDITY
EXERCISE
WIND
DESIRED CORE TEMPERATURE
PORTION OF BRAIN FOR TEMPERATURE CONTROL
SWEATING
EVAPORATION
BLOOD FLOW
VASOMOTOR. CONTROL
CELLS
METABOLISM
MUSCLES
SHIVERING
BODY (CORE, MUSCLES, SKIN, BLOOD)
CORE TEMPERATURE
SKIN SENSORS
SKIN TEMPERATURE
CORE SENSORS

Fig. 23 Thermal regulating system for temperature of body core

The primary input to our system is the desired core temperature (normally 98.6°F). In case of illness, this input is probably increased to produce a fever. Medical research has not yet indicated clearly how this change is accomplished. In the brain, the measured core temperature is compared with this desired value to yield an error signal to start correction of the output. It is interesting to note that the human system shown in Fig. 23 also includes a measurement of skin temperature, in order to anticipate heat demands when the external environment changes rapidly (as it does when one enters a hot or cold shower or moves from the inside to the outside of a house during cold winter weather). In our analogy to the home heating system, the skin sensors correspond to outdoor thermometers connected to the thermostatic control system to anticipate sharp changes in the outside temperature.

The block diagram of Fig. 23 includes the primary elements and signals which are now believed to constitute the temperature control system of the human being. What is the value of a diagram of this sort? How does such a description of the system aid the researcher in learning about the operation of the system? How might the model help in developing improved medical procedures?

The answers to these questions are difficult unless we develop a much more detailed model. We should consider each block in Fig. 23 and attempt to determine an appropriate mathematical representation (from experimental measurements on human beings or from an understanding of the physical laws explaining the behavior of the element). Such an effort would require, however, an entire chapter or more, for stating the scientific knowledge available today on these topics -- and we would have a book on psysiology rather than the man-made world. Certain comments can, however, be made just from this block-diagram model.

First, the importance of feedback is apparent. The internal body temperature changes very little (typically less than a degree) when the human being passes through radically different environments: feedback almost eliminates completely the effect of disturbance signals. Second, the core temperature is almost entirely independent of changes in the body properties. Major changes can be effected by amputation or surgery, for example; minor changes, by injury or normal activites such as changing clothers. Feedback yields a system in which performance is almost independent of system characteristics over wide ranges.

The model also indicates the scientific research required for a more complete understanding of the system and can be used to guide further experiments. As we mentioned above, detailed understanding depends on a detailed model; such understanding can be achieved only if experiments can be devised to permit determination of the mathematical relations represented by each element. The model indicates basic questions: for example, how is the input signal changed (if it is) to cause a fever in a patient who is ill. Further study of this question may indicate that the input is not changed, but rather that the system loses effective control during illness. If the latter is true, perhaps major efforts should be made to hold down the temperature of an ill patient.

Furthermore, our model is admittedly approximate. As science advances and detailed experiments are performed, interactions and interrelationships not represented in the model will be discovered. When the model is then revised new experiments will be suggested to improve further our understanding of the system. Thus, the feedback model is a key tool in the development of scientific understanding.

Finally, the model of this aspect of the human being may well indicate improvements in analogous physical systems. We have already examined the close similarity between this human control system and the house heating system, and the suggestion of a set of outdoor thermometers to improve internal temperature control. The vasomotor and sweating actions suggest similarity with the air conditioning complement of the heating system.

Final feedback question: In hibernation, what changes in the system would you expect to occur (if the model represented a bear rather than a human being)? Which elements of the system would be operative during hibernation? Normally, a bear's internal core temperature drops to about 42°F in this period; why is it desirable to reduce the bear's body temperature to this lower value?

C-1.7 INSTABILITY IN FEEDBACK SYSTEM

Feedback is, as we have seen, primarily of interest in goal-seeking applications: systems in which the quality of performance is measured by how close we approach the design goal. In the example of navigating through the reefs, the goal is to place the ship in the center of the channel; in the human temperature control system, the goal is a constant core temperature of 98.6°F, regardless of ambient temperatures or physical or emotional activity.

For the production of a system which achieves a desired goal, feedback serves as an important engineering or scientific concept. Feedback permits the successful design.

The impressive success of feedback in controlling both disturbances and changes in the system component characteristics involves certain inherent disadvantages. We have already seen that feedback normally requires a more complicated system (the output signal must be measured and compared automatically to the desired value of the output in order to calculate an error signal which can be used to correct the output). In this section, we discuss a second, major disadvantage associated with the use of feedback: the possibility that the system may be unstable.

In very general terms, a system is unstable if its output goes out of control. Perhaps the most dramatic example of instability is the hydrogen bomb; here a small detonation rapidly grows into a major explosion. There are many other examples. Visitors to Bermuda observe the abundance of lizards on the island. Some years ago, a few lizards were brought to the island to control the mosquitoes; the lizards rapidly multiplie , the mosquitoes disappeared, and the islanders are now worrying about controlling the lizard population.

Perhaps a more familiar example is the instability occurring when a car goes from an understeer to an oversteer condition. Understeering means that, as the car traverses a curve, there is a tendency for the car to pull out of its curved path, that is to increase its radius of curvature (Fig. 24). To follow the road, the driver must insert more and more turning of the steering wheels. In the oversteer case, the automobile characteristics tend to decrease the radius, to "tighten" the turn; to compensate the driver must ease up on the steering wheel during the turn.

The study of an automobile behavior during turning is an exceedingly complicated engineering problem since the motion depends on the road angle and surface and on the rapid actions of the driver. The dangerous situation occurs when the automobile at a certain speed passes from an understeer to an oversteer condition; the driver, if not aware of the change, tends to turn the steering wheel to the wrong extent and to try to correct car position improperly.

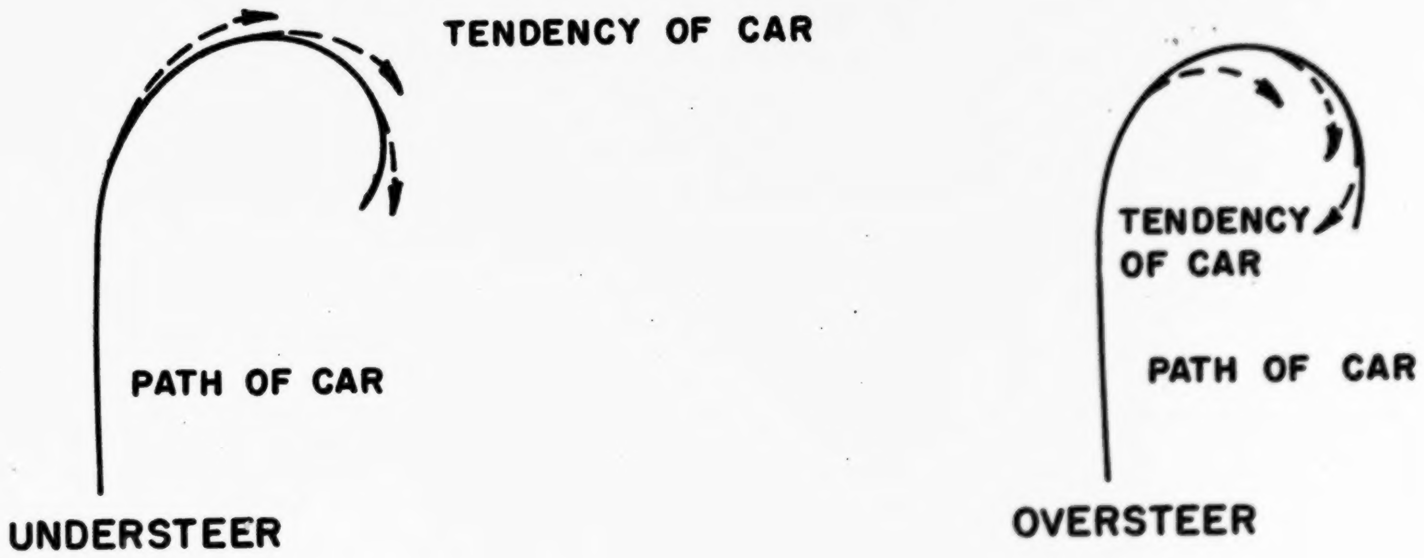


Fig. 24 Path of a car around a curve

A most important and difficult problem in automobile design is to ensure that such a transition does not occur abruptly during operation under any conditions.*

An Example

In order to emphasize the nature of instability in a feedback system, we return to our earlier navigation example, shown again in Fig. 25. The ship is 10 nautical miles (n. m.) south of the channel center; the navigator sights the channel markings and orders a heading due north; the fog now settles in and no further navigation is possible.

The problem is complicated by a current, which in the preceding example was assumed to be 2 knots toward the east. In the present example, we change the current to a more interesting signal as shown in Fig. 26. Here the current is not constant, but reverses and changes in strength every fifteen minutes. Admittedly, this particular type of current is not very probable (a sudden change from +2 to -1.6 knots taxes one's imagination), but we wish to avoid complicated mathematical analysis. This particular signal illustrates what can happen when the current is not constant.

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* The hundreds of law suits against General Motors in connection with the Corvair car manufactured during a few years in the early 1960's were based in large part on the claim that poor engineering resulted in such a transition from understeer to oversteer. In the first case brought to trial with outstanding engineers testifying for both sides on the technical aspects, the judge decided for General Motors. The engineering problem in such a case is so complex that scientific, mathematical, or computer analyses and experimental tests often yield no clear conclusions.

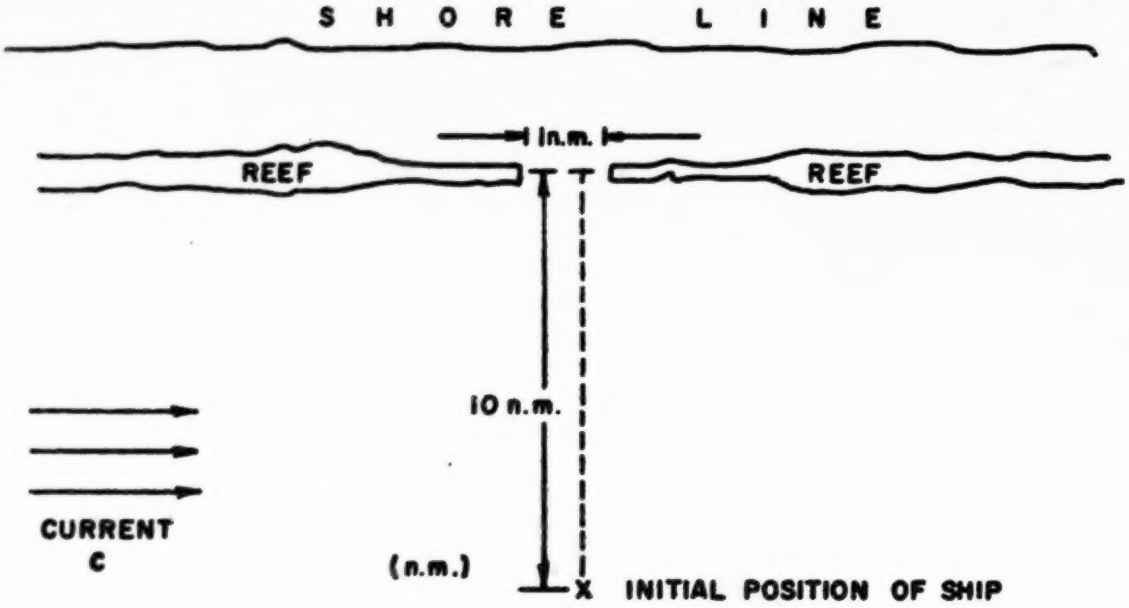

Fig. 25 The channel navigation problem.

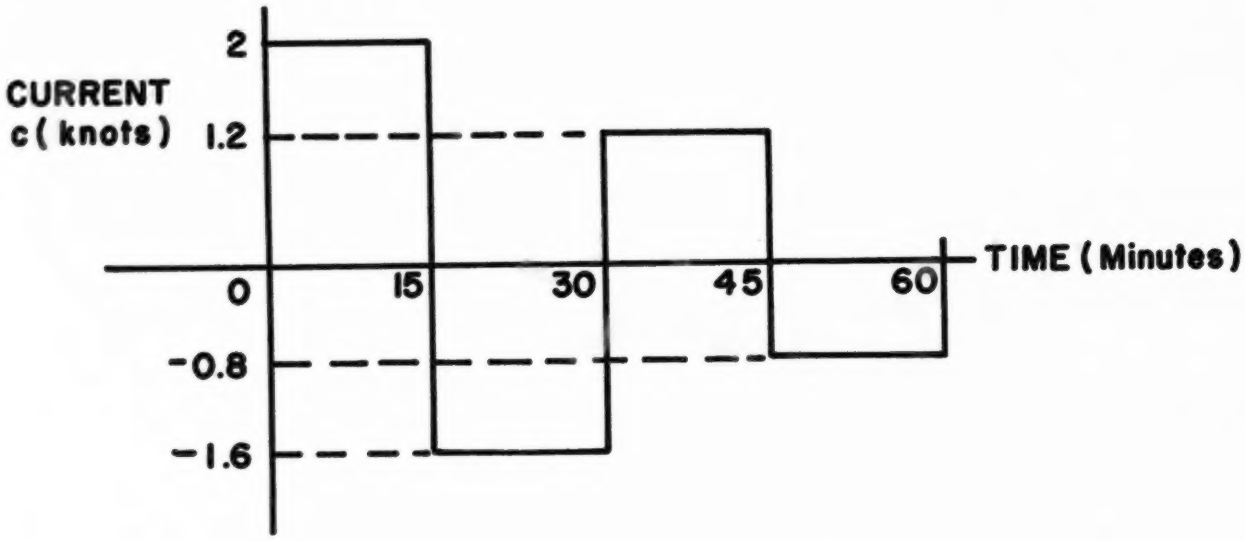

Fig. 26 Possible current signal

If we accept this current as possible (even if not probable), the next step is to determine the motion of the ship through the fog. During the first quarter hour, the ship travels north 2.5 n.m. and (because of the current) east 0.5 n.m. During the second quarter hour, travel is again 2.5 n.m. north, but also 0.4 n.m. west (the current is now flowing toward the west at 1.6 knots). The third fifteen minutes the ship moves 2.5 n.m. north and 0.3 n.m. east; the fourth quarter hour, 2.5 n.m. north and 0.2 n.m. west. Thus, the ship reaches the reef line in one hour and only 0.2 n.m. east of the center of the channel. Indeed, the maximum deviation of the ship from its desired course is only 0.4 n.m., occurring at both points A and C in Fig. 27. The system is certainly stable:

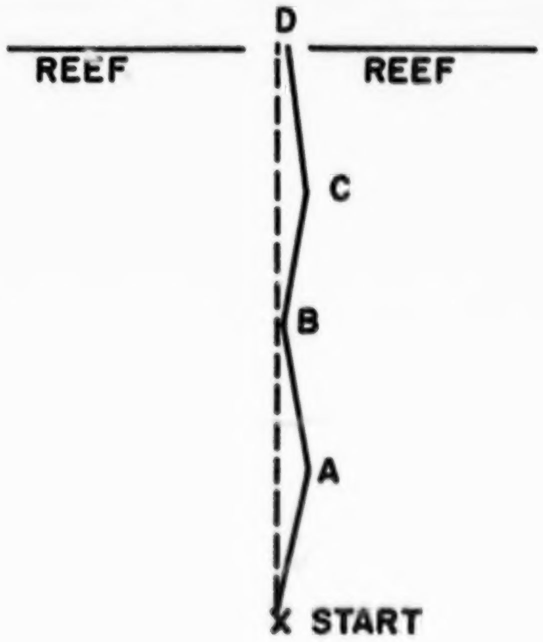

Fig. 27 Path of ship with no feedback and varying current.

there is no loss of control over the ship position. The actual path travelled by the ship differs only slightly from the desired northerly direction. Thus, even without feedback, the channel is successfully negotiated.

Will feedback improve the system? Certainly if the feedback is continuous (with the navigator continually observing the channel markers), common sense tells us the path followed by the ship will have the general appearance shown in Fig. 28. Each time the current changes abruptly (at points A, B, and C),

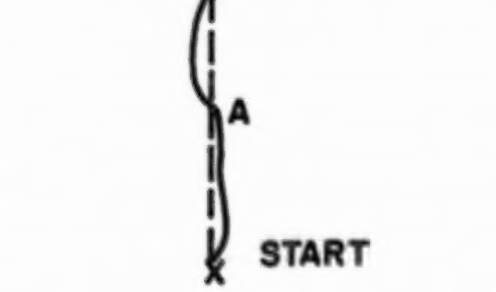

Fig. 28 Path of ship with continuous feedback and varying current

the helmsman requires a little time to estimate accurately the new value of the current. During this period of measurement and estimation, the ship departs slightly from its desired course. As soon as the helmsman determines the new value of current, however, he adopts a course accurately set to bring the ship to the middle of the channel if the current does not change again. Thus, the system with continuous feedback behaves admirably and, indeed, has all the advantages associated with feedback (including the proper system behavior for any reasonable variations in current).

If the feedback is intermittent, however, performance may be quite different. For example, we consider the case in which the fog lifts momentarily every fifteen minutes and the helmsman is an intelligent individual who tries to estimate the current in order to compensate in the heading he adopts. The ship starts its motion due north; in fifteen minutes it reaches A (Fig. 29) 2.5 n.m. north of the starting point and 0.4 n.m. east. At this time, the fog lifts; the navigator measures his position and calculates the current as having been 2 knots toward the east. He assumes this current will continue; consequently, to hit the channel center he must turn the ship so that it will move 7.5 n.m. north while travelling 1.9 n.m. west (0.4 to return to the center and 1.5 to compensate for the anticipated current during the next 45 minutes).

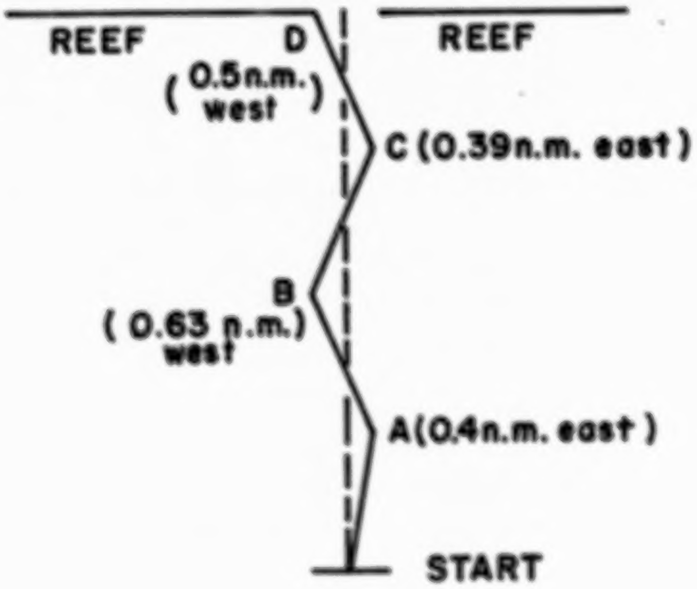

Fig. 29 Motion with intermittent feedback and varying current

During the second quarter hour, therefore, the ship would move 2.5 n.m. north and 0.67 n.m. west if there were no current; the current of 1.6 knots toward the west adds 0.4 n.m. to this motion. Point B in Fig. 29 is therefore 5 n.m. south of the channel center and 0.63 n.m. west.*

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*We have simplified the calculation by assuming the ship moves 2.5 n.m. north each 15 minutes. Actually, this corresponds to a speed through the water of slightly more than 10 knots.

The poor navigator now sees the channel again and realizes the current has been 1.6 knots to the west. He assumes he will reach the channel center by heading the ship so as to move 5 n.m. north while going 1.43 n.m. north eastward (0.63 to return to the centerline, 0.8 to compensate for the current during the remaining 0.5 hour). As a result, travel from B to C results in 2.5 n.m. north, and an eastward motion of 0.72 n.m. (with no current) plus 0.3 n.m. (due to the current of 1.2 knots for 1/4 hour). Hence, C is 0.39 n.m. east of the centerline.

Similar calculations show D is 0.5 n.m. west of the channel center, on the reef. The intermittent feedback results in a system in which the output is not controlled. While the distance from the desired path does not increase steadily (instead it rises and falls alternately), there is only a weak tendency for the ship's course to approach the desired path as time progresses. Indeed, the deviations would actually tend to grow if the current became stronger rather than weaker as time passed -- and the system would not be stable during the short period required to move toward the channel.

Thus, the use of intermittent feedback results in a total breakdown in system performance. With no feedback, the system operates satisfactorily; when feedback is used every 15 minutes, the system is unsatisfactory. We then not only fail to realize the advantage of feedback, but also fail to achieve the performance of the system without feedback.

Another Example

The possibility of instability is a basic limitation on the use of feedback. In most systems, when feedback is inserted we are concerned with the possibility of instability. While the topic of stability is discussed again in a subsequent chapter, it is desirable here to consider an additional example of instability.

Perhaps the most familiar use of feedback occurs in the manual operation of touching a pencil which is on the top of a table. In this system, the output is the position of the pencil. The man observes the difference between the actual output and the desired value, and corresponding electrical signals from his brain to the muscles produce the required change in the output (Fig. 30). In addition to

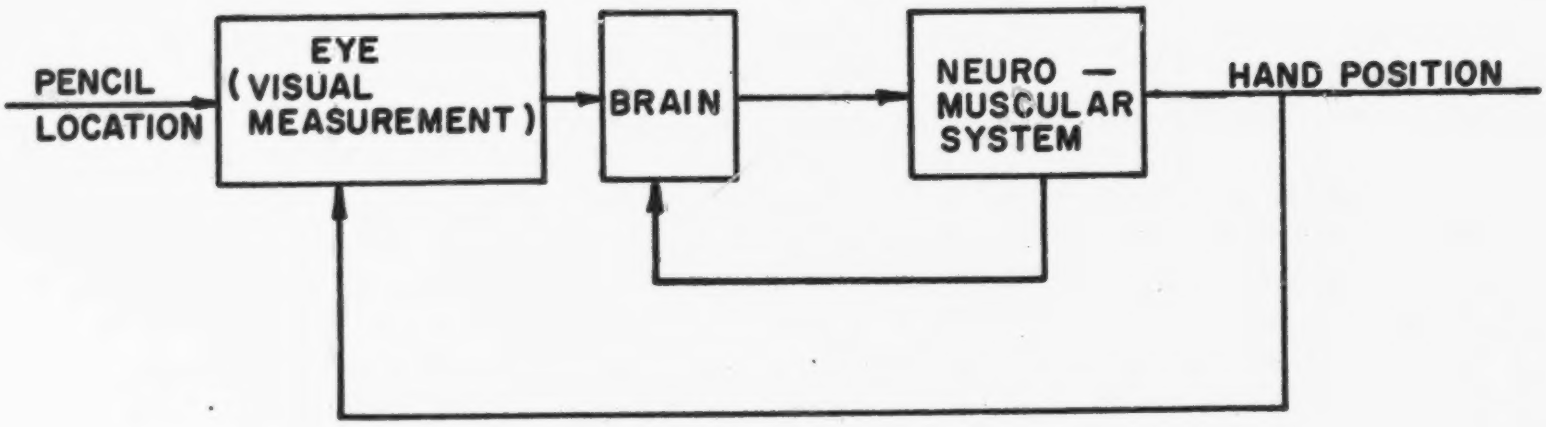

Fig. 30 Feedback system involved in picking up a pencil

the main feedback through the eye, there is an additional feedback path directly from the muscular system to the brain: as the muscles are actuated, the individual senses the forces and motion resulting.* As soon as the hand touches the table top or pencil, there is an additional feedback path (not shown in the figure) which results from the sense of touch.

The above feedback system is normally superb. It operates satisfactorily even when disturbing signals are present (e.g., motion of the system when the task is performed within an airplane or moving vehicle) or when system characteristics change (the individual is tired, weak physically, or distracted mentally by other events).

The possibility of instability exists, however, when individuals suffering from an illness called ataxia attempts this task, the hand starts to oscillate as it approaches the pencil. The man is unable to pick up the pencil because of the violent shaking of his hand.´The loss of system control results from oscillation or instability within the feedback structure.

Instability and Delay

The determination of the point at which instability occurs in a feedback system is a difficult task mathematically and analytically. Intuitively, we can understand some aspects of the problem by a consideration of the simple system of Fig. 31. Here the input is compared with the output, and the error signal

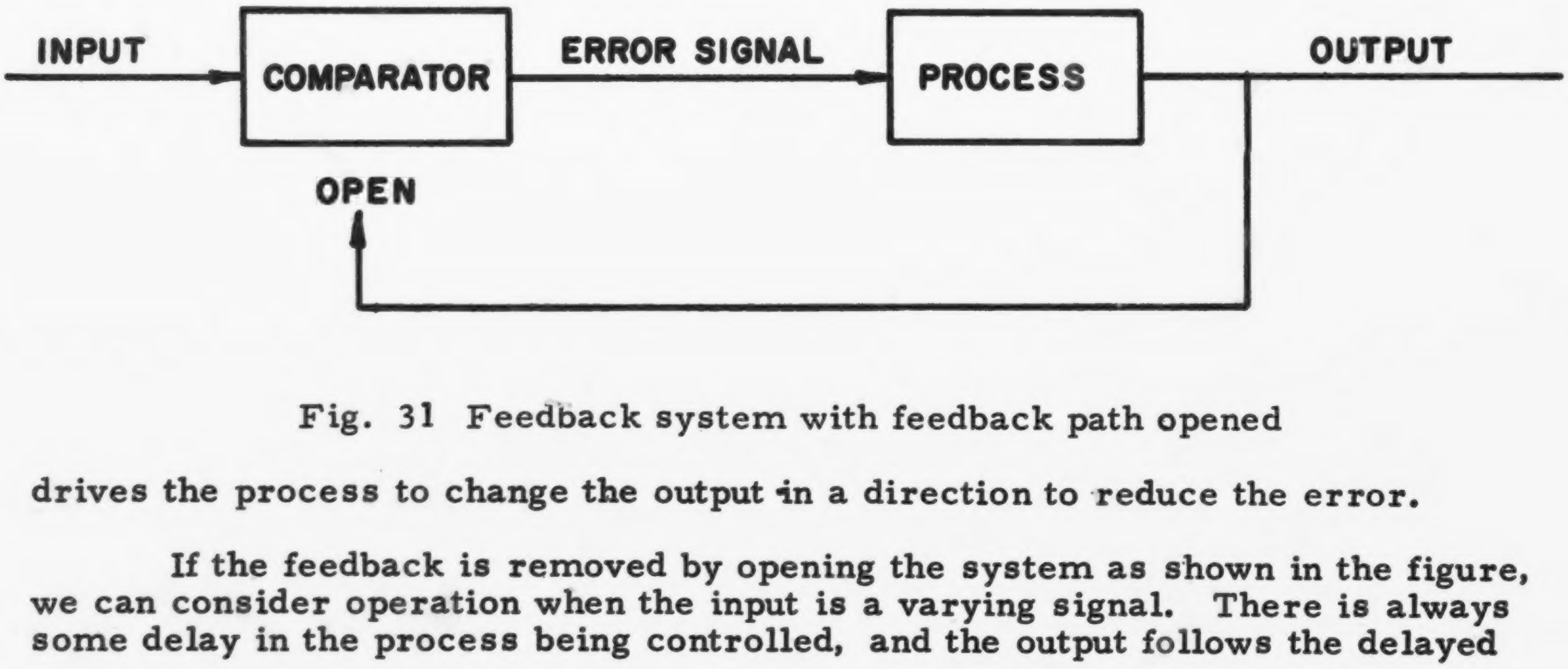

Fig. 31 Feedback system with feedback path opened

drives the process to change the output in a direction to reduce the error.

If the feedback is removed by opening the system as shown in the figure, we can consider operation when the input is a varying signal. There is always some delay in the process being controlled, and the output follows the delayed input. This output is fed back to the input at the comparator. If the time delay is such that this output reinforces (or adds to) the input, the system may become oscillatory: the input causes a certain output, this is fed back and augments the input, the even larger output is in turn fed back, and the output signal grows larger and larger.

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*The existence of this internal feedback path is apparent if the man closes his eyes after initially observing the pencil location. He still can sense how far his hand has moved and approximately where it is located.

The above description is certainly not rigorous or, indeed, any real help in determining whether a feedback system will be stable or unstable. The paragraph does, however, indicate that delay in the system is a key to the instability phenomenon. The delay (which we find in every real system) is essential to instability in a feedback system. In general, the greater the delay, the more tendency there is for the system to become unstable.

In the two examples considered previously, the source of the delay is apparent. In the navigation example, the helmsman sets his course on the basis of knowledge about the current in the past. In the system for positioning the hand to touch a pencil, the delay arises from the time required for the brain to reach a decision and the time needed to move the muscles.

Final Examples

The possibility of instability in feedback structures can be illustrated by many other examples: here we mention only a few:

(1) In inventory control, a company observes its sales of a particular product and then decides on manufacturing schedules accordingly. If the company wishes to keep its customers happy, it must have enough completed products in its warehouses to ensure its ability to satisfy rush orders. On the other hand, completed and unsold products in the warehouse represent dollars not working for the company. Thus, decisions are required in an attempt to realize an optimum policy.

The feedback system is complicated by two primary delays: the delay in manufacturing (if the demand rises sharply, time is required to increase the factory output because of the delays in obtaining raw materials and the time required for manufacture), and the delay in measuring changes in orders and demand for the product. Instability (or loss of control) results in very large variations in factory activity (and the consequent costs of large changes in the work force, the required training of new employees, and so on), and may lead to warehouses bulging with unsold products or to major inability to meet customer orders (with the dissatisfied customers then turning to a competitor's product).

(2) As another biological example of instability, we can consider the population growth of hydra in experimental environments. Hydra are very small, fresh-water animals which increase rapidly in numbers when the food supply is ample. One class of hydra can be fed exclusively on water fleas. As the hydra population in a closed container grows, the individual animals tend to become smaller (an automatic adjustment over generations to the population explosion). Indeed, this regulating or control system may be unstable: the hydra become so small they are unable to eat the water fleas, and the entire population dies of starvation. In this case, the feedback mechanism (yielding the decreasing individual size with population growth) leads to system annihilation.

Such an extreme effect of feedback in a biological system probably results from the artificial environment in the laboratory. In a natural environment, small food sources would be available in at least small quantity, and some hydra would survive, even though the total population might decrease. This decrease in population would then result in larger animals, which would in turn thrive on the water fleas, and the total system would tend to oscillate around an average population and size.

Thus, in biological systems feedback tends to cause severe or catastrophic instability only when the natural environment is radically changed -- e.g., by human intervention. It is this possibility of unstable behavior, however, which is the basis of much of the national concern today over the impact of technological development. With the rapid increase in recent years of our technological capabilities, we are today able to effect major changes in the natural environment.* What feedback system in nature will then be driven into instability?

(3) As a final example of feedback oscillation, we cite a case in which feedback is used to realize a constant-amplitude oscillation. In the pendulum shown in Fig. 32, the displaced ball (hung on a string attached to the ceiling at 0) is moved to A and released. The motion thereafter is familiar: the ball swings down to B, then on to C where it reaches a maximum height, then back to B and on toward A. As the ball oscillates back and forth, a little

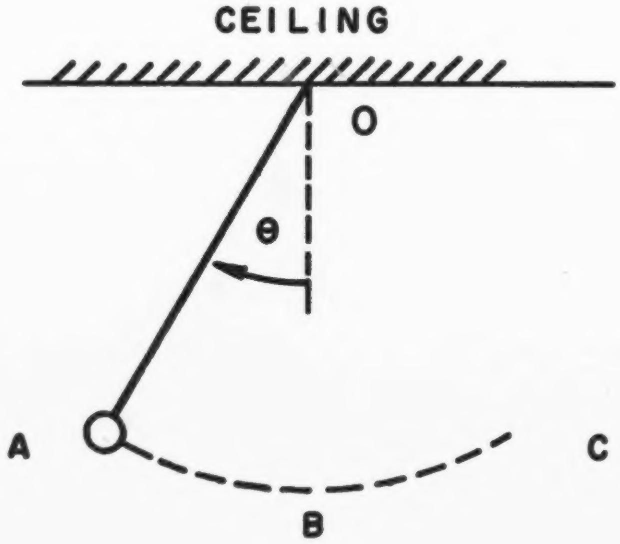

Fig. 32 A swinging pendulum.

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*The possibility of major success in weather control (particularly rainfall) is a particularly timely example. A more dramatic example is the current technological feasibility of placing in orbit at an altitude of 22,500 miles (so that it sits over the same spot on earth) a large reflector to reflect back to the earth's surface sunlight during hours when the sun is below the horizon. The darkness of night would thereby be avoided -- with the consequent, undetermined effects on nature as well as on man's living habits.

energy is lost each trip because of air friction and friction at the pivot. Consequently, each return to the left is a little lower than the preceding, and the pendulum ultimately comes to rest at point B. If we plot the angle θ versus time, we find a signal of the form shown in Fig. 33.

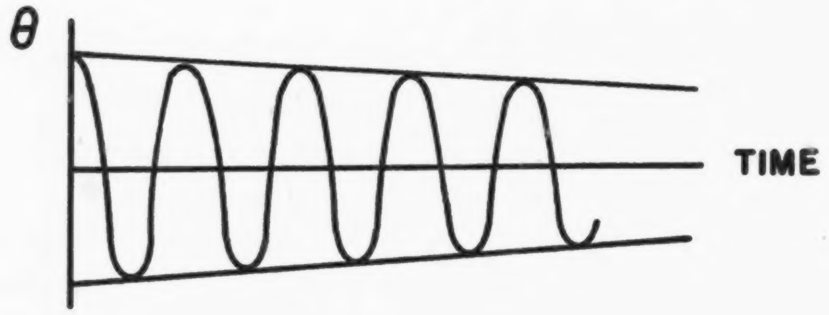

Fig. 33 Gradually decaying oscillation of a free pendulum.

If we desire to keep the oscillations going indefinitely, we can give the ball a very slight push each time it returns to the extreme left, just as a slight amount of pumping in a swing, is required to keep the swing moving. Specifically, we need to impart to the ball the same energy the system has lost in each cycle because of friction. With this slight push in each cycle, the θ signal takes the form shown in Fig. 34.

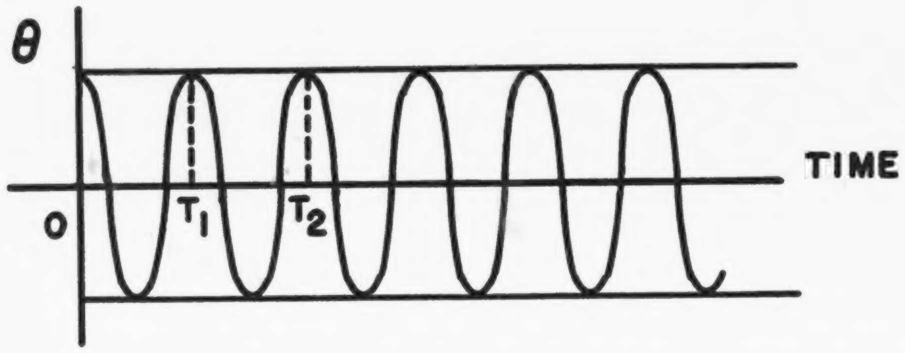

Fig. 34 Pendulum oscillation with regular excitation

If this were indeed a ball swinging from the ceiling, manual control by a human being continually tapping the ball would obviously be a tedious task. Instead, we can automate the system by surrounding the area at A by an electromagnet, which is energized in such a way as to repel the ball at the top of each swing (the ball would have to be magnetized also, of course, since we are using the force between two magnets). Thus, at instants t_1, t_2, and so forth in Fig. 34, a pulse of current through the electromagnet gives the ball a slight push as it starts its downward swing: just enough of a push to compensate for the energy lost during the preceding cycle. If we wish, even the timing of the electromagnet current can be controlled by the motion of the ball -- so that the current starts slightly after the ball enters the electromagnet near the end of the up-travel toward A. The system is then entirely automatic and maintains constant-amplitude oscillation of the pendulum.

The above system is essentially the basis for an accurate clock operated by a battery (rather than by winding or from the electric power lines). The clock requires no winding and operates for the life of the battery (typically a year). In this case feedback is used to yield an oscillation; the desired output signal is one which is constantly varying, and we are really using feedback to obtain a controlled type of instability.*

The system described above is called an oscillator: a system in which the desired output is a constant-amplitude oscillation or a continuing variation. Electronic oscillators are important components of many electronic systems. In commercial radio and television, the audio and video information can not be efficiently radiated from antennas of practical size; hence the information is used to vary the amplitude or the frequency of a high-frequency signal (near 10^6 cycles per second for radio and 10^8 for television). Electromagnetic energy at these high frequencies can be radiated from the antenna and received at the listener's home. Thus the transmitting system depends on the availability of electronic (feedback) circuits which produce voltages oscillating at millions of cycles per second.

The primary purpose of this section is to emphasize that feedback is often associated with instability, or a least a susceptibility of the system to instability. The amount of feedback which can be used in a system design is often limited by the fact we must realize a stable system; otherwise, control is meaningless. Although the mathematical analysis of instability is a complex problem (which even in the most advanced engineering work can only be studied rigorously for relatively simple feedback systems), we can often proceed with the system design by representing our tentative design on an analog computer to determine whether it is stable.

Final Question

Determine block diagrams for the three feedback systems discussed in the last part of this section: the inventory control, the hydra population, and the clock pendulum. In each case, decide first on the system output, the desired output, and the input signals, then determine the various blocks by considering the successive cause-effect relationships. It is noteworthy that there is often no unique solution to the problem of determining a block diagram for a system.

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*We should emphasize that stability is of course not necessarily desirable. A stable system is inherently one which is well controlled, one in which changes in the output are directly and logically related to the input signals. For example, perhaps major changes in the poverty status of so many people throughout the world can be achieved only by a system at least temporarily unstable -- but with an instability which can be arrested at a new level of operation.

two people may choose to emphasize or to neglect quite different aspects of the system.

Conclusions

Feedback is an engineering concept or viewpoint. To understand and to design complex control systems, it is often convenient to think of the system in terms of a block diagram in which one can emphasize the various signals (the output, the desired output, the input, and the error) and the separate cause-effect relationships inherent in the separate parts of the system. In a block diagram, feedback is represented by measurement of the output and comparison of this measured value with the desired value. The error is then used to change the output in a direction designed to reduce the error.

Feedback has, of course, been used by engineers for centuries and appears in natural systems. Perhaps the earliest engineered feedback system was the plumbing device developed by the Romans and still used in much the same form in many homes: the water level control in the reservoir at the back of the common toilet. Here a ball floats on the water, as the level rises with water supplied from the main water line. When the water level reaches the desired value (the error is zero), the rising ball shuts a valve which stops the incoming water. When the water in the tank is released by flushing, the tank empties, a rubber stopper covers the outlet, and the cycle repeats.

The first major work on feedback-system engineering was carried out at Bell Telephone Laboratories during the 1920's, as the telephone engineers attempted to develop a system for long-distance telephony. With the availability of electronic amplifiers (based on vacuum tubes), telephone conversations were possible across the United States if amplifiers were used every few miles to amplify the voice signals. Once a larger number of amplifiers was included, however, satisfactory system performance required that the gain of each amplifier should not change markedly; otherwise, the volume for the listener might vary over a wide range. In an attempt to build amplifiers with characteristics which were constant over long periods of time, the engineers utilized feedback. The success of the endeavor is indicated by the quality of the American telephone system today.

Feedback control engineering achieved another major advance during World War II, primarily because of the importance of feedback systems in aiming large guns and radar antennas. In earlier wars, guns were smaller and targets moved very slowly, so that guns could be aimed manually. In World War II, anti-aircraft gunfire required exceedingly rapid aiming, which demanded force levels beyond the human capabilities. Feedback systems were used to achieve satisfactory control.

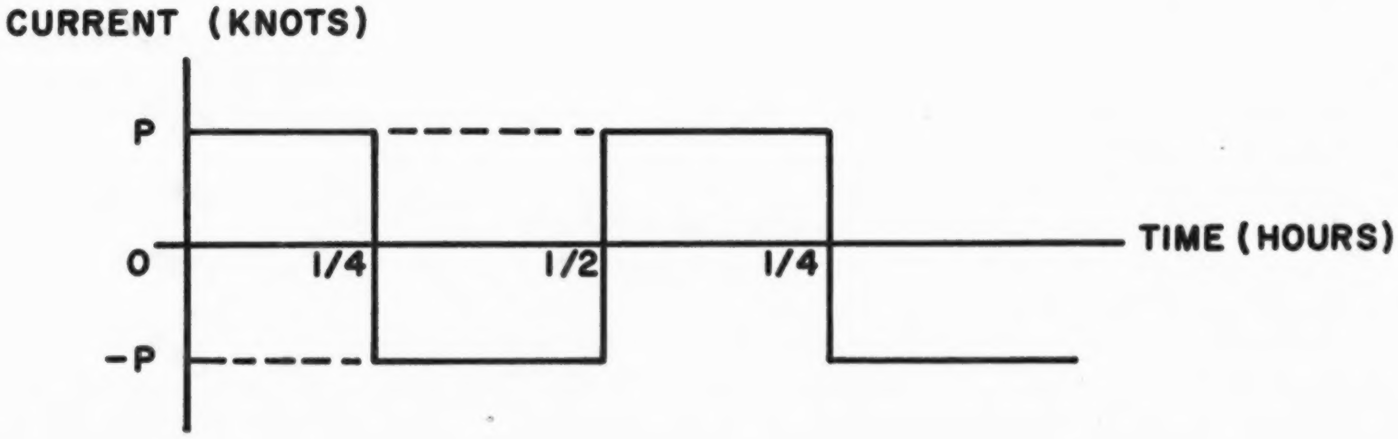
Since the Second World War, feedback engineering has continued to be an integral part of modern technology. Automation essentially involves automatic feedback control of decision-making processes, whether in factories, in traffic control, in the control of anesthesia during operations, or in the navigation of space vehicles. In addition, during the past two decades the concept of feedback has been used in the study of biological, social, and economic systems.

Problem for Sec. 5 of C-1

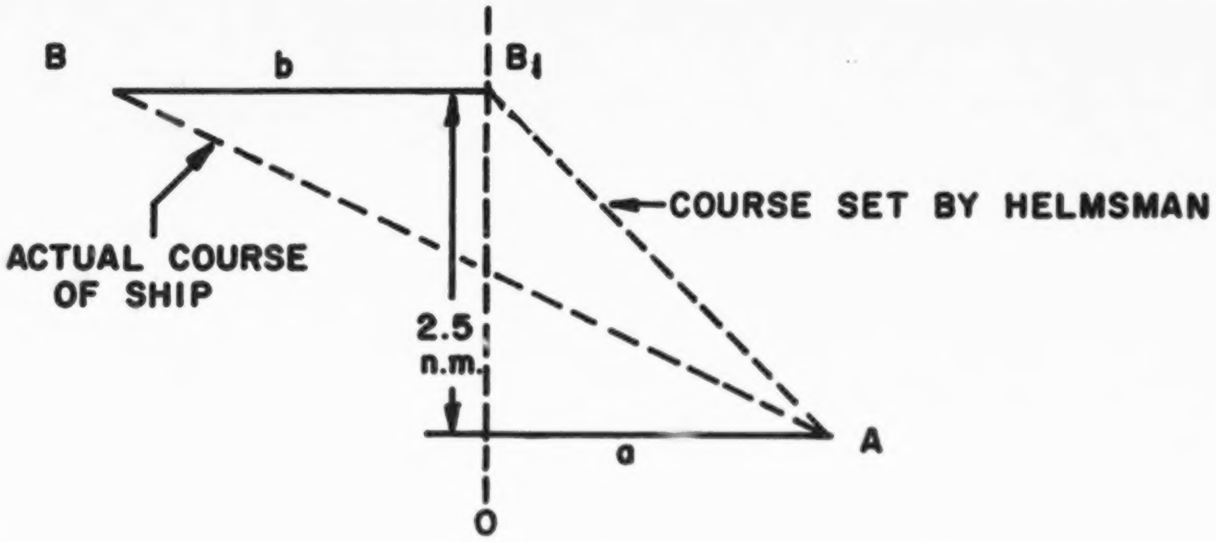
Consider the navigation problem described in Sec. 5 (the problem in which the ship starts 10 n. miles south of the channel). Using a graphical construction, determine the path of the ship when sightings are taken every 25 minutes. Repeat if sightings are taken every 10 minutes.

Sample Problem: (Try to solve this problem before you read the answer)

For the ship navigation problem discussed in Sec. 8, the current signal has the form



The current alternates between the values +P and -P. Each quarter hour the ship moves north 2.5 n.m. At the beginning of each quarter hour, the helmsman measures his location and sets a course which (if the current continued as during the last quarter hour) would return the ship to the centerline in 15 minutes. Thus, in the figure, if we are at A at the start of the 15-minute period, the course set would bring the ship to B_1 a quarter hour later if the current were +P. Actually, the current changes to -P and the ship ends up at B rather than B_1. Determine the values of a, b, c, d, ... when the ship starts from 0.



Solution:

(1) The value of a: In 1/4 hour the ship moves P/4 n.m. eastward; hence

$$a = P/4$$

(2) The value of b: The course is set for B_1 if the current is +P. Actually the current is -P, or we must add -2P to the motion set by the helmsman. Hence in 1/4 hour the ship moves 2P/4 or P/2 to the left of B_1. Hence

$$b = -P/2$$

(3) The value of c: The course is set for the centerline, but the current is +2P more than expected. Hence

$$c = +P/2$$

and the ship continues indefinitely to oscillate back and forth between +P/2 and - P/2.

Problem:

(a) The temperature control for a conventional shower is an interesting feedback system. Construct a block diagram for the system if we assume that the man standing in the water spray attempts to control the temperature of the water reaching him by adjusting the hot-water control knob only. The knob is at water level on the line leading to the shower head.

(b) The simple block diagram probably derived above can be modified to include the fact that, when the man changes the knob setting, he estimates the time delay before the temperature change will be detectable at the surface of his skin. Show this modified block diagram if your answer to (a) did not include this part of the system.

(c) The statement is made in the text that time delay is often associated with instability. Explain briefly how instability is likely to result if the man underestimates the time delay between turning of the knob and an observable change in water temperature. In other words, indicate the probable sequence of events in this case.

(d) The feedback-system stability problem is often complicated by parts of the system in which a change in input results in no change in output over significant ranges (this is one example of what the mathematician and engineer call nonlinear behavior). For example, in many shower systems there is appreciable backlash in the knob: as the knob is turned clockwise, the amount of hot water decreases. If we try to reverse direction (and move counterclockwise), the first 15° of motion result in no change in hot water flow. It is only after we have turned the knob through 15° that control is reestablished. This "slippage" or backlash occurs every time the direction is reversed. Describe briefly how such a common malfunction may make effective temperature control more difficult (and may actually lead to instability).

Chapter C-2

AMPLIFICATION

C-2.1 INTRODUCTION

The word, amplification, as used in everyday conversation, has many different meanings. One can "amplify" a fishing story by exaggeration or "amplify" the size of a small object by magnification. A minor occurence sometimes distorts our point of view and we may amplify the importance of the event thereby making a "mountain out of a mole hill". In this chapter we will not be interested in these broad implications of the word amplification; rather we will discuss amplification as an important concept which is extremely useful for building and studying the man-made world.

We frequently describe the development of the man-made world as an evolution of man's ability to control his environment. In the beginning man was severely limited in this task by a lack of available energy for such control. The only energy available to him was that provided by the human body. Man had to learn to harness other sources of energy in order to accomplish greater changes which were desirable and to accomplish this faster. In the earliest days, this harnessing was accomplished quite literally with beasts of burden such as oxen and horses. Later man learned to harness the energy of more powerful sources such as water, wind, fossil fuels, steam, electricity, and finally nuclear power. In all these instances, the implication that the energy has been "harnessed" is an important one, for man's concern is to control the flow of energy in order to achieve a useful result. In particular, he seeks to exercise this control with a minimum expenditure of energy on his part. In some cases, as in turning on a light bulb, crude control over the flow of energy is adequate; but often, as in landing men on the moon, precise control of energy is essential.

To help achieve a controlled flow of energy, man has developed the process of amplification: the application of a small amount of energy, a signal, supplied directly by an operator or a sensor to control (or modulate) the flow of a greater amount of energy supplied by another source. This control is exercised in a manner such that the output signal of the amplifying device provides much more power than could be provided by the initial signal.

Amplification has many applications which justify its importance. We have already been introduced to one of its important applications as a vital element in a feedback system (see Fig. C-1.7). We will study several other applications in this chapter. But to understand any of these applications, the meaning of the terms energy and power must be made clear. We have used these words throughout the preceding chapters, and we have intuitive ideas of their meanings. In order to use these terms perceptively, we must develop quantitative descriptions for them.

C-2.2 A DESCRIPTION OF ENERGY

Energy is the primary resource which keeps our man-made devices running. If energy were not stored in the main springs, watches could not operate. If electric batteries contained no stored energy flashlights or pocket radios would be inoperative. If gasoline with its internal chemical energy were not available our automobiles would not run.

Energy may be contained in a system in many different forms. The energy stored in the battery of our transistor radio or automobile is in chemical form; it is released through a chemical reaction. The energy of the water in the reservoir above a hydroelectric station is stored as energy of position, and it can be reclaimed by allowing the water to flow to a lower position. The energy stored in an ionized cloud is stored in electrical form, and it is released by the discharge of lightning. The energy stored in a stem-wound clock is elastic energy, and it is released through the unwinding of the spring. The energy in a moving billiard ball is energy of motion, and it may be reclaimed by stopping the ball.

To become effective, energy must be converted from one form to another, or transferred from one place to another. The stored chemical energy in an automobile battery is converted to electrical energy by means of a chemical reaction and then transferred to the starter motor where it is converted into the mechanical energy of motion. The energy of flowing water is converted into electrical energy with generators. This energy is then transmitted through a network of power lines across great distances to such devices as toasters in our homes where it is converted into heat energy, or perhaps to lamps where it may be converted into light energy. The chemical energy of gasoline is first converted into heat energy by combustion in the engine and then into the mechanical energy required to turn the wheels of our car. The Sun, the dominant source of energy in the solar system, converts the nuclear energy of hydrogen to electromagnetic energy which is then transferred to the earth as radiation. Lightning can produce heat energy and acoustical energy. The billiard ball, even if it hits no other ball, will eventually come to rest, and its energy converted into minute amounts of heat energy as it slides and rolls across the surface of the table, strikes the cushions, and perhaps as it heats the webbing of the pocket.

In this discussion we have implied that the energy which is initially contained within a system is all transformed into other forms of energy. We may transform only a part of the available energy into other forms. The real principle behind all energy transformations is that the difference between the initial energy stored in the system and the energy which finally remains in the system is equal to the energy converted into other forms. Thus we account for all of the initial energy: the energy initially available is equal to what remains after a change has taken place plus the energy which has been converted into other forms. (This is the basic idea of the principle of conservation of energy which will be discussed in Chapter C-4.)

The task of the engineer is to use these energy sources and these various means for converting and transporting energy in an efficient and useful manner to operate or produce all the things which he designs. He is concerned with the efficient control of energy conversion. This desire to control energy involves the need for adaptability and ingenuity in the choice of alternatives. The energy source, the converter, and the transfer system must be matched to the required application. The effective conversion of energy requires stable control. Lack of control or instability usually produces undesirable results. (This aspect of energy will be discussed in Chapter C-3.)

Mechanical Energy

How do we describe in a quantitative fashion what we mean by the term "energy"? We can attempt to relate it to something in our daily lives. We recognize that when work is done, energy is expended; when a machine does work, it expends energy; and when a system does work, it expends energy. We think of energy loosely as the capacity to do work. If energy is contained in a system, and if some way to release the energy is available, work can be done.

In the man-made world the ideas of force and motion are related to the concept of mechanical energy or work. A force must be exerted and its application must result in the motion of some object, if work is to be done. As we look back through the preceding pages of the text, this statement becomes more and more credible, for we realize that the most interesting thing about each force we discussed was the motion that it produced. For example, in Chapter B-4, the study of the spring-mass model is given in terms of how the displacement of the mass is changed by a force; resonance is the condition which exists when the displacement is much greater for a certain time-changing force than for other forces. In Chapter B-3, we saw that the force on the spring scale is measurable because it compresses or extends the spring; the force on a phonograph needle causes a piezoelectric crystal to bend; the force caused by interacting magnetic fields causes a compass needle to turn. In fact, from the very definition of the unit of force given in Chapter B-3; that which defines the newton as the force necessary to give a one-kilogram mass an acceleration of one meter per second per second, we sense that the most interesting property of a force is the motion it can produce.

In each of these cases a force is applied over some distance, and mechanical energy is expended. Since force and displacement are the quantities involved in calculating work or mechanical energy, it is instructive to plot one versus the other. As a first example, let us consider a constant force f_1 applied to a mass through a displacement x_1. This situation is plotted in Fig. C-2.1 and might physically represent the situation where a vehicle is being pushed with a constant force f_1 over a distance of x_1 meters. For a quantitative description of energy, we can use the area under the f-x curve. Thus, we define work as the energy expended by a moving force and set it equal quantitatively to the area under f-x curve.

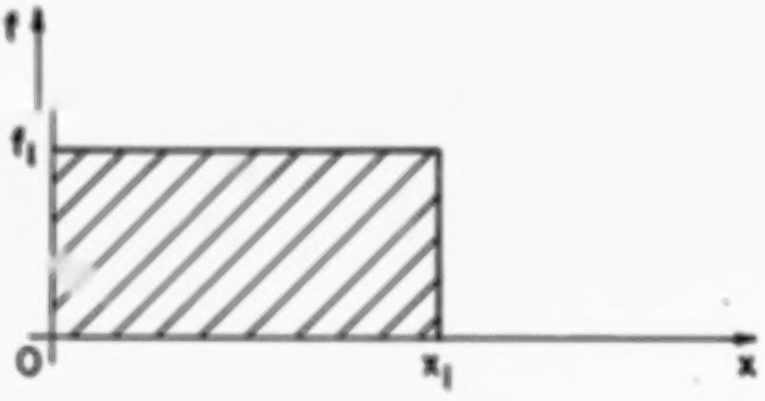


Fig. C-2.1 The plot of a constant force f_1 applied through a displacement x_1. The work done (i.e., the energy expanded) during this motion is defined as the area under the f-x curve.

In Chapter B-3 we saw that the gravitational force on a mass m on Earth at sea level is 9.8m newtons. Thus, if we lift this mass to a height of h meters, we must apply a constant force of 9.8m newtons through a displacement h meters and we therefore do an amount of work:

$$w = 9.8mh \text{ newton-meters}$$

As an example, if $m = 2$ kg (about the mass of a good-sized textbook), then the area under the curve of Fig. C-2.1 could represent the work done in raising this mass 2 meters if we let $f_1 = 9.8(2) = 19.6$ newtons and $x_1 = h = 2$ m; thus, $w = 39.2$ newton-meters. The unit, newton-meter, has been given the shorter name joule in honor of the famous British engineer of that name.

The use of area under the f, x curve to define work is sensible because it illustrates a very important property of energy. As shown in Fig. C-2.2, the work done by a force of 10 newtons through a displacement of 2 meters is exactly equal to the work done by a force of 1 newton through a displacement of 20 meters. Though the plots are different, the areas are identical and indicate that 20 joules of energy have been expended. Thus, it requires the same amount of work to move a mass through 2 meters with a 10 newton force as it does to move the same mass through 20 meters with a 1 newton force.

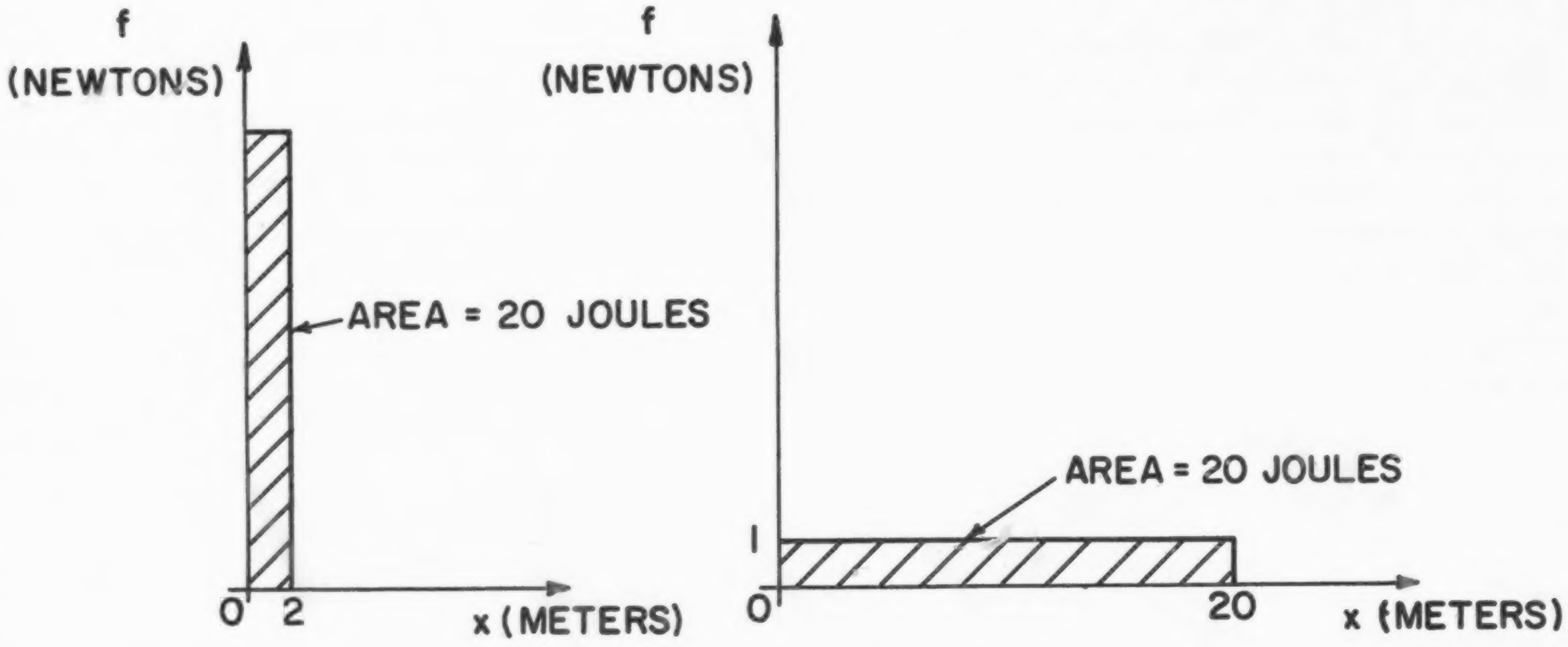


Fig. C-2.2 The same amount of work represented by different force-displacement situations.

In most practical situations the applied force is not constant with displacement, in which case the computation of area under the f, x curve may be accomplished by the approximate methods studied in Chapter B-2. In a general sense then, work is found by integration; that is, according to the equation:

$$w = \int_{x_1}^{x_2} f dx,$$

as illustrated in Fig. C-2.3.

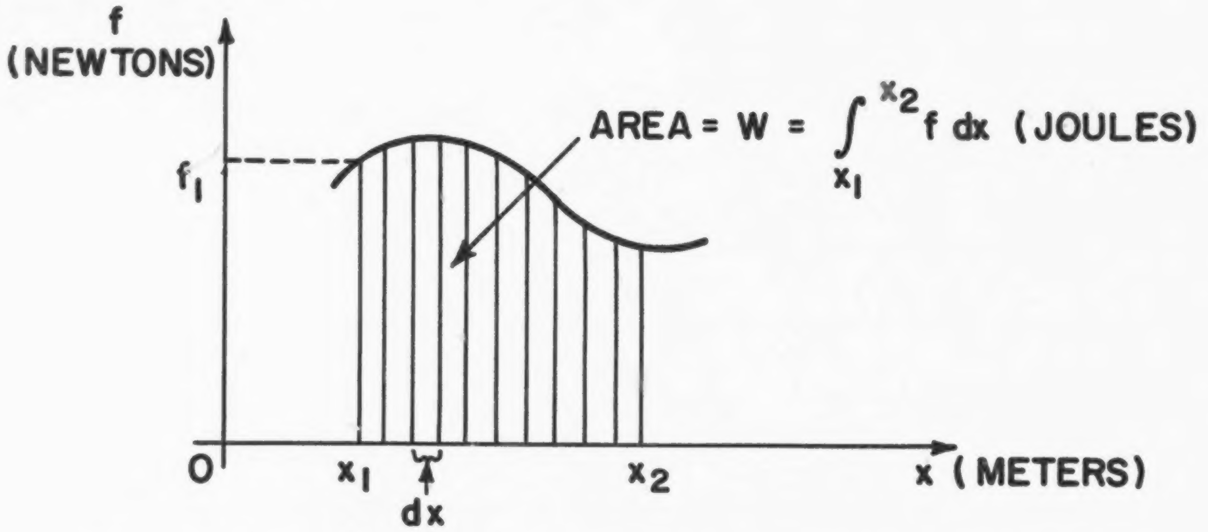


Fig. C-2.3 The general definition of mechanical energy of work.

In addition to the computation of energy or work as a product of the force acting on a mass and the displacement of the mass, the engineer is often concerned with the product of the force acting and the speed imparted to the mass as a result of this force. This product is a quantitative statement of the rate at which work is done or the rate at which energy is expended. In Fig. C-2.4 we again see a plot of force versus displacement. Also displayed is an increment Δx, thru which the mass has been displaced by the force which varies according to the graph.

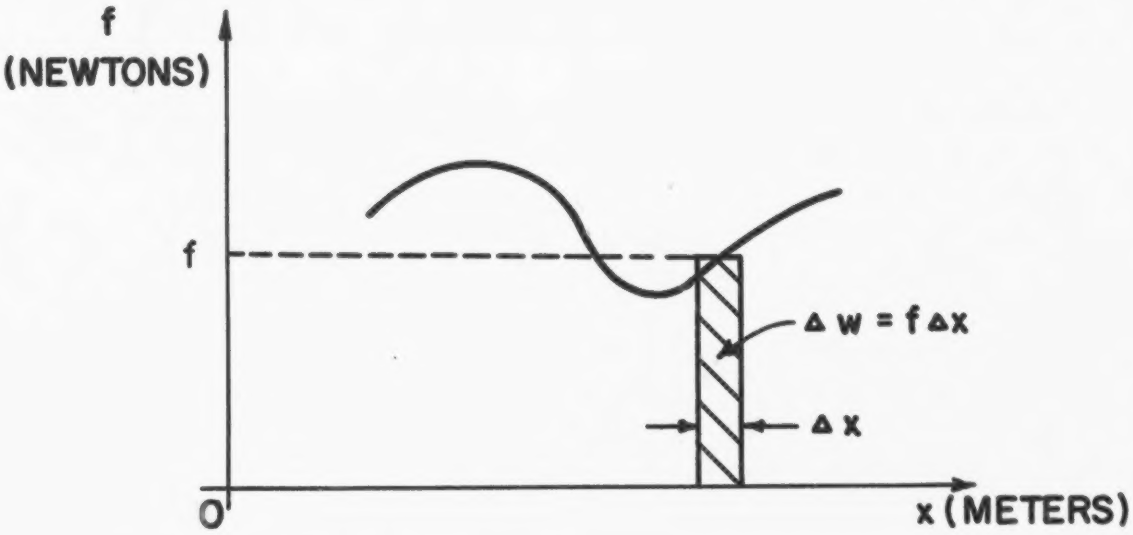


Fig. C-2.4 In moving a mass an incremental distance Δx the incremental energy expended is given by the area of the shaded section.

Let us assume that this displacement occurs in a time interval Δt seconds. The transfer of energy during these Δt seconds is thus given approximately by the area of the narrow rectangle.

$$\Delta w = f \Delta x$$

and the rate at which energy is being delivered to the mass is

$$\frac{\text{energy}}{\text{time}} = \frac{\Delta w}{\Delta t} = \frac{f \Delta x}{\Delta t}$$

However, the ratio of Δx to Δt is the velocity at that point, as we saw in Chapter B-2. Hence,

$$\frac{\Delta w}{\Delta t} = fv$$

We define power as the rate of expenditure of energy, or as the rate of doing work. Therefore,

$$\text{Power} = p = fv$$

The unit of power is of course, the joule per second, and this expression is shortened to watts in honor of the British engineer, James Watt, who designed the first successful steam engine. If we apply a force of 680 newtons to raise ourselves a distance of 3 meters (about one flight of stairs), the energy expended is (3) (680) = 2040 joules. Doing this work in 4 seconds signifies an average power of 2040/4 = 510 watts during that time interval. This is about two-thirds of a horsepower. (one horsepower is equivalent to 746 watts)

Power is thus defined as the time rate of change of energy. It may be helpful to recognize that the relationship between power and energy is the same as that between velocity and displacement, for velocity is the time rate of change of displacement. Given a displacement versus time graph, the slope of the curve at any point gives the velocity; similarly, given a plot of energy versus time, the power at any instant is given by the slope of that curve at that instant.

Moreover, if we start with a graph of velocity as a function of time, we know that we can relate the displacement at a time t_2 to that at t_1 by the area under the v, t curve,

$$x_2 = x_1 + \int_{t_1}^{t_2} v \, dt$$

The analogous relationship for power and energy is important. If we have a curve of power versus time, as shown in Fig. C-2.5, the energy expended at time t_2 may be found in terms of the energy expended at t_1 and the area under the power versus time curve:

$$w_2 = w_1 + \int_{t_1}^{t_2} p \, dt$$

Thus, given the curve of power versus time, we can find the total energy supplied between any two instants, and, conversely, given the energy versus time curve, we can find the power or rate at which energy is being supplied at any instant.

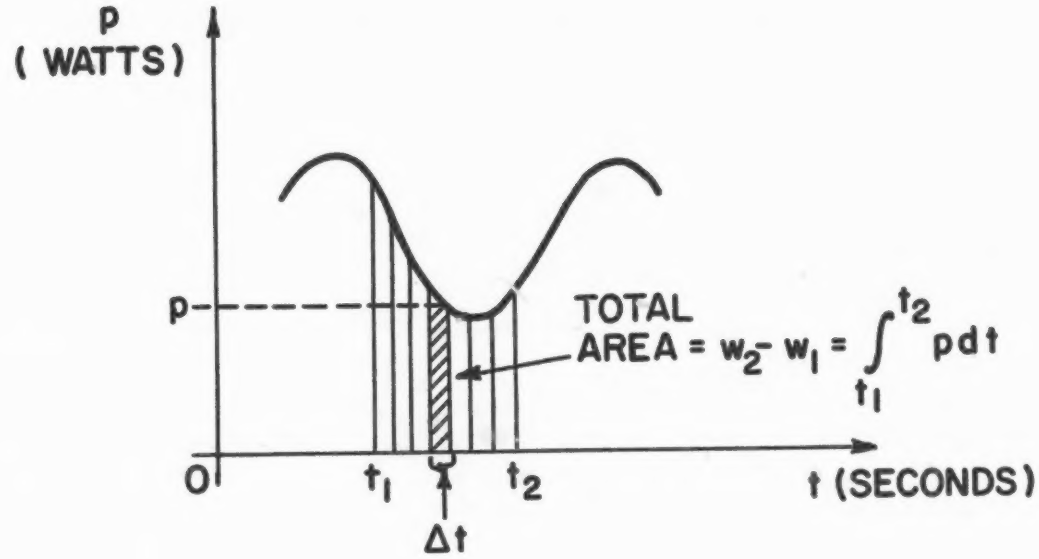


Fig. C-2.5 The area under the power versus time curve gives the energy supplied by the power source during that time interval.

Electrical Energy

Before any quantitative statement of electrical energy and power can be made, there must be a definite meaning to the terms "voltage" and "current". Although we have used these words previously, we have only referred to them in general terms.

Current, for example, was defined as the rate at which "electricity" is moving along a wire, and it is easy to establish a similarity between current and velocity. Voltage, on the other hand, is closely analogous to force, for we often consider it to be some form of electrical force that pushes electricity along the wire. If this analogy is carried one step farther, power, which in mechanical terms, is calculated as the product of force and velocity, should be the product of the voltage and the current. Let us however define voltage and current more carefully and show that power is truly the product of these two electrical quantities.

The study of electrical phenomena begins with the concept of electric charge and the experimentally demonstrable fact that a force always acts between any electrical charges. If two charges are both negative, like electrons, or both positive, like protons, the force tends to separate the charges. If charges are of unlike sign, the force is one of attraction. If one positive charge is fixed in position and a second positive charge is brought to it, a force must be continuously applied through the entire displacement. A force acting through a displacement represents work or energy so that energy is thus expended to bring these two like charges closer together.

This energy, that we expend in moving each unit of charge from one point to another is really the voltage between these two points. Thus, if it takes 12 joules of energy to move one unit of charge from one terminal of a battery to another, we say that the voltage between the battery terminals is 12 volts. A 12-volt automotive battery, for example, will deliver 12 joules of energy to each unit of charge that leaves one of the terminals, to flow through the headlights or the car radio, and back into the other terminal. In starting a car on a cold morning, the battery may deliver 200 units of charge to the starting motor; it therefore supplies 2400 joules of energy to the starting motor.

Although we have defined the voltage as the energy required to move one unit of electrical charge between two points, there is nevertheless an analogous relation between force and voltage. Since work or energy is calculated as force multiplied by distance ($w = fd$), if a unit distance is assumed, then the numerical values of w and f must be equal. It is therefore possible to compute the energy as numerically equal to the force required to displace an object through one unit of distance. The electrician usually interprets the concept of voltage as a force which acts on electrical charges to produce a displacement of these charges. This mechanical analogy is helpful and sufficient for his needs.

Just as it is possible to think of voltage as analogous to force, it is also possible to establish an analogy between electric charge and displacement. The analogy arises from our definition of current as the quantity of charge flowing through a wire in each second. A current of one ampere represents a flow of one unit of charge per second. While starting our cold engine, the current may be as high as 200 amperes. Thus, 200 units of charge leave one terminal of the battery every second and flow through the starting motor and back to the other battery terminal.

The analogy between current and velocity appears from the definitions of these two quantities. Current is charge per unit time, and velocity is displacement per unit time. Again we see that charge in an electrical system is analogous to displacement in the mechanical system.

Now let us turn our attention to power. Since voltage is the energy transferred for each unit of charge and current is the number of unit charges transferred in each second, then the product of current and voltage must be the energy transmitted in each second, or the electrical power which has been converted. In our 12-volt battery supplying 200 amperes to the starting motor, we see that the battery must deliver 12 joules to each of 200 units of charge every second. In other words, it is supplying energy at the rate of 2400 joules per second, or 2400 watts, since we have defined one watt as the equivalent of 1 joule per second.

We can express these results symbolically. We shall use the symbol e for voltage (in volts), and i for current (in amperes). Power is thus given by the product of the current and the voltage:

$$p = e\,i$$

The analogy between mechanical and electrical quantities that we have developed is summarized in Table C-1. The following pairs of mechanical and electrical quantities are analogous: force and voltage, velocity and current, and displacement and charge. Mechanical power is given by the product of force and velocity; electrical power is given by the product of voltage and current. Since energy in a mechanical system is given by the product of force and displacement, our analogy provides the information that energy in an electrical system is found from the product of voltage and charge.

Table C-2.1 The Mechanical - Electrical Analogy.

Mechanical Quantity		Electrical Quantity
f, force	⟷	e, voltage
v, velocity	⟷	i, current
displacement	⟷	charge
$p = f v$, mechanical power	⟷	$p = e\, i$, electrical power

The energy delivered by an electrical power source may be found from the area under the curve of power versus time, as illustrated previously in Fig. C-2.5. The watt-hour-meter found at the point where the wires from the electric utility company enter a house is a meter which measures electrical energy. The speed at which an aluminum disk in the meter turns is proportional to the power being supplied at that instant. The small dials register the number of revolutions which the desk has made in any given time. The power and time factors are thus related to the speed and time factors of the rotating disk, so that the number of revolutions indicated on the dials is related to the total electrical energy which has been transferred to whatever electrical device is attached to the circuit. The dials record the electrical energy in kilowatt-hours. One kilowatt-hour represents 1000 watts of power flowing through the circuit for one hour. Each watt is equivalent to 1 joule per second, so that one kilowatt-hour is equal to 3,600,000 joules. The average cost of this quantity of energy is between two and three cents in many areas.

C-2.3 COUPLERS AND THE CONCEPT OF MATCHING

The energy required to elevate one corner of an automobile high enough to change a tire is about the same as the energy required to walk up a flight of stairs. Thus, the total energy required to lift the car is well within the capability of a man. Yet we know that to grasp the bumper of the car and to attempt to lift the car will be of little consequence and will get us nowhere. We know that a man has the necessary energy to perform this act! What is the problem?

The answer to this question is illustrated by the f, x curves of Fig. C-2.2, where 2 identical quantities of energy are represented by two different force-displacement situations. The shape of Fig. C-2.2a is typical of the f, x relationship required by the car (large force, small displacement) while that of Fig. C-2.2b is typical of the f, x capabilities of the man (small force, large displacement). The engineer describes this situation by saying that although the energies are equal there exists a poor "match" between the small force a man can exert with his muscles and the large force required to lift the car. To lift the car a device is required which will transform the force a man can normally develop into a force large enough to raise the automobile; a device which will match or couple the force of the man to the force required to lift the car. Such devices are called couplers or transformers, and for the automobile, the coupler is obviously a bumper jack. The jack provides a link or couple between the man and the car and transforms the force the man can exert (the input force) into a force which matches the force necessary to elevate the car (the output force). Notice, however, that in this operation of "force transformation" or "energy coupling" the man himself must provide all the energy needed to lift the car. In other words, there is no external source of energy added to the coupled man-jack system.

C-2.4 AMPLIFIERS AND THE CONTROL OF ENERGY

Civilized man has built many devices to transform the forces that his muscles can exert. In addition to the mechanical jack (a form of lever) which was mentioned in the preceding section, the wedge, screw, pulley, hydraulic jack, and gear train are additional examples of man-invented force transformers or couplers. These devices, though certainly useful, have an important limitation. Although the energy which a simple machine can supply at its output is ideally equal to the energy the man can deliver to the input, energy losses resulting from friction cause the actual energy output to be less than the energy input. Any energy required to assist man must come from somewhere, and all these simple devices provide no source of energy other than that supplied by man.

We have already seen that the force exerted by human muscles is quite limited; a coupler or transformer is needed to enable us to lift a car. Similarly, the power, or rate at which our muscles can expend energy, is also limited. For example, suppose we attempt to produce the electrical power required to light a 250-watt light bulb by turning a crank of an electric generator. Two hundred and fifty watts is about 1/3 horsepower, about half the amount of power we expend in running up a flight of stairs. How long could we maintain this output? Most likely, we would collapse after running up a few flights of stairs and the light bulb would go out. The rate at which energy is required for continuous operation is not within the capabilities of human muscles. What can we do to overcome this difficulty?

Man has discovered methods of utilizing energy sources beyong his own muscle system. The theory upon which the devices that make these energy sources available is simple. All these devices require the use of a small amount of energy to control the flow of energy from a large energy source. Such energy sources are available in the flow of streams and rivers, in the movement of winds, in tides, in fossil fuels or directly from the sun. Recently the nucleus of the atom has been tapped as a source of energy.

But all of these sources of energy are of little value unless some form of control is devised to insure useful outcomes. Otherwise danger rather than benefit will develop. One aspect of control of energy is the appropriate transfer of energy from a source to a place where it can be used effectively. The terminus of this energy flow is called the load; in other words, the load is the converter of the energy. We speak of this entire process as "controlling the flow of energy". A very important consideration in energy control is man's desire to exercise control with the least expenditure of control energy.

In some cases, such as with the electric light switch, a very crude control over the flow of energy is achieved. We merely exert enough energy to "flip" a switch. Often, however, a very high degree of precise control is required (as in a high-fidelity amplifier, for example). In any case, a device which controls the flow of energy and produces a large energy flow with a small amount of control energy is called an amplifier.

A block diagram of a model of controlled energy flow is shown in Fig. C-2.6. As indicated in Section C-2.1, this model illustrates that all amplifiers have three properties in common: (1) In addition to the energy of the signal at the input, they require an external source of energy, (2) they exercise some degree of control over the flow of this external energy, and (3) the power delivered to the load is greater than the power of the controlling signal.

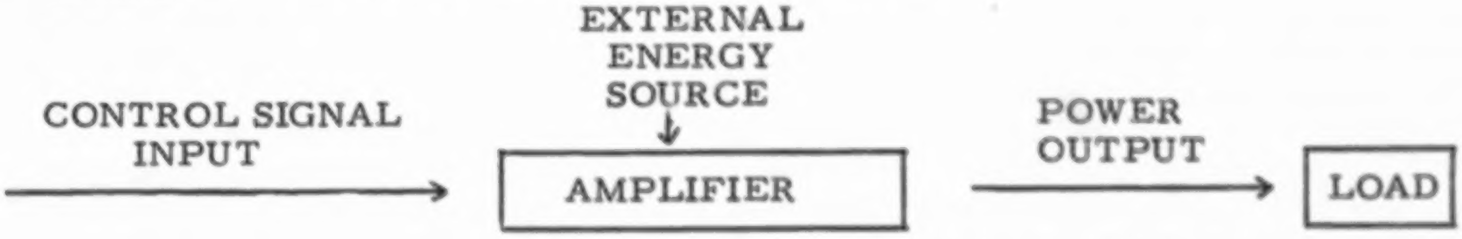


Fig. C-2.6 Block diagram model of an amplifier showing the control signal, the external energy source, and the amplified power output to the load.

C-2.5 AMPLIFIER CONTROL BY ON-OFF LOGIC

The control signal in Fig. C-2.6 has a variety of forms. It may result from the movement of a lever, the pressure on a button, the opening of a valve, the turning of a knob, etc. Of these, one of the simplest ways in which man controls the flow of energy is by an "on-off mechanism"; that is, an ordinary switch. This form of energy control was applied usefully in Section A on Logic and Computers, where relays were used to turn other relays on and off. In some cases the operation of several relays and lights were controlled by a single relay. Thus, the energy transferred to the load, consisting of the several relays and lights, was greater than the energy needed to operate the single control relay. Such a logic circuit, then, has an external source of energy in the power supply operating the other relays and lamps. The flow of this energy may be controlled by a single relay or switch, and the power delivered to the load is greater than that required to operate the single relay or switch; it is therefore an amplifier, although in rudimentary form.

As another example of the rudimentary control of energy, consider the man-made system illustrated in Fig. C-2.7. This system contains an electric power supply as the source of energy, a water pump as a load, and an electric switch as a means of control. (Such systems may be used to pump water from the basements

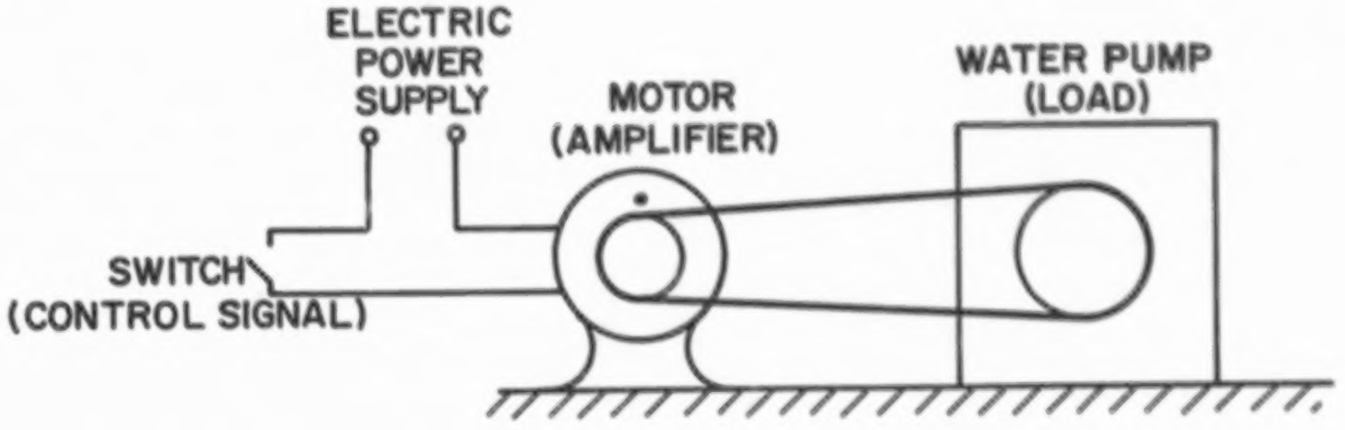
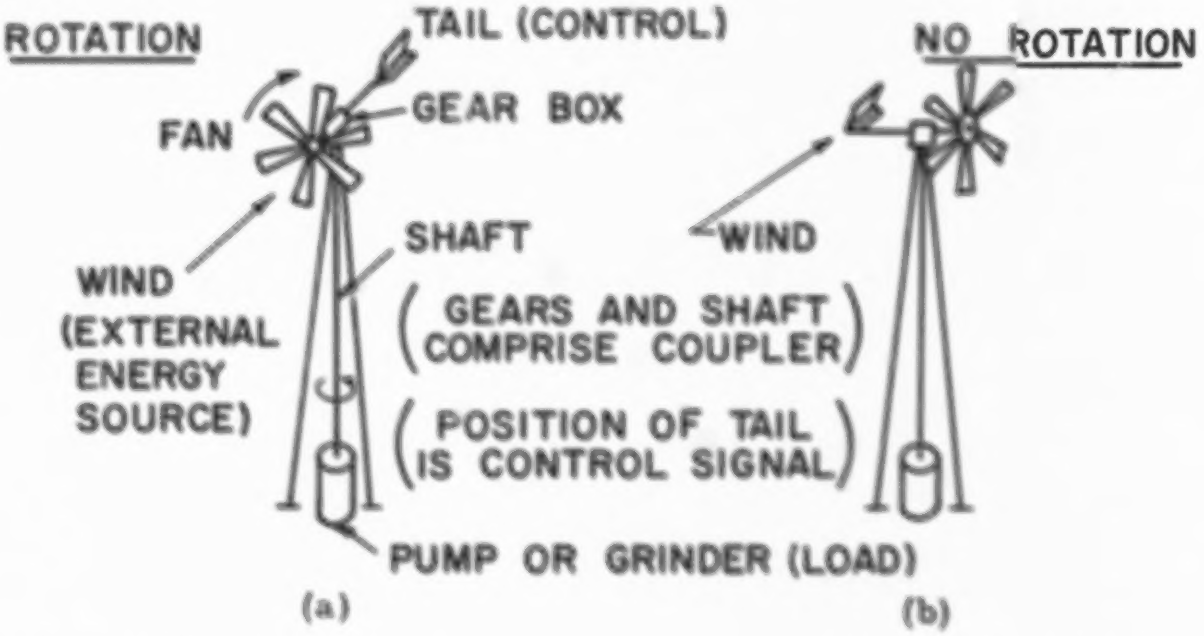


Fig. C-2.7 A crude form of control is illustrated by an electric motor, a load, and a switch which is either on or off.

of homes built in low-lying areas.) When the switch is in the off position no energy is transferred to the load by the motor; when it is on, the energy available from the electric power supply is applied to the pump. Although this amplifier utilizes a very crude from of control, it is certainly very useful; we can control the flow of large amounts of energy from the electric power supply to the load with the expenditure of little energy on our part. However, the only means by which we can exert some degree of precision in our control is by the adjustment of the period of time during which the motor is in operation. Thus, we cannot control the power being transferred to the load but we can control the amount of energy delivered to the load in any given time interval. The power is a fixed quantity determined by the external source and the characteristics of the motor and pump. Our control is limited to transferring this fixed amount of power or no power at all.

Simple on-off mechanisms can be viewed as diverting the flow of energy. We can construct many simple control devices which are useful for diverting other sources of energy. For example, the windmill illustrated in Fig. C-2.8 is a simple on-off type of amplifier used to "harness" the energy of the wind. Control of the energy flow to the load is exercised by changing the direction of the tail control with respect to the direction of the wind.



(a) Tail perpendicular to plane of windmill fan.

(b) Tail parallel to plane of windmill fan.

Fig. C-2.8 The windmill portrayed as a simple on-off type of amplifier.

In Fig. C-2.8a, the tail of the windmill is oriented perpendicular to the plane of the windmill fan. For this configuration, the force exerted by the wind pushes the tail and aligns it with the wind's direction. This action causes the fan to turn until the blades are perpendicular to the wind and thus rotate. Through a coupling system of gears, the shaft rotates as the fan blades turn and water can thus be pumped or grain can be ground. When the windmill tail is oriented parallel to the plane of the fan, as shown in Fig. C-2.8b, the wind flows parallel to the plane of the fan and its rotation stops. In this simple amplifier the external source of energy is the wind, the control signal is the setting of the direction of the tail, and the load is the water pump.

The production of water power can be controlled in a similar fashion. A trough or ditch can be used to divert the flow of water to a water wheel as shown in Fig. C-2.9. Here the control signal is the positioning of the trough to carry water to the blades of the wheel, the external source of energy is supplied by the water, and the load is represented by the generator used to produce electric power.

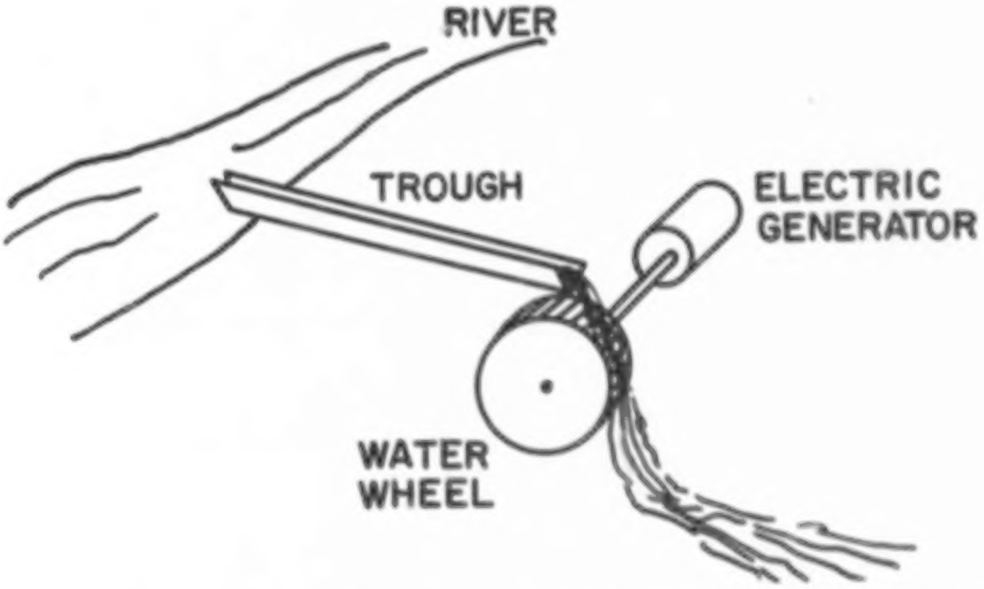


Fig. C-2.9 A water wheel portrayed as a simple on-off type of amplifier.

C-2.6 OTHER AMPLIFIERS

The Automobile

A more versatile example of the control of the flow of energy is provided by the automobile, which may be considered as an amplifier. The source of energy in the system is the gasoline-air mixture burned in the engine. The load is the vehicle itself, and control of energy from the engine is exercised by the driver. By means of the accelerator pedal, we can control the energy output of the engine and the speed of the vehicle smoothly and continuously over a wide range, as indicated by the motion model of Chapter B-2. Moreover, we can adjust the energy output to meet the demands of variations in the load, such as one encountered in driving over hilly terrain. It is possible to exercise a much greater degree of control over the flow of energy from the automobile engine than is possible with the use of an on-off switch for the control of an electric motor.

Once an automobile is in motion it must be capable of coming to a stop. This can be accomplished with a force transformer or coupler: the braking system. A moving automobile carries a large amount of energy of motion or kinetic energy (almost a million joules for a large car at 60 mi/h). To bring the vehicle to a stop, this energy of motion must be transferred or converted in some way. The energy may be expended by hitting another automobile or smashing into a tree. A more humane technique involves the use of the brake pedal to couple this kinetic energy source to the frictional load that develops between the brake linings and brake drums. In this manner, the energy of the moving car is transformed to heat energy.

Since the control of amplification usually involves feedback it is interesting to examine the automobile system for this function. The block diagram of Fig. C-2.10, which models the control system involved in keeping an automobile in a turnpike lane, clearly illustrates that the driver is an important link in the safe operation of the automobile.

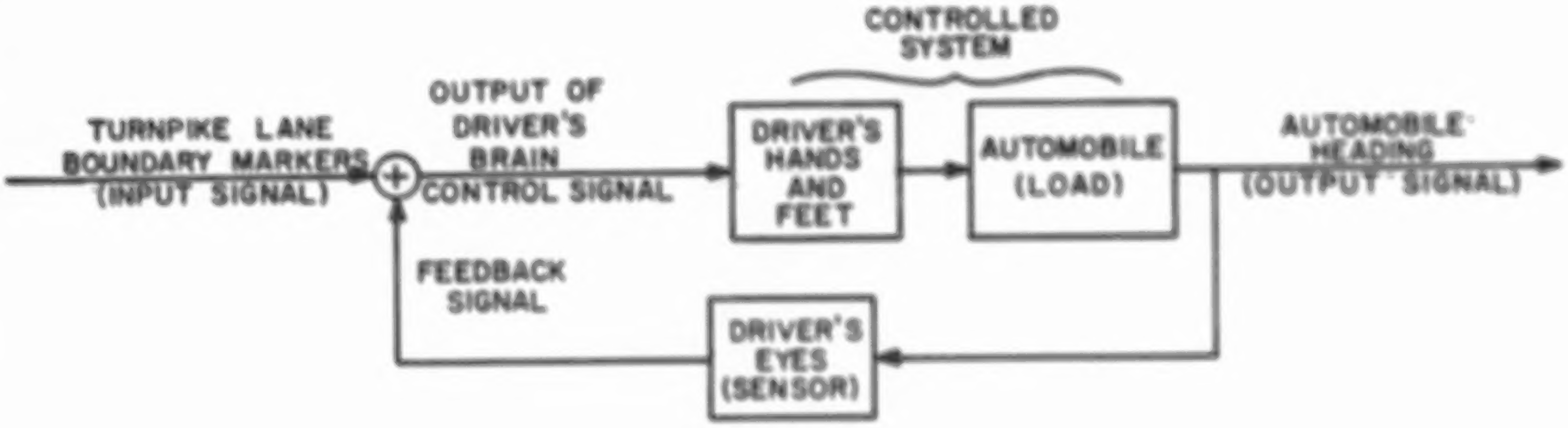

CONTROLLED SYSTEM
TURNPIKE LANE BOUNDARY MARKERS (INPUT SIGNAL)
+
OUTPUT OF DRIVER'S BRAIN CONTROL SIGNAL
DRIVER'S HANDS AND FEET
AUTOMOBILE (LOAD)
AUTOMOBILE HEADING (OUTPUT SIGNAL)
FEEDBACK SIGNAL
DRIVER'S EYES (SENSOR)

Fig. C-2.10 A block diagram modeling the control system involved in keeping an automobile in a turnpike lane.

The Human Muscle

Not all amplifiers are man-made. Consider, for example, the muscles in the human body and how they operate. A control signal is sent from the brain in the form of small electric impulses. These impulses control the release of energy which is stored in chemical form in the muscle cells. The muscles move the arms with sufficient control to assure their proper positioning with respect to the objective which is sought, with a fair degree of accuracy. Feedback is definitely involved here and is channeled through the sensors in the body that respond to touch, sight, smell, and taste. When a feedback mechanism (that is, a sensory nerve) is damaged or destroyed, control is affected.

In positioning an object manually, the final outcome (or output) is the position of the object. When placement of an object is accomplished manually the same degree of accuracy is achieved despite a wide range in size and weight. This is an example of an important property of feedback discussed in Chapter C-1: feedback causes the amplifier performance (muscle control here) to be relatively insensitive to fluctuations in loading.

The Windlass

Another arrangement for controlling the flow of energy is the versatile windlass shown in Fig. C-2.11. The source of energy in this system is either an

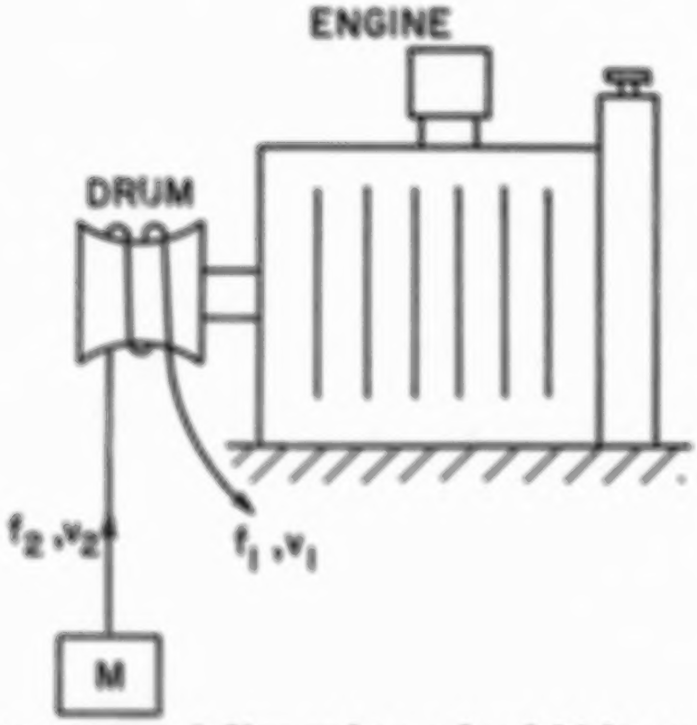


Fig. C-2.11 The energy delivered to a load M is controlled smoothly by a windlass.

electric motor or a gasoline engine. Control of the energy flow is provided by a manilla or nylon rope which connects the engine to the load. One end of the rope is connected to the load, the middle part is wrapped around the revolving pulley, or drum on the engine shaft, the operator holds the free end. If the rope is slack, the drum rotates freely inside the loose turns, and no force or power is delivered to the load. When a pull is exerted on the free end of the rope, the turns tighten on the drum, and friction between the rope and drum produces a tension in the line which exerts the force on the load. This force can be much larger than the force which is exerted on the free end of the line. Windlasses are widely used on ships, sail-boats, and motor yachts for lifting heavy anchors, trimming sails, etc. They are also used for digging wells and driving pilings.

To show that the windlass acts as an amplifier, we must determine the relationship between the input force and velocity, f_1 and v_1, and the output force and velocity, f_2 and v_2, both shown in Fig. C-2.11. To begin with, we notice that the two velocities must be equal since there is only a single rope involved; i.e.:

$$v_2 = v_1$$

If v_1 is greater than v_2, the rope must stretch, the tension in the line will increase, the force applied to the mass M thus increases, and v_2 increases until it is equal to v_1. On the other hand, if v_2 is greater than v_1, the line becomes slack, the friction force between the line and the drum decreases, the tension in the line decreases, the force applied to the mass decreases, and v_2 decreases until it is equal to v_1.

To qualify as an amplifier, f_2 must therefore be greater than f_1 since the velocity of the rope is the same at both ends. Fortunately, this relationship is satisfied because power from the engine is transferred to the load via friction between the rope and the drum. This output force f_2 is actually derived from the engine through the rotation of the drum. If the rope were connected directly to the drum, the power developed by the engine would be applied completely to the load. But this would represent a condition in which no variation in power transfer is possible, either full power or no power would be delivered.

When the rope is wrapped around the surface of the drum, the force which can act on the load is transferred from the engine through the friction between the drum and the wound section of the rope. The greater this friction the greater will be the force transmitted. As the frictional force is varied, the force f_2 will vary.

A change in the frictional force between the drum and the rope wound on the drum is developed by the force f_1 which is applied to the free end of the rope. Increasing f_1 pulls the rope more tightly against the drum. This increases the friction between the rope and the drum and thereby permits a larger fraction of the force developed by the engine to be transferred to the load. Usually the force f_2 will be directly proportional to the applied force f_1. Thus, we can write

$$f_2 = Af_1 \quad \text{or} \quad A = f_2/f_1$$

where A is the constant of proportionality. In a practical windlass, A may be as big as 50 or 100. The constant A depends on the number of degrees of arc through which the line is in contact with the drum and on the friction between the line and the drum. Thus, in principle, very small forces applied to the free end of the rope may become very large forces at the load. There is, of course, a limit to the force available at the load. This limit is imposed by the force and power developed by the engine. The windlass, acts as an amplifier, because the large quantity of power available from an engine can be controlled precisely with a small amount of power applied at the free end of the rope. With a large number of turns of rope around the drum, the frictional force may be very great, despite an extremely small force at the free end. If we place too many turns on the drum, the line grips the drum because of its own weight and stiffness without any input force at all being applied. The frictional forces may cause the line to bind, or adjacent turns may become entangled. Thus, with too many turns, the operator loses control of the machine. (A means of obtaining greater values of power amplification by using two or more amplifiers is described in Section C-2.7 below.)

As with all amplifiers the function of the windlass is to control the energy flow from the engine to the load. If we pull on the line with a constant force f_1, then the rate at which we expand energy (called the input power) is

$$p_1 = v_1 f_1$$

and the rate at which energy is delivered to the load (called the output power) is

$$p_2 = v_2 f_2$$

By division, the last two equations yield:

$$\frac{p_2}{p_1} = \frac{v_2 f_2}{v_1 f_1}$$

but we have seen that $v_2 = v_1$, therefore

$$\frac{p_2}{p_1} = \frac{f_2}{f_1} = A \text{ or } p_2 = Ap_1.$$

Thus, the output power is A times as large as the input power and, as mentioned above, A may be as great as 50 or 100. Moreover, the output power is at every instant directly proportional to the input power applied by the operator. Thus, the windlass has the effect of amplifying the power we apply by a factor of 50 or 100. With the help of this amplifier we can therefore exert a high degree of control over the power delivered to the load. As shown by equation $f_2 = Af_1$ the force applied to the load is at every instant directly proportional to the input force we apply. Thus, the windlass is also a force transformer. This point of view is useful when we are interested in the force applied to the load rather than the power delivered tothe load. As the input force is varied in time, the output force varies in exactly the same manner but is A times as great. The constant A often is called the power amplification factor of the device.

The windlass is just one of the many devices capable of producing power amplification. Devices which permit a very small energy input, or signal, to control the flow of a much larger output energy to some load. Another example of an amplifier is the electronic amplifier that is the basis of a record-player system. This amplifier permits the very small power available from the pick-up in the tone arm to control very much greater power flow to the loudspeaker. Other examples are discussed in the paragraphs that follow. At this point we can review briefly the three features common to all amplifiers:

1. A source of energy that is separate from the input, or controlling, signal.
2. Some type of control mechanism by means of which the input signal controls the flow of energy from the energy source to the load.
3. A ratio of load power to controlling power that is greater than unity.

By the first and third criteria for an amplifier as stated above, it is evident that a device such as the bumper jack is not an amplifier. We say that the bumper jack acts as a force transformer or coupler, for the output force may be much greater than the input force. However, the output power must be equal to or less than the input power, since the bumper jack does not contain any separate source of power. The hydraulic press, which is used to shape metals, is a similar device. It may be arranged so that the output force is greater than the input force, or the output displacement is greater than the input displacement, or the output velocity is greater than the input velocity, but for any of these arrangements, the output and input powers are equal if frictional forces in the oil and pistons are neglected.

On the other hand, the electric motor with its on-off switch does satisfy all three criteria. It is a very crude amplifier, however, for the control mechanism is a simple switch.

The Hydraulic Amplifier

Another important amplifying device is pictured in Fig. C-2.12. Various forms of this hydraulic amplifier are used for power steering and power brakes in automobiles. The source of power is a reservoir of oil, maintained under high pressure by a pump. The control mechanism in the hydraulic amplifier is a valve, as shown in Fig. C-2.12. When the valve piston is in the position shown, it blocks the flow of high-pressure oil, in either direction, and the entire system is stationary. If the valve control moves the valve piston to the right of its neutral position, high-pressure oil can flow from the valve through the connecting tube to the left side of the power piston. The high-pressure oil thus forces the power piston to move to the right. Oil on the right side of the power piston flows back to the valve through the other connecting tube, through the drain, to atmospheric pressure, and then it is forced back into the high-pressure reservoir by the pump. The power piston continues to move to the right until it closes the right exit port and oil can no longer circulate. If the control valve is displaced to the left of the neutral position, movement in the opposite direction takes place; high-pressure oil is admitted to the right side of the power piston, and as the power piston moves to the left, oil on the left side of the piston flows back to the valve and out through the drain.

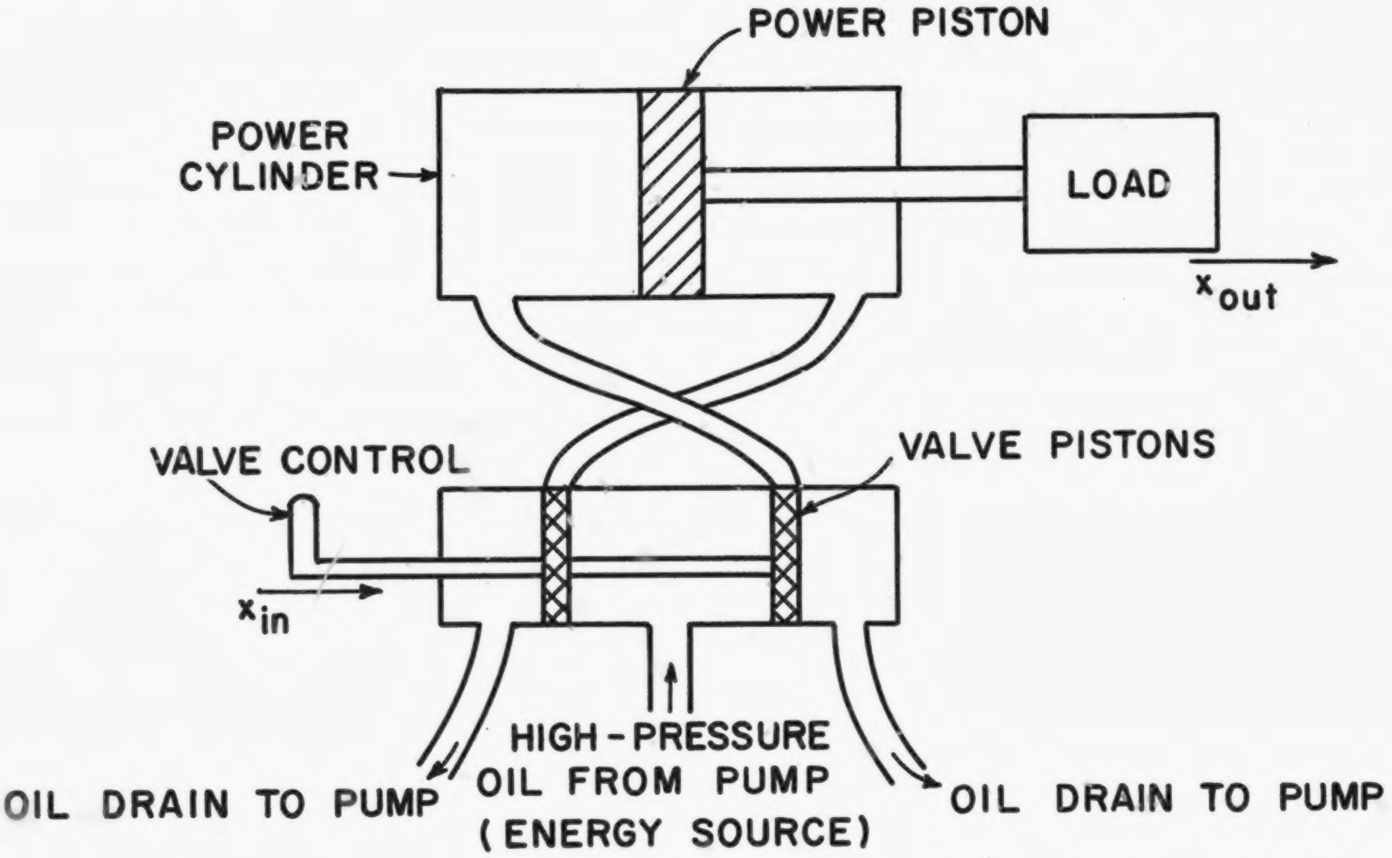


Fig. C-2.12 The basic hydraulic amplifier.

The input power required to actuate the control valve in the hydraulic amplifier is just the power required to overcome the friction associated with the motion of the valve. This friction can be kept very small, and thus a very small amount of input power can control the very large power made available through the power piston. However, the nature of the control that can be exerted on the output is rather crude; it is the same as the control available with the electric motor controlled by an on-off switch, and it is quite inferior to the control provided by the windlass.

The performance of the hydraulic amplifier can be improved and made comparable to the performance of the windlass by adding a feedback linkage, as shown in Fig. C-2.13. When the input lever is displaced to the right, the heavy load does not move immediately. The vertical bar pivots around point A, and the valve moves to the right. Thus, high pressure oil enters the left side of the power piston, the piston moves to the right, and oil is exhausted from the right side of the piston through the valve and out of the drain, as before. But if the position of the control handle is held fixed as the power piston moves; the vertical bar will pivot around point B as the power piston moves to the right, and this action pulls the control valve to the left to close the ports. Since this sequence of events occurs almost instantly, we now have a system in which the load moves back and forth in synchronism with the motion applied to the input lever. A small amount of signal power applied to the input lever controls a very large amount of power applied to the load. The source of the energy transferred to the load is the engine which operates the oil pump. The greater the energy source at the pump, the greater will be the energy delivered to the load.

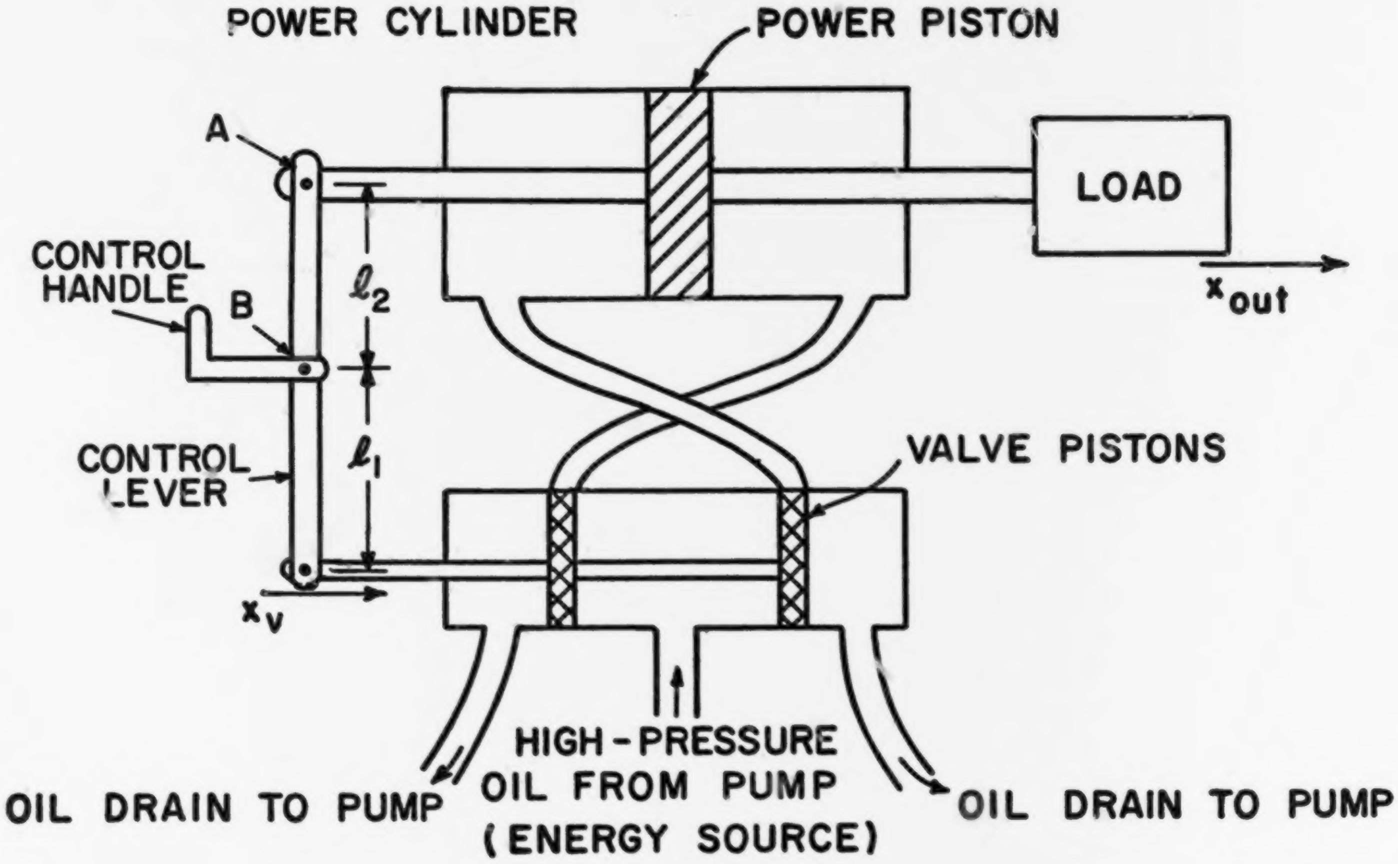


Fig. C-2.13 An improved form of the hydraulic amplifier with feedback linkage.

To be an effective amplifier, not only must a large energy output be available for a small energy input, but the energy output must vary in direct proportion to the input energy by a constant factor. Thus if the amplification factor of the amplifier is twenty five, then whatever the energy we apply at the input, the output will always be twenty five times as large. In the hydraulic amplifier above, the force acting on the power cylinder is fixed by the pressure of the oil pump and the area of the power piston. The energy delivered or work done by the power piston can therefore vary only as a result of this fixed force acting through a variable distance. If the output energy must be an enlarged replica of the input energy the distance moved by the load x_{out} must be related to the distance moved by the control x_{in} by an unvarying or constant ratio. Under such conditions, the large available force operates through a variation in distance which imitates precisely the input variation.

The relation between the motion of the input lever and the motion of the load can be deduced by noting that under static conditions the valve must always be closed. Thus, for static conditions we must always have $x_v = 0$ in Fig. C-2.13. Then, it may be proven that:

$$\frac{x_{out}}{x_{in}} = \frac{\ell_1 + \ell_2}{\ell_1} = \text{Constant} = C \text{ (Since } \ell_1 \text{ and } \ell_2 \text{ remain fixed)}$$

under static conditions. Multiplying the above equation by x_{in}, we have

$$x_{out} = C\,x_{in}$$

Under dynamic conditions (i. e., when the system is in motion) the valve must be displaced somewhat from the neutral position, and equation $x_{out} = C\,x_{in}$ does not hold exactly. However, if the displacement of the valve is small compared to the displacement of the input lever, the above equation is a useful approximation even when the system is in motion.

This is the type of hydraulic amplifier that has been used, with various modifications, in some power steering systems for automobiles. You may find it interesting to analyze the action of this amplifier when the control valve is connected to point B on the vertical bar and the input lever is connected to the bottom of the bar.

In the case of the windlass there is a simple relationship between the input and output forces, as given by equation $f_2 = Af_1$. There is no such simple force relationship for the hydraulic amplifier, however. The input force is the force required to overcome friction in moving the control valve. The output force is the difference in oil pressure on the two sides of the power piston multiplied by the area of the piston. No simple relation between these forces normally exists, so that the calculation of the power amplification provided by the hydraulic amplifier is usually difficult; however, experience does indicate that the amplification can be made quite large.

The Electronic Amplifier

The electronic amplifier satisfies a great need in the man-made world because, in constrast to the mechanical amplifiers discussed above, it is capable of amplifying input controlling signals which fluctuate with great rapidity. If the controlling signal is an electrical voltage or an electrical current, amplification can proceed directly. For other signals, a transducer is used prior to amplification in order to provide the electric signal which is necessary to control the electronic amplifier.

By analogy to Fig. C-2.6, the model of an electronic amplifier is shown in Fig. C-2.14. Notice that the external energy source is a battery (or an electronic power supply), the control signal is a voltage or a current, and the load is a resistor. (Recall from Figure B-3.4 that a resistor has the property that if a battery is connected to it, an electric current will flow in inverse proportion to the resistance in the circuit.) The amplifier serves to control the flow of electric energy from the battery or external energy source to the load resistor.

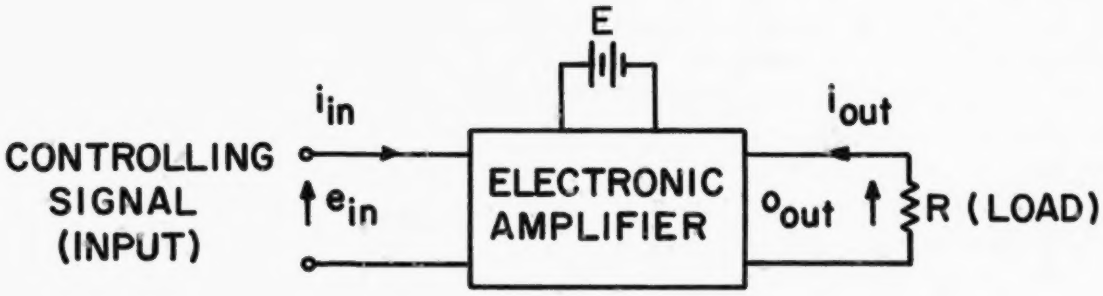


Fig. C-2.14 Model of an electronic amplifier.

Common electronic amplifiers are the vacuum-tube amplifier and the transistor amplifier, and there are many others. The theory of all these electronic devices has been worked out in considerable detail, and excellent models are available for them. Unfortunately, however, it is not possible to present a satisfactory explanation of these specific devices without devoting a substantial amount of time to the study of electricity. Since the study of amplification, and not electricity, is our present purpose, we can gain sufficient insight into electronic amplification by resorting to experimental measurements made at the input and output terminals of the amplifier. This is a technique frequently used by engineers for complicated problems: rather than to try to understand the device itself, we measure its significant properties or behavior at the input and output of the device, and we use these measurements to devise a model illustrating how the device works. These measurements are called the "characteristics" of the device.

For example, let us suppose that a transistor amplifier is inside the electronic amplifier block of Fig. C-2.14. Measurements on a typical device of this sort would result in the graph of Fig. C-2.15 which shows how the output load current i_{out} varies with changes in the input controlling current i_{in}. (The nature of the transistor makes it impossible for either of these currents to be negative; therefore, only positive values are shown on the graph.) This curve is called the input-output current characteristic of the transistor amplifier.

The curve in Fig. C-2.15 is clearly not linear; the curvature depends on the type of transistor and on the range of currents used. However, for most devices the curvature is not extreme for small range of variation and we can

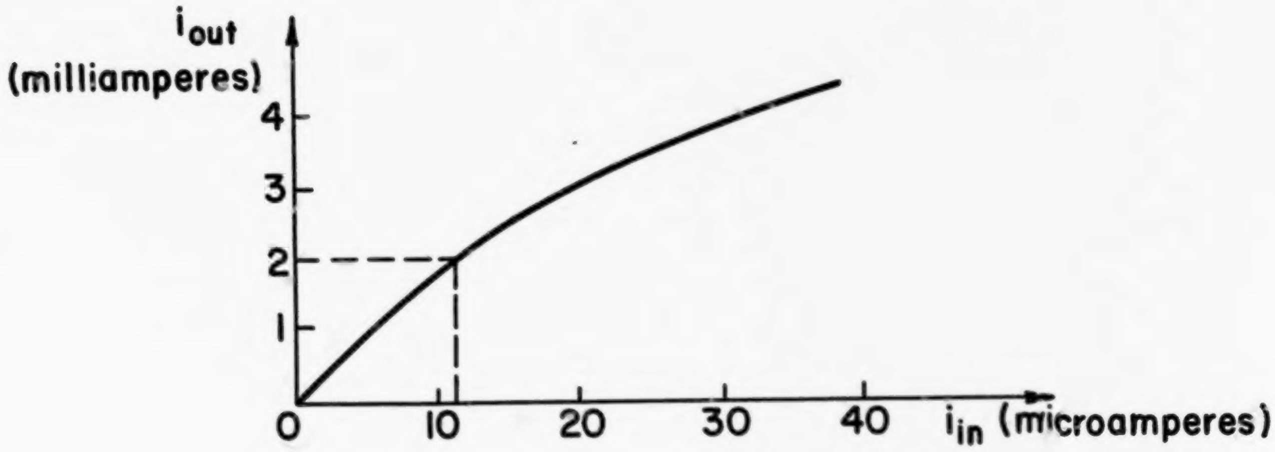


Fig. C-2.15 A typical relationship between output current and input current in a transistor amplifier (the input-output current characteristic).

therefore assume an approximate model with a straight line characteristic. We can then write

$$i_{out} = A\, i_{in}$$

where the current transformation factor A (usually spoken of as current gain) is typically between 50 and 100, although it may be as high as 300 or 400 in some transistors. Thus, the graph of Fig. C-2.15 and the approximation of the above equation both show that the transistor amplifier produces current transformation. But electrical power is a product of current and voltage. Is there power amplification?

To answer this question we must, of course, consider the other component of electric power: voltage. Let us turn our attention first to the relation between input voltage and input current. In order to get some feeling for the magnitude of the input voltage e_{in} we obtain a second experimental curve, the input voltage-current characteristic, shown in Fig. C-2.16. Here, the input current i_{in} is

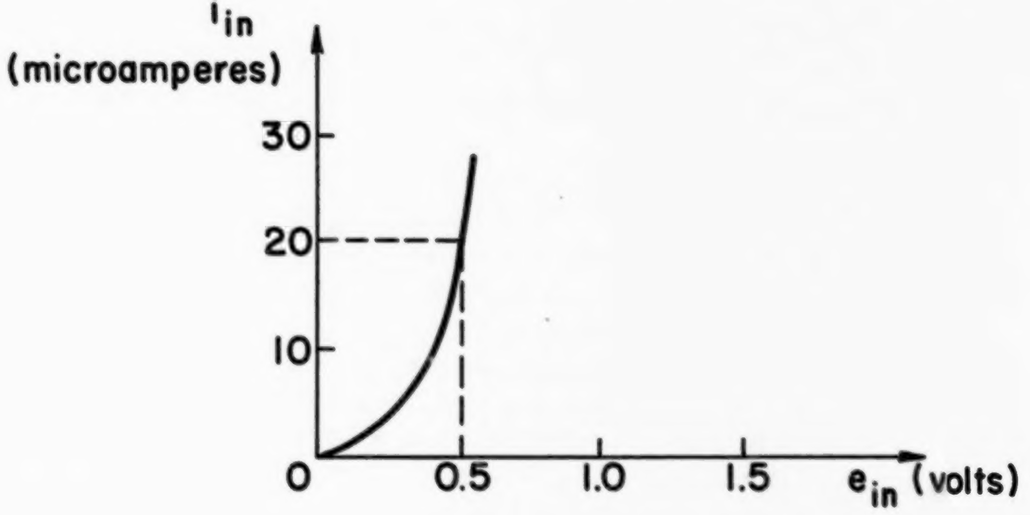


Fig. C-2.16 A typical relationship between the input current and input voltage in a transistor amplifier.

shown related to e_{in}. An inspection of the scale on the e_{in} axis shows that normally, the input voltage, when taken with the small value of the input current it produces in normal operation, means that the input power is very small.

$$(p_{in} = e_{in}\, i_{in} \text{ from } p = e\, i)$$

On the other hand, the voltage available at the load is directly proportional to the current in the load (see Fig. B-3.4); therefore, since i_{out} is much larger than i_{in}, it is certainly possible to make p_{out} ($= e_{out}\, i_{out}$) larger than p_{in}, through the selection of appropriate values of the load resistor R. As a numerical example, we can examine a typical output current-voltage characteristic shown in Fig. C-2.17. Here, the load resistance selected is 5000 ohms. If i_{in} = 20 microamperes, then from Fig. C-2.16, e_{in} = 0.5 volt and, therefore

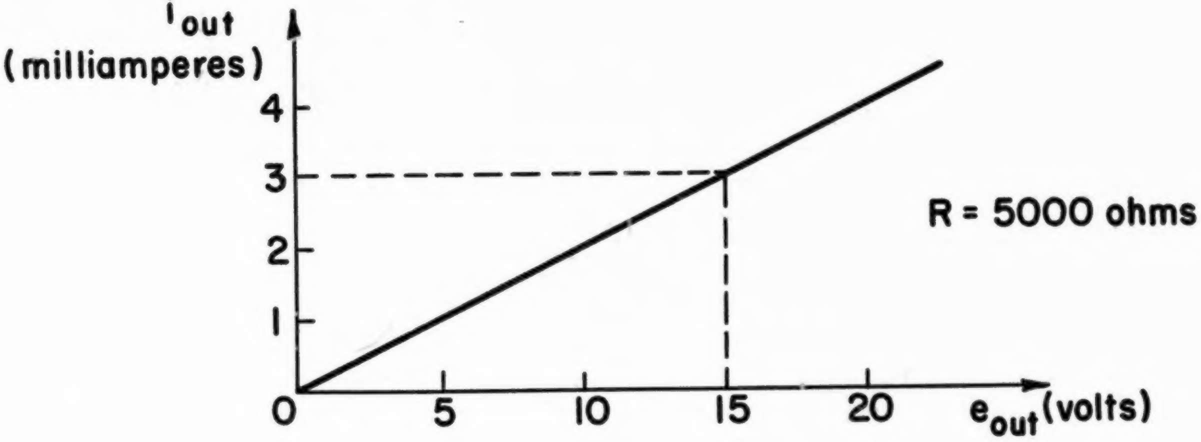


Fig. C-2.17 A typical relationship between the output current and output voltage at the load of an electronic amplificer. (R = 5000 ohms)

$p_{in} = (0.5)(20 \times 10^{-6}) = 10$ microwatts. From Fig. C-2.15 for i_{in} = 20 microamperes. Entering Fig. C-2.17 with this value of i_{out} we find that e_{out} = 15 volts and, therefore, $p_{out} = (15)(3 \times 10^{-3}) = 45$ milliwatts. Hence, p_{out} is 4,500 times as great as p_{in}, showing that this device does indeed produce power amplification!

The assumption that the i_{out} vs i_{in} curve is a straight line ($i_{out} = A\, i_{in}$), implies that the load-current variations will be a replica of the input-current variations. In practice the i_{out}-i_{in} curve is slightly nonlinear, so that some distortion is always found in the output-current waveform. Much effort is devoted to designing electronic amplifiers to reduce this distortion to negligible values. A so-called hi-fi amplifier, or high-fidelity amplifier, is merely an amplifier in which distortion is reduced to a minimum so that the output is a faithful copy of the input waveform. Thus, when a hi-fi amplifier is used in a sound system, the output from the loudspeaker is a faithful copy of the sound which has been recorded. One of the most important methods for reducing distortion and obtaining fidelity in electronic amplifiers is the use of the feedback principle discussed in Chapter C-1. This application of feedback is used extensively in radio and telephone systems, as well as in record players and tape recorders.

Distortion does not arise solely through a non-linear input-output characteristic, however. Another source of distortion results from the finite speed with which the output of the amplifier can respond to a change in the input waveform. Although it has already been stated that one of the great advantages of the electronic amplifier is its great speed of response when compared with a mechanical or a hydraulic amplifier, phenomena that fluctuate more rapidly than the electronic amplifier is capable of reproducing, will not be reproduced faithfully.

The lack of instantaneous response in the transistor amplifier is illustrated by the two waveforms shown in Fig. C-2.18 a and b. The first figure shows the input current jumping abruptly back and forth between two levels to provide a square waveform of input current. In response to this input current, the load current also jumps back and forth between two levels, but, as illustrated in Fig. C-2.18b, the output change does not take place as abruptly as the input change. Because of the delay in the response of the amplifier, the output waveform is not an exact, and magnified copy of the input waveform. Distortion is present. When a good, high-speed, low-power transistor is used in the amplifier, the time required for the load current to make the transition from one level to another may be less than 0.1 microsecond. For many applications, this may be considered to be instantaneous. When a transistor which is designed to control large power is used, the time required for the transistion may be 100 times longer that this. In comparison, the hydraulic amplifier has a response time which may be 1000 times as long as that of the high-power transistor.

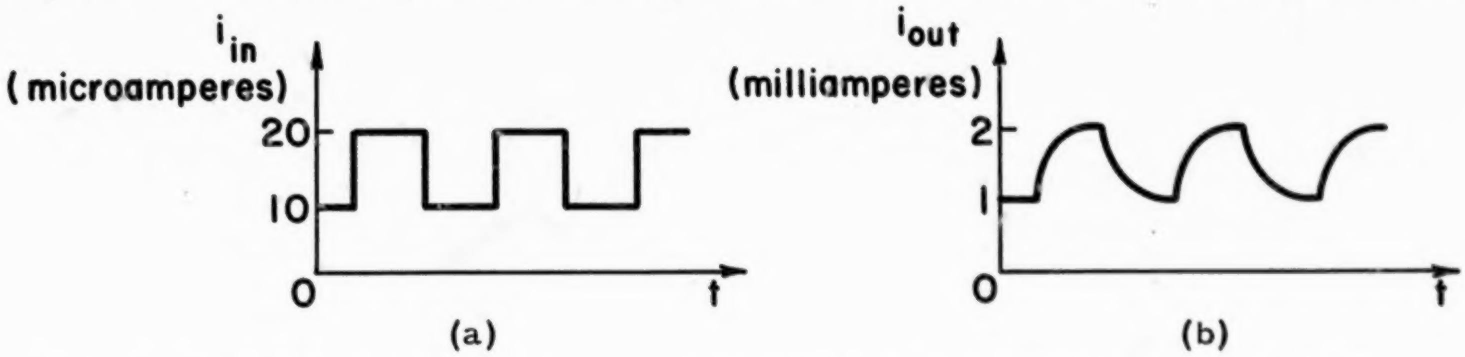


Fig. C-2.18 (a) An input square wave applied to a transistor amplifier may produce
(b) A distorted output waveform because of the finite speed of response.

C-2.7 MORE ABOUT MATCHING

Frequently we refer to a young couple as being well-matched, poorly-matched, or even perfectly matched. It is generally accepted that the better the match, the more compatible the couple will be, and the better will be some result. A similar situation exists when a load is coupled to the output of a power amplifier or when a control signal is coupled to the input of an amplifier. Transformers of various types are used to couple devices so that they are better matched to produce the effects for which they were designed. The bumper jack, discussed in Section C-2.3, is a good example of such a device. Coupling devices are not power amplifiers since they have only an input and an output with no provision for an external source of energy.

Mechanical Transformers

If we return for a moment to our discussion of the problem of lifting the corner of an automobile we recall that the energy available to accomplish this task could be developed by a man's muscles. But despite the availability of the necessary energy an automobile jack was necessary to match or couple the force exerted by the man to the force required to lift the corner of the automobile. In other terms, although the energy required is available, it must be in a form to match the load characteristics. From the smaller force that could be developed by the man through a fairly large distance, it is necessary to produce a large force which need act only through a small distance. In both cases, the product of the force and the distance through which it acts - the energy - is the same. Suppose that we wish to determine the best design for a jack so that a man can lift the car with the minimum movement of his arm. From experience we know that to raise the car a small amount we must move the jack handle up and down many times. In other words, we exert our muscular force through a large displacement in order to lift the heavy car through a small displacement. This is a direct application of the principle illustrated in Fig. C-2. 2, that equal areas may be found under many differently-shaped force-displacement curves.

The bumper jack can be modeled as a lever, which is a simple device for balancing unequal weights or forces. Whenever we apply a force to a wheel to make it turn, or use a wrench to tighten a nut, we apply the principle of the lever. The key elements of a lever are a center of rotation, some resistance to rotation, and a point at some distance from the center where a force can be applied which tends to produce rotation to overcome the resistance. It is familiar experience that the greater the distance from the center of rotation, the greater the effectiveness of applied forces in overcoming the resistance to motion. For example, a bigger wheel is easier to turn by hand than a small one of the same mass; a longer wrench makes it easier to turn the nut. In other words, the turning effectiveness, or torque, as it is termed technically, is proportional to the distance for a given force. This principle can be formulated quantitatively as shown in Fig. C-2. 19. Here the lever consists of a long rod supported at a point called

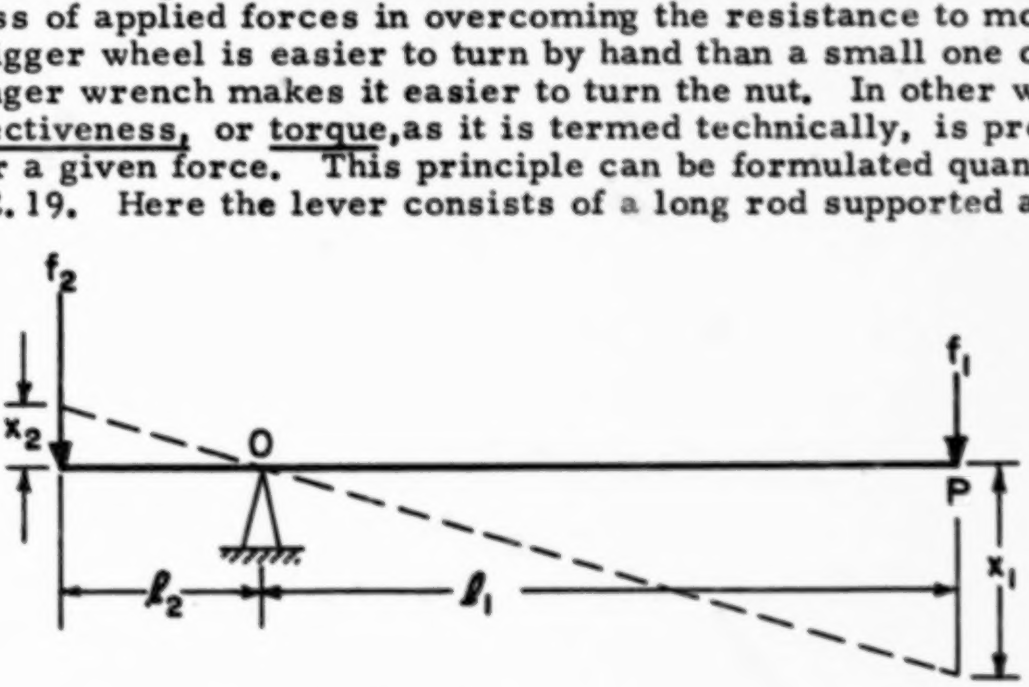


Fig. C-2. 19 Model of the lever.

its fulcrum, or center of rotation. If we relate this model to the bumper jack, resistance is provided by the force f_2 due to the weight of the automobile and rotation is attempted by the force f_1 exerted by the hand of the human operator at point P and is easy to observe experimentally that if a relatively small force f_1 is to be capable of lifting the car, the distant ℓ_1 must be greater than the distance

ℓ_2. If the mass of the car is doubled as when trying to lift a Cadillac instead of a foreign sports car, the point of application of force f_1 must be moved twice as far from the fulcrum. Thus the torque, is proportinal to the distance from the center of rotation. The same turning effectiveness for a mass twice the original size can also be obtained by doubling the applied force and thus doubling the torque. Thus torque is also proportional to applied force. Applied force and distance are the only variables available. The torque, denoted by T, is defined quantitatively by the equation

$$T = f_1 \ell_1$$

where ℓ_1 is the distance from the point of force application to the fulcrum 0 and f_1 is the force applied perpendicular to the bar (Fig. C-2.19).

The direction of application of the force is very important. It is common experience that in turning a body about a fixed axis, such as in loosening a bolt as in Fig. C-2.20, the given force produces a maximum effect when it is applied

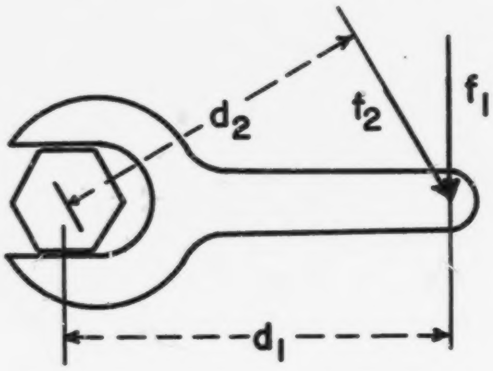


Fig. C-2.20 Torques applied to a wrench.

perpendicular to a line from the axis to the point of application. In Fig. C-2.20 the force f_1 will be more effective than the force of equal magnitude f_2. Force f_1 produces a torque $f_1 d_1$, while force f_2 produces the torque $f_2 d_2$. Since d_1 is greater than d_2, the torque $f_1 d_1$ represents greater turning effectiveness.

Note that torque has a direction as well as magnitude. Thus both $f_1 d_1$ and $f_2 d_2$ tend to turn the wrench in a clockwise direction. When applied to the wrench they are resisted by a torque applied to the wrench by the nut in a counterclockwise direction.

Returning to Fig. C-2.19, for the lever to be balanced, the torque $f_1 \ell_1$ must be equal to $f_2 \ell_2$; that is

$$f_1 \ell_1 = f_2 \ell_2$$

or

$$f_2 = f_1 \frac{\ell_1}{\ell_2}$$

Thus, a small force applied at a large distance from the fulcrum can develop a large force nearer the fulcrum. To raise a weight at one end of the lever (for example, the car), we simply exert a downward force on the other end, and unless $\ell_1 = \ell_2$, the upward displacement of the weight will differ from the downward displacement of the end being pushed. Fig. C-2.19 illustrates how these displacements x_2 and x_1, are related to ℓ_1 and ℓ_2 by

$$x_2 = x_1 \frac{\ell_2}{\ell_1}$$

The distances are corresponding sides of similar triangles.

The last two equations indicate quantitatively (that is, in numbers) how a bumper jack transforms or converts the force exerted on the jack hangle and its associated displacement, into a force required to lift the car to a specific height. With these equations we can now determine the best design for the jack. It is clear from:

$$f_2 = f_1 \frac{\ell_1}{\ell_2}$$

that for a maximum force f_2 the ratio ℓ_1/ℓ_2 must be made large enough to transform f_1 into the force necessary to lift the car. Simultaneously, however, the vertical displacement of the car x_2 depends on the inverse ratio ℓ_1/ℓ_2. If we make the ratio ℓ_1/ℓ_2 too large, we will have to move the jack handle farther and if we make this ratio too small we will not be able to lift the car at all. The "perfect match" occurs when ℓ_1/ℓ_2 just transforms our maximum force to equal the force necessary to lift the car. For example, if it takes a force of 2,500 newtons to lift one corner of a car weighing 10,000 newtons (the weight of a compact car), then a jack designed with $\ell_1/\ell_2 = 5$ would serve as a good matching device for a man who can only exert a force of 500 newtons. Of course, since most real bumper jacks are designed to be used by different individuals, they will not develop a perfect match at all times. They are normally designed to be useful for persons of below average muscular ability.

Notice that in lifting the car, the energy we expend is calculated from the relation

$$w = f_1 x_1$$

Substituting

$$f_2 = f_1 \frac{\ell_1}{\ell_2} \quad \text{or} \quad f_1 = f_2\left(\frac{\ell_2}{\ell_1}\right)$$

and

$$x_2 = x_1 \frac{\ell_2}{\ell_1} \quad \text{or} \quad x_1 = x_2\left(\frac{\ell_1}{\ell_2}\right)$$

into this equation gives

$$w = f_1 x_1 = \left(f_2 \frac{\ell_2}{\ell_1}\right)\left(x_2 \frac{\ell_1}{\ell_2}\right) = f_2 x_2$$

Thus, all of the energy expended in pushing down on one end of the lever is equal to the energy in raising the car at the other end of the lever.* It is clear, therefore, that the lever is not a power amplifier: it has an input and output but no source of external energy. Sometimes we are tempted to say that the small f_1 has been "amplified" to the large f_2, but this is incorrect because the term amplification pertains only to power. The correct phraseology is: f_1 has been "transformed" to f_2.

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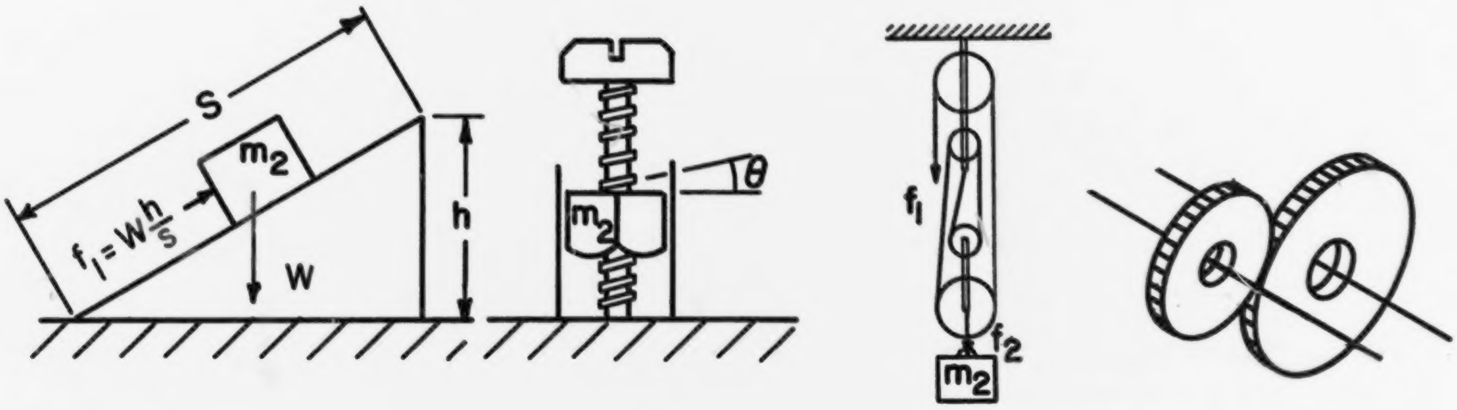
*In an actual bumper jack, of course, f_2x_2 is always less than f_1x_1 because energy is lost in friction between the rod and fulcrum and in the slight bending of the rod.

There are other types of transformers, all of which are capable of transforming forces. As with the bumper jack, they are all used to match equal energies so that the force provided by one is efficiently transferred (or coupled) to the force required by the other. In all of these,the energy output is, of course, always less than the energy input. Some are illustrated in Fig. C-2.21.



INCLINED PLANE SCREW PULLEYS GEARS

Fig. C-2.21 Mechanical force transformers or energy couplers.

The inclined plans, or wedge, is one of the first force transformers used by man. When we are unable to exert a force W on a mass m_2 to lift it vertically we can move it to the desired height by pushing it up an inclined plane with a force f_1 whose magnitude is W h/s, where h is the height and s is the slant length of the incline. W represents the weight of the mass m_2. The equation with which shows the relationship between the applied force and the resisting force is:

$$\frac{f_1}{W} = \frac{h}{s}$$

the ratio h/s performs the same function as the ratio ℓ_1/ℓ_2 in

$$f_2 = f_1 \frac{\ell_1}{\ell_2}$$

It provides a means for matching force capability to force requirement.

A screw is simply a wedge wound around a cylinder. Its transformation factor is $1/\sin\theta$, where θ is the angle its grooves make with the plane of the nut.

A pair of pulleys can also be used for force conversion. Here the force f_2, exerted by the gravitational force on the mass m_2, is developed from a smaller required lifting force f_1 according to the formula

$$f_2 = nf_1$$

The transformation factor n, is equal to the number of ropes which support m_2.

Gears represent another type of coupler used to match a power amplifier to its load. We shift into "low gear" when going up a steep hill, and this permits the engine to rotate through a number of revolutions for each revolution of the wheels of the car. The car moves slowly but probably would not move at all in any other gear position. On level reads shifting into high gear permits the wheels of the car to rotate more rapidly for the same engine speed. Whenever the force necessary to move the load increases, we must shift to a lower gear. Some racing bicycles have as many as 9 gear ratios to match the various load conditions. And, of course, our sports car has "four on the floor". The use of gears for energy coupling is illustrated in Fig. C-2. 22.

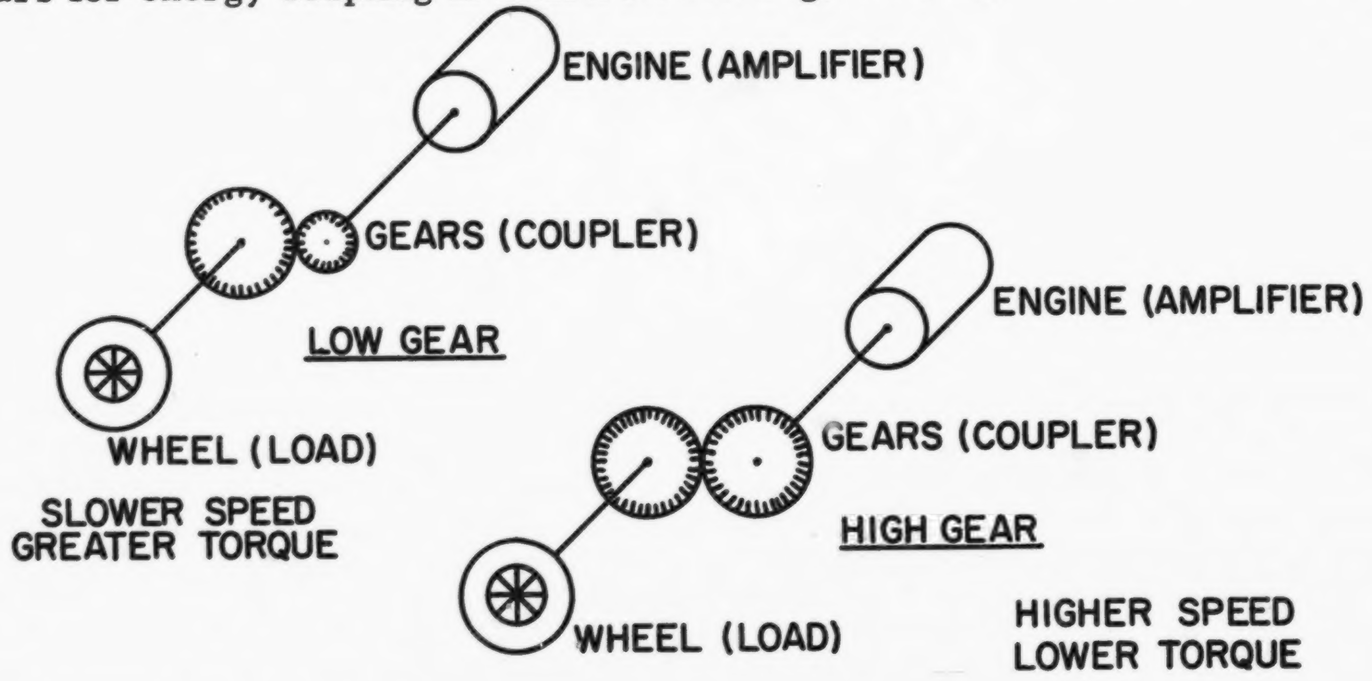


Fig. C-2. 22 The use of gears for energy coupling in an automobile.

The Electrical Transformer

The concept of matching is also used in many electric circuits where it is desired to obtain the greatest transfer of power from an amplifier to a load. A common example of this is the coupling of electrical power from the amplifier to speakers in radio, television, and high fidelity equipment. Electrical transformers are used for this purpose and their transformation properties can be deduced from the principle that the power into a coupler is equal to the power out of it. Thus, if we denote the inpute power as $e_1 i_1$ and the output power as $e_2 i_2$, then if we assume ideal conditions with no losses in the transformer:

$$\text{Power input} = \text{Power output}$$

$$e_1 i_1 = e_2 i_2$$

or

$$\frac{e_2}{e_1} = \frac{i_1}{i_2} = n$$

where n is a constant defining the two ratios, e_2/e_1 or i_1/i_2. If n is larger than unity the transformer is called a step-up transformer, since the output voltage is larger than the input voltage. A step down transformer produces a smaller output voltage than the input voltage. The above equation indicates that an electrical transformer can transform voltage or current, subject to the condition that, if voltage is increased, current must decrease, and vice versa.

The electrical transformer is essentially an arrangement as shown in Fig. C-2. 23. Here a number of thin soft iron laminations are stacked to form a core on which two coils are wound. The electrical power source is connected to one of the coils, called the primary coil, while the load is connected to the other coil called the secondary coil. It is interesting to note that the transformation ratio n is fixed by the ratio of the turns of the two coils. Thus if the secondary coil contains three times as many turms as does the primary coil then

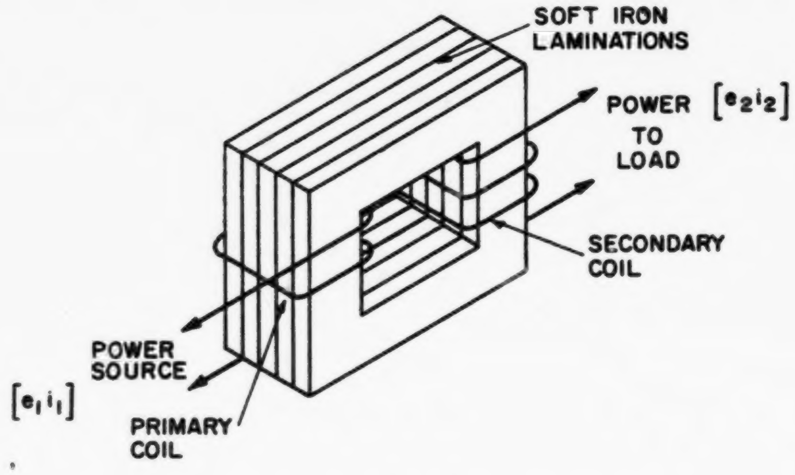


Fig. C-2. 23 The electrical transformer

$$n = \frac{\text{turns on secondary}}{\text{turns on primary}} = \frac{3}{1} = 3$$

With this transformer, power applied to the primary coil in which the voltage is 110 volts, will become, under ideal conditions, electrical power in the secondary but with a voltage of 330 volts. To compensate for this increase in voltage so that the power input and the power output are equal, the current which will flow to the load, will be one third of the current which flows into the primary coil from the electrical source.

In summary, a transformer whether it is mechanical or electrical, transmits power while changing the physical quantities which describes power. Voltages and currents are changed to suit the needs of a particular application, but their product (power) is unaltered. Similarly, in a mechanical coupler velocity and force are changed but their product (power) is fixed.

C-2.8 CASCADING AMPLIFIERS

We have seen in Section C-2.5 that the power delivered to the load of a typical electronic amplifier is thousands of times greater than the input power needed to control this flow of energy. Although this may seem like a considerable amount of amplification, there are many applications for which it is not enough. For example, the power produced by a radio wave in the antenna of a radio receiver is exceedingly small. In order to produce a suitable volume of sound from the receiver's loudspeaker, it is necessary to amplify this power by a factor of several million. This is far more amplification than can be provided by any single electronic component.

In order to meet this need for large amplification, we may interconnect two or more amplifiers in the manner shown in the block diagram of Fig. C-2.24.

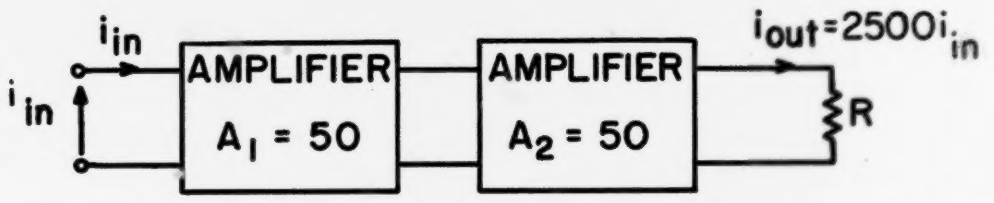


Fig. C-2.24 A cascade interconnection of amplifiers.

We call this a cascade interconnection and we refer to each of the amplifiers as one stage of amplification. With this arrangement, the input control signal current i_{in} is transformed to a larger current by some amount (shown, for example, as a factor of $A_1 = 50$ in Fig. C-2.24) by the first stage of amplification. The output current of the first stage then serves as the input signal current for the second stage, where it is transformed by another factor of 50. Thus, the overall current gain for the two stages is 50 x 50 = 2500. If this is not enough gain, additional stages can be added in cascade. Typical amplifiers used in good record-playing systems or small radio receivers have four or five stages cascaded in this way.

This method of cascading stages to obtain large values of amplification raises the question of whether any limit exists for the number of stages that may be connected in cascade. Experience indicates that a limit does exist, and it results from the unavoidable noise that is introduced along with the signal at the input or introduced by the amplifiers themselves. In addition to the noise problem, the tendency of a multistage amplifier to become unstable or oscillate is greater as the number of cascaded stages is increased. Great care must be exercised in the design and construction of multistage amplifiers. This problem of stability is discussed in the next chapter.

C-2.9 NOISE IN AMPLIFIERS

In order to study the effect of noise on the limitations of the number of stages that can be cascaded, let us consider a public address system. These systems use electronic amplifiers similar to those discussed above. With an increase in the amplification in such systems, a faint sound at the microphone will produce an audible sound from the loudspeakers. As we increase the amplification, a point is reached at which the room noise begins to produce a noticeable sound from the loudspeakers. If we increase the amplification further, we reach a condition where the amplified noise becomes loud enough to mask the useful sound coming from the loudspeakers. Thus, it is not useful to increase the amplification beyond a certain value. This value depends on the amount of noise in the room in which the microphone is placed.

In addition to the background noise discussed above, a similar effect arises from another source whenever large amplification is sought. This is the electrical noise, which arises from the fact that the atomic particles that make up the wires, the resistors, and the transistors (or other electronic components) of an amplifier, are in a constant state of agitated motion which is dependent on their temperature. As a result of this motion there are very small currents, present throughout the amplifier. The power associated with these random currents in the wires at the input to the amplifier are amplified the same as the power of the control signal. Indeed, the noise currents appear to the amplifier to be the same as if they were control signal currents. If enough amplifier stages are connected in cascade, the noise currents produce a large signal at the output. These noise currents can be heard readily on any FM radio receiver; they produce the s-h-h-h noise that is heard when the FM radio receiver is not tuned to a station. (The noise heard on an AM radio receiver when it is not tuned to a station is generally electrical noise generated in the Earth's atmosphere.) Thus, all electronic amplifiers have a built-in electrical noise source that limits the amount of useful amplification obtained by cascading stages.

There are many areas of scientific and engineering importance in which it is essential to secure the greatest possible amplification. One example is the field of radio astronomy in which it is desired to detect the very faintest radio signals from extremely large distances in space. For such applications special low-noise amplifiers are used for the first stages of amplification, often called the front end of the cascade of amplifiers. Since electrical noise increases in direct proportion to the temperature of the amplifier, the front ends are often immersed in a bath of liquid helium which is at a temperature near absolute zero. At this low temperature the electrical noise is reduced almost to zero.

C-2.10 SUMMARY

In this chapter we have studied the methods man has developed to control the flow of energy from the many sources available to him. In particular, we developed the concept of amplification as the process of utilization of a small amount of energy, supplied directly by a sensor to control the flow of a greater amount of energy supplied by another energy source. An important objective of this control process is the achievement of an output signal from the amplifying device which is a replica of the input controlling signal.

In order to discuss amplification satisfactorily, we developed quantitative descriptions of both mechanical and electric energy. With the observation that the most interesting property of a force is the type of motion which it can produce, mechanical energy was described in terms of the area under an f, x curve. We developed the important concept that equal energies can be represented by many different force-displacement situations (Fig. C-2.2). Further analysis of mechanical energy revealed that the rate of expenditure of energy, called power, is equal to the product of force times velocity, and that the total energy expended during some time interval is simply the area under the p, t curve. The voltage between any two points in an electrical circuit was described as the energy per unit charge expended in moving an electrical charge from one of these points to the other, and the electrical current was described as the measure of the amount of electrical charge flowing through a wire per second. We were then able to show these quantities to be analogous, respectively, to force and velocity. By this analogy, we developed the relationship for electric power to be $p = ei$.

With these tools in hand, the remainder of the chapter discussed amplifiers and couplers and the distinctions between them. We defined a coupler (or force transformer) as a device used to match an energy source to a load by transforming an input force (or current or voltage) to match a required output force (or current or voltage). We indicated that due to losses the power output from these devices is always less than the power input, they cannot be termed amplifiers. In the case of both mechanical and electrical couplers (or transformers) we found that power was transmitted through these devices without any power amplification. In an ideal electric coupler, e and i are changed to suit the needs of a particular application but their product (power) does not increase. Similarly, in a mechanical coupler, f and v are changed but their product (power) does not increase.

Many different types of amplifiers were discussed, from the very simple on-off variety to the more refined control of the windlass, the hydraulic, and the electronic amplifiers. Distortions caused by the nonlinearities and response times of these amplifiers were also discussed and the limitations that stability and noise impose on the cascading of amplifiers were introduced.

PROBLEMS

C-2.1 Assume that the force needed to push an automobile up a slight incline at a constant velocity is 1500 newtons.

(a) What energy is needed to push the car 30 meters along the slope?

(b) What is the average power required if the force is applied for 1 minute?

C-2.2 The force required to stretch a spring in proportional to the displacement of the free end. For a certain spring, $f = 2000x$.

(a) Sketch a curve of force versus displacement and find the work required to extend the spring 8 cm.

(b) If 12.1 joules of work are done in stretching the spring, what displacement would result?

C-2.3 A constant force of f newtons is applied to an automobile by its engine (through the tires and the road).

(a) If the mass of the car is m kilograms, use the information of Section B-3.4 to determine the acceleration, a.

(b) Sketch a curve of acceleration versus time and then determine the velocity v_1 at any time t_1 seconds after the force is applied.

(c) Sketch a curve of velocity versus time and also a curve of power versus time.

(d) How much energy has been delivered to the car when the velocity is v_1? Eliminate t_1 from your result by using the relationship between t_1 and v_1 found in (c) above.

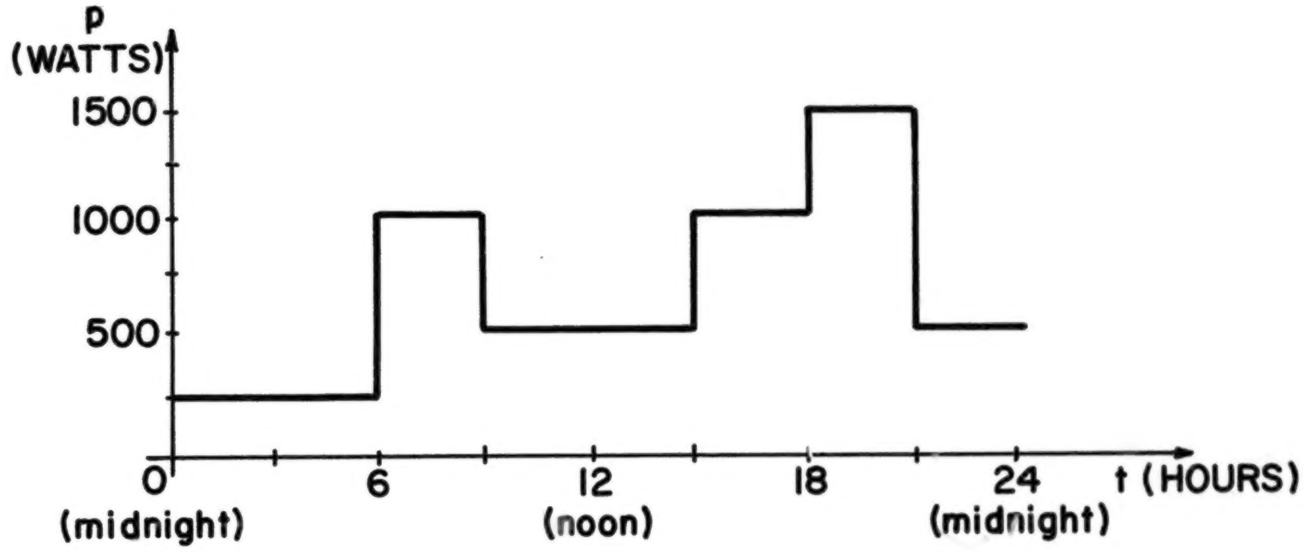
(e) What is the numerical value of this energy for a 2000-kilogram automobile having a velocity of 30 m/s?

C-2.4 (a) How much energy is needed to raise 70 kilograms 10 meters?

(b) How much energy could be obtained by allowing 10^5 kilograms of water to fall 100 meters? If energy is valued at 1 cent per kilowatt-hour, how much is this energy worth?

(c) How long would a 1-kilowatt pump have to run to empty a 1-meter depth of water from a rectangular 10m by 15m basement if the water must be lifted 3 meters to ground level? The density of water is 1000 kg/m^3.

C-2.5 If a mass m is given an initial push and then allowed to coast freely across a horizontal surface until frictional forces bring it to a stop, the exponential curve shown below represents the frictional force tending to slow it down as a function of displacement from the starting point, $x = 0$. How much work is done in stopping the mass?



C-2.6 A 12-volt automative electrical generator delivers a current of 4 amperes to the headlights from 9:00 to 9:30 p.m., and a total current of 10 amperes to the headlights and radio from 9:30 to 10:00 p.m.

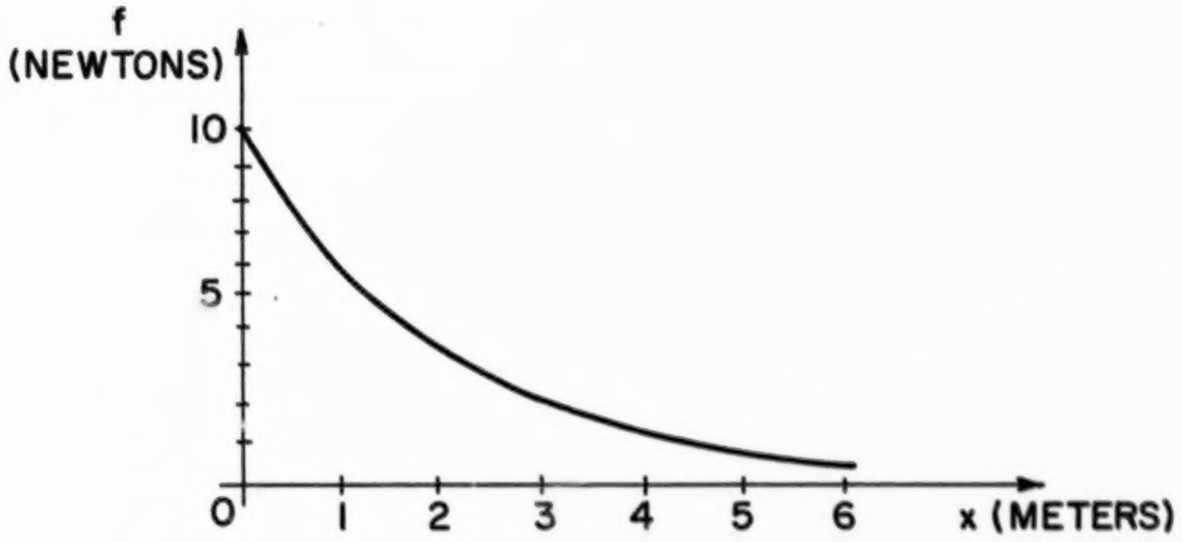
(a) How many units of charge are furnished by the generator in the first half-hour?

(b) What is the total energy supplied by the generator during the 1-hour interval?

(c) What power is delivered to the headlights?

(d) What power is delivered to the radio, assuming that the headlights are just as bright with the radio on as they are with it off?

C-2.7 The power used by a household during a certain day is shown in the graph below. If the cost of electrical energy is 3 cents per kilowatt-hour, how much did that day's electricity cost?



C-2.8 A typical residential furnace is rated at 136,000 Btu per hour. In mks units this is 40 kilowatts. At the existing outdoor temperature, the house loses heat by radiation at the rate of 15,000 joules per second.

(a) For how many minutes in each hour should the furnace operate to maintain a constant temperature?

(b) The temperature of the interior increases 1°C when there is a net increase of heat energy of five million joules, how long should the furnace operate to raise the house temperature 2°C?

(c) If the cooling rate is also 5 million joules for a 1°C drop, how long would it take the house to cool down 2°C if the furnance were shut down?

(d) Assume that the interior temperature is 20°C at $t = 0$. Draw a curve of temperature versus time if the temperature is maintained between 20°C and 22°C by on-off control of the furnace.

C-2.9 A windless similar to that shown in Fig. C-2.11 is used as a small pile driver. A mass of 100 kg is lifted 5 meters and then dropped on the pile to drive it into the ground. It is found by experimental measurement that a force of 5 newtons pulling on the line causes a force of 125 newtons to be applied to the lead. The weight of the line can be neglected in this problem.

(a) Using the information given above, calculate the power amplification factor of the windless.

(b) When the load has been lifted 5 meters, how much line has been pulled in by the operator?

(c) How many joules of work have been done on the load?

(d) How many joules of work have been expended by the operator?

C-2.10 A windlass similar to that shown in Fig. C-2.11 is used on a cabin cruiser for lifting the anchor. It is to be designed so that a small boy capable of pulling steadily with a force of 40 newtons can lift an anchor having a mass of 25 kg. (The mass of the anchor line usually can be neglected, except when anchoring in water, deeper than 10 meters.)

(a) Neglecting the mass of the line, what is the minimum amplification factor that the windless can have and still meet the specifications given above?

(b) Suppose that the windlass is designed to have an amplification factor of 15 to give the boy "extra strength" for breaking the anchor out of a mud bottom. If he raises the anchor 6 meters from the bottom to the deck how much work is done on the 25-kg anchor?

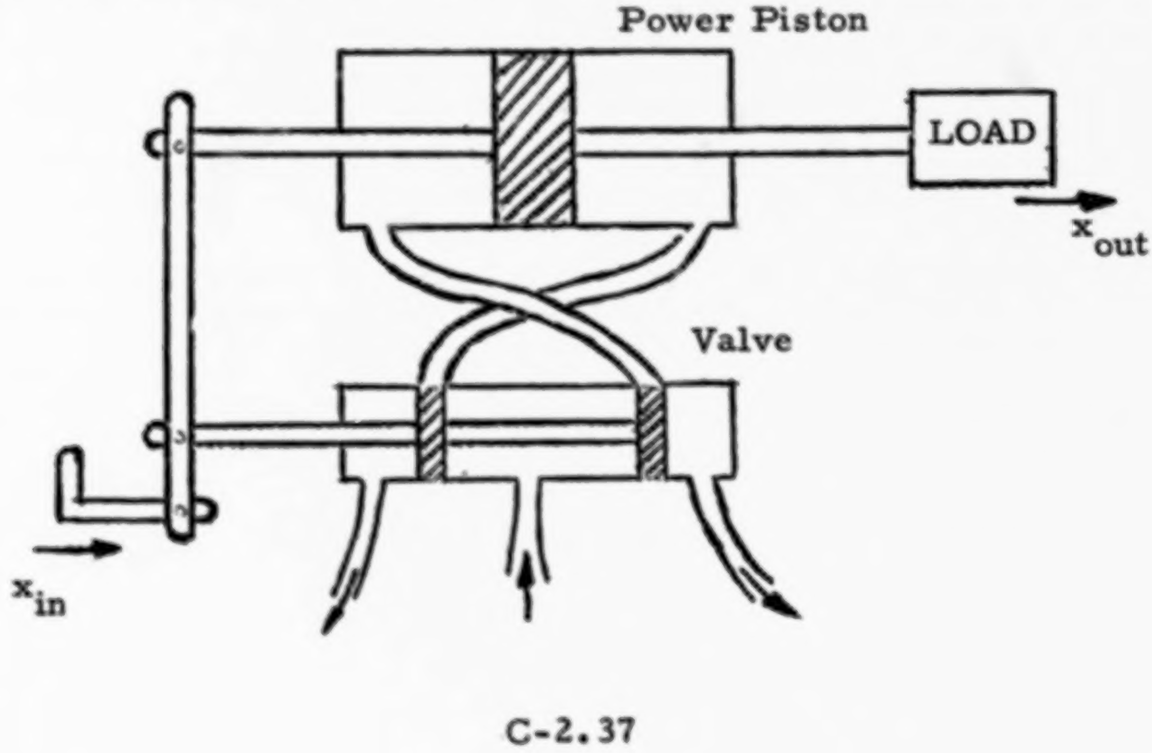
(c) How much work does the boy do in the process?

(d) If the anchor is raised with a constant velocity in 15 seconds, how much power, in watts, does the boy deliver?

C-2.11 When the operator of a windlass pulls on the line with a constant force of 20 newtons for one-half minute, it is found that a mass of 100 kg can be lifted 1 meter in that time. What is the amplification factor of the windlass?

C-2.12 A hydraulic amplifier similar to that shown in Fig. C-2.13 to be designed so that $x_{out} = 3x_{in}$. If $\ell_2 = 10$ cm, what is the required value of ℓ_1?

C-2.13 The mechanical linkage in the hydraulic amplifier of Fig. C-2.13 is reconnected as shown in the figure below. The amplifier is initially at rest with the valve closed. Then the input lever is given a small displacement to the right, opening the valve slightly, and it is held fixed with this small displacement from its original position. Describe briefly the subsequent motion of the power piston and the valve.



C-2.14 Two children have masses of 20 and 30 kg.

(a) If the smaller sits 3m from the fulcrum of a see-saw, where should the larger seat himself?

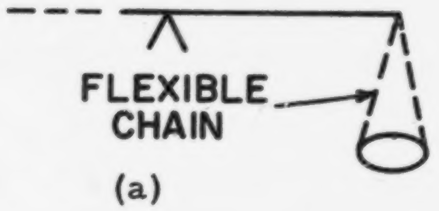
(b) If the smaller child goes through a vertical excursion of 1.5m, through what displacement will the larger one travel?

(c) Which child will move faster?

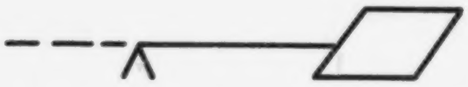
(d) Which child will be subjected to the greater acceleration?

(c) Who has all the fun?

C-2.15 Why is the platform arrangement for a mass balance shown in (a) below superior to that shown in (b)?



(a)



(b)

C-2.16 A meter stick is placed on a fulcrum at the 50-cm mark. Masses of 500gm and 200gm are located at the 20-cm and 90-cm marks, respectively. Where should a 300-gm mass be placed to achieve balance?

C-2.17 (a) How much force is required to pull an 80-kg skier up a slope for which $h/s = \frac{1}{2}$, if friction between the skis and the snow if neglected?

(b) What is the length of the slope in Fig. C-2.21 along which this force must be applied?

(c) If the trip requires 5 minutes, what average power in kilowatts is required to pull 10 skiers up the slope simultaneously?

(d) How much power would be required to lift the same 10 skiers the same 500-m distance vertically in the same time?

C-2.18 (a) Make a sketch showing how a rope may be used with two pulleys in order that a man can raise an engine having a mass of 200kg by exerting a force of about 400 newtons.

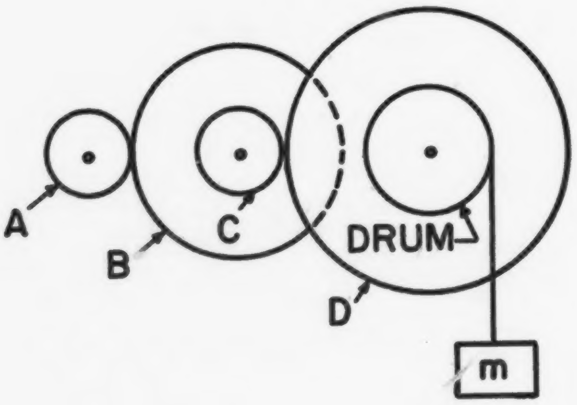
(b) If the man lifts the engine 1 meter in 15 seconds, with what average velocity is he pulling the rope?

(c) What average power does apply to the rope?

C-2.19 The gears shown in the figure below have the following numbers of teeth: A, 10; B, 50; C, 10; D, 80. Gears B and C are mounted on the same shaft and turn at the same angular velocity. Gear D and the smooth drum also turn at the same angular velocity. The drum has a diameter of 20 cm.

(a) If mass m is to be raised with a constant velocity of 0.314 m/s, determine the direction of rotation and the angular velocity of each gear and the drum.

(b) If m represents a compact car of 2000-kg mass, what power must be delivered to gear A?



C-2.20 An electrical transformer has an input voltage of 2300 volts, an output voltage of 115 volts, and an input power of 10,000 watts. Find the output power, output current, and input current. (assume no losses)

C-2.21 Electrical transformers are available having voltage ratios of two-to-one and six-to-one. In an emergency, how might these transformers be used to deliver 40 volts to a motor from a 120-volt source?

C-2.22 The electronic amplifier of Fig. C-2.14 has the characteristic shown in Fig. C-2.15 for i_{out} versus i_{in}.

(a) What is the approximate value of the current transformation factor A when i_{out} = 2 milliamperes?

(b) Under certain operating conditions the input current to the amplifier is i_{in} = 20 + 10 sin 100t microamperes. Assuming that equation: $i_{out} = Ai_{in}$ holds with the value of A found in (a), what is the expression for i_{out}?

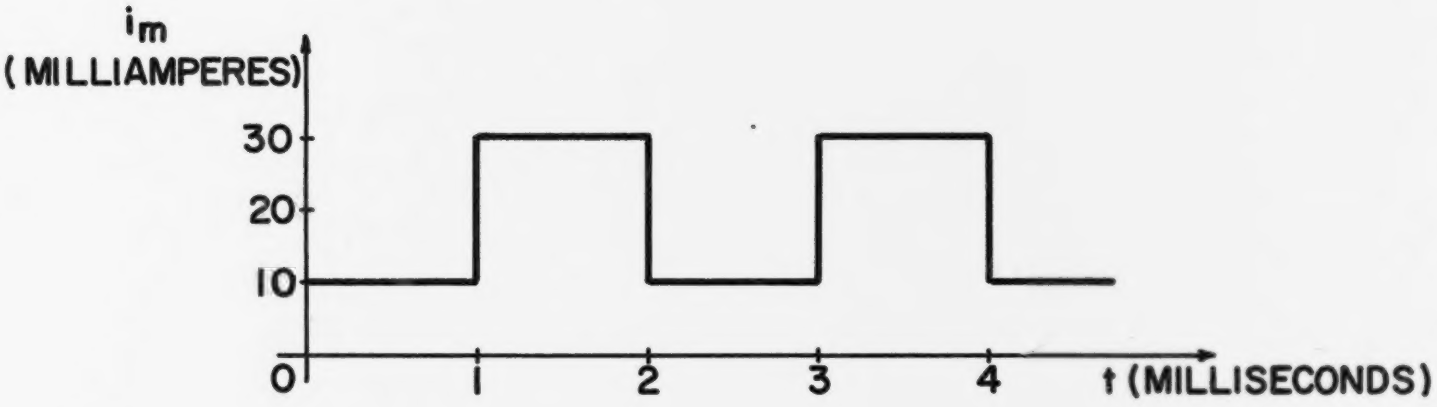
C-2.23 A high-power transistor with a current transformer factor A = 100 is used in the electronic amplifier of Fig. C-2.14 with R = 500 ohms and i_{in} = 20 miliamperes. (use the graphs in Fig. C-2.17).

(a) Calculate the load current i_{out}.

(b) Calculate the voltage across the load R.

(c) Calculate the power delivered to the load R.

(d) If the input signal current has the waveform shown below, draw a graph showing the waveform of the load current i_{out}. (Assume that the load current changes abruptly when the input current changes.) Show the maximum and minimum values of i_{out} on this graph.



C-2.24 A signal current of 0.5 microamperes must be transformed to a value of at least 30 milliamperes. If a current gain of 50 can be obtained from each stage of amplification, how many stages must be interconnected in cascade to achieve the desired result?

C-2.25 Show why the systems described below do or do not satisfy the three criteria of an amplifier. In each case you should identify the controlling signal, the energy source, and the power output to the load.

(a) A miner using a large hose to wash away the bank of a river;

(b) A gardener using a wheelbarrow to move a pile of topsoil 58 feet;

(c) A transistor radio;

(d) A tube;

(e) A photoelectric device used to turn street lights on at dusk and off at dawn;

(f) A public-address system;

(g) Power brakes;

(h) A man speaking into a telephone in New York that is connected to another person in California.

C-2.26 The motor in a certain vacuum cleaner is rated at 690 watts. This is the power that it requires from the household electric system. When it is operating, frictional losses in the motor and fan bearings involve 40 watts, 80w are used to move the air inside the casing of the motor, 150w are lost in heating the wires of the motor because of the resistance, 80w are converted to heat in the magnetic materials of the motor, and 50w are spent in heating the fan blades. The remaining power is delivered to the air by the fan. However, half of this power is lost in heating the air as its pressure is increases.

(a) How much power is effectively used to provide the "vacuum"?

(b) If the efficiency is defined as the useful power output divided by the total power input, find the efficiency of fan and motor system.

(c) If the force required to operate the switch is 1 newton and the switch moves 1 cm, what energy is required to turn the machine on? to turn it off?

(d) In use, assume that the cleaner is turned on for 5 minutes and then off for 25 minutes to answer the telephone, and that this pattern is repeated for 2 hours. The movement of the machine across the floor requires a mechanical energy input from the operator of 25 w. What is the average power used to control the machine for 5 minutes of operation?

(e) What is the power amplification?

(f) Sketch curves of useful power output and average power input versus time.

Chapter C-3

STABILITY

C-3. 1 INTRODUCTION

In designing systems that help us cope with the world in which we live, we usually have in mind some desired operating state or condition. This desired state may be very simple; we may wish to place a book on a table so that it remains stationary. Or the desired state may be quite complicated; we may wish to place an astronaut into a precisely calculated orbit.

Sometimes, the specification of the operating state may be easy; we want the book to remain in a fixed and unchanging position. At other times the specification of the operating state may be difficult; we must compute the astronaut's precise orbit with the aid of higher mathematics and the use of high-speed digital computers. But whether the specification of the operating state is simple or complex, the design must involve additional information. We must know if the operating state toward which we design is stable or unstable. An unstable design may produce disastrous effects. A man in a satellite sent into an unstable orbit would invite catastrophe.

The idea of stability is deeply rooted in the popular language. We talk of a political leader who does not maintain his equilibrium is unstable. His behavior is unpredictable and erratic. The man who is a good provider and whose behavior is as "stable as the Rock of Gibralter" is looked upon favorably by his neighbors. In this chapter we shall go beyond popular usage and probe into the concept of stability. We shall find that unstable and stable systems differ in behavior if a disturbance is applied to them when they are in a state of equilibrium. In many systems we can deal with the phenomenon of stability in a quantitative manner by applying our old friends: modeling and analysis techniques. This offers great advantage for we can then determine quantitatively when a system changes from a stable to an unstable state.

A phenomenon which is useless under one set of conditions may often prove useful in another context. Instability is this type of phenomenon. We shall give some examples where man has exploited instability for good use.

C-3. 2 SKYSCRAPERS BEGET SKYSCRAPERS

Consider the adage, "skyscrapers beget skyscrapers, " What does this mean? Until the beginning of this century, office buildings were generally a few stories in height, (often only three or four stories). The advantage of a tall building is that one could centralize many operations. Workers could reside near their place of business, thus cutting down transportation costs. With taller buildings, land could be used much more efficiently. Buildings such as the Empire State Building with more than 100 stories in height occupy only about 1/100 the land needed to house all of its tenants if they were to be arranged on a single floor.

Some disadvantages, however, are associated with increased vertical construction. As a building is made taller, the foundations must be made sturdier to support the added weight of the additional stories. With the all-masonry buildings that existed to the end of the last century, gigantic foundations were required for tall buildings. More important, elevators were non-existent. People simply would not or could not walk up more than a few flights of stairs.

A natural competitive operating state of equilibrium existed. Every office building owner realized approximately the same profit, and all office buildings in a large city were generally about the same in height. The profit equilibrium was stable. Any attempt to disturb this equilibrium with the construction of a taller building proved wasteful since no one would walk up many flights of stairs. This would mean that the income necessary to pay for the additional cost of the foundations could not be realized. A building of fewer than the customary stories would not, of course, realize the income of the buildings of the normal height.

But toward the end of the century two changes occurred. First, the cheap fabrication of steel beams made possible tall, light-steel frame buildings. Secondly, Mr. Otis invented the elevator and electrical motors became available for their operation. Despite these inventions, the equilibrium state for three or four story buildings could have continued to exist. The new inventions could simply be ignored. But such equilibrium was now obviously unstable.

The first skyscraper brought its owner large profits from a small plot of land. This produced an increase in the value of the land on which the skyscraper was built. Owners of vacant land were no longer willing to sell at the price that prevailed when buildings were only a few stories in height. They could wait for a skyscraper builder who would be willing to pay more for the same plot because his building would realize a larger return for his investment. With increasing land prices, only a skyscraper would be profitable and each new skyscraper created pressures for succeeding office building to be skyscrapers. Each skyscraper encouraged new, and for a time, higher skyscrapers. In New York, for example, the instability was especially severe and in addition the bedrock of Manhattan made skyscraper building easy. Recently, in cities such as London, where the foundation conditions for tall buildings are much less favorable than in New York, the "skyscraper begetting skyscraper" effect has become quite noticeable.

This simple example illustrates several characteristic features associated with the stability or the instability of a system.

(1) In a stable system, a disturbance which moves the system operating state will evoke forces that tend to return the system to the operating state.

(2) In an unstable system, a disturbance will evoke forces that tend to drive the system even further from the operating state.

Before the invention of the elevator and the availability of cheap steel beams, tall buildings were thus impractical. Later, buildings taller than a few stories created pressures for yet taller buildings. A "snowballing" effect occurs which is typical of instability. This effect is readily observed when a snowball rolls down a hill; each new layer adds more snow to the ball.

We shall meet the above characteristics; the tendency of a stable system to restore itself and the tendency of an unstable system to "snowball" away from the equilibrium or operating state again and again. A knowledge of these characteristics is essential for the control and creation of many systems in the man-made world. Although a qualitative appreciation of these characteristics is sufficient in many instances, in others we prefer a quantitative mastery; a knowledge of the numerical values of the quantities that may change the system from a stable to an unstable state. We shall apply modeling and analysis to some examples in order to illustrate the method of approach to such problems.

C-3.3 EPIDEMICS

Some years ago a person suffering from smallpox was accidently permitted to enter the United States. To prevent the outbreak of an epidemic, an immediate search was instituted to locate all persons that had been in contact with this individual since his arrival. The possibility of a rapid spread of the disease from a limited initial beginning was well understood. This need to quarantine or isolate infected individuals to prevent widespread infection is really a process for preventing an unstable situation to develop from an initial disturbance. The single infection acts as a disturbance of the entire community which is originally in equilibrium. The city of Aberdeen in Scotland, for example, was affected by a typhoid fever epidemic in 1964. Strict measures were taken to quarantine the entire city to prevent a further spread of the disease.

This suggests that an epidemic has the mark of an instability phenomenon, with its sudden and abrupt change from one state to another precipitated by a disturbance. In the smallpox case, the disturbance was a single infected person. In the typhoid fever case, the initial disturbance was traced to an imported shipment of tainted fish carrying the typhoid fever germ. Indeed, for a number of years public health officials have developed mathematical models of epidemics. They hope to obtain deeper insight into the epidemic mechanism. Given the fact that many people, in a community, are immune to smallpox by virtue of vaccination, what would happen if a single infected person were to appear? He may infect some of the others. But if these in turn come in contact only with immune persons, no epidemic will develop. On the other hand, if each of these infected persons infects three or four others the epidemic will grow exponentially. Another factor is related to the fact that people who recover become immune. If they recover quickly enough, they may not infect enough additional people to expand the epidemic. How important are these various factors relative to one another in the growth of an epidemic? Should the public health officials concentrate their resources on quarantine or on vaccination? Such are the questions that one wishes to answer with the help of a model. In many situations definitive answers are difficult to secure because the phenomena are complex and influenced by unpredictable events. Nevertheless, the insights that the models have provided have been useful and have spurred further research.

We will consider the simplest models of an epidemic. Let us first analyze the following, somewhat artificial, situation. We assume that in a total population of N persons, everyone is either infected with a certain disease or is susceptible to catching the disease from those who have been infected. This model thus omits the possibility of individual recovery and subsequent immunity, death, or isolation. This is an oversimplification for most diseases, although for certain mild upper respiratory infections with a long period of infection, the model does have validity.

We denote the number of infected persons by i and the number of susceptible persons by s. We are interested in the rate of increase in i, that is how many new people will become infected each day. Let us denote the rate at which new people become infected by R_i. We expect this rate to depend on the number of persons that are already infected. The greater this number, the greater the probability that they will transfer the infection to others. But we can also expect that the number of newly infected persons on any day will depend on the number who are susceptible. Again, the greater the number of susceptible persons, the greater the number of people who will be infected.

Hence a reasonable model should have the number of newly infected increasing both with the number of presently infected, i, and the number who are susceptible, s. But what is the precise form of this dependence? We could, for example, assume that a certain fraction (1%) of the infected and 2% of the susceptible, will produce newly infected persons each day. Symbolically, the number of newly infected each day, or the rate of infection could then be represented by the equation:

$$R_i = 0.01i + 0.02s.$$

But would such an assumption make sense? Suppose we have no infected, so that $i = 0$ and $s = N$ (the total population). Our proposed equation would lead to

$$R_i + 0.02s = 0.02N$$

or 2% of the total population would predictably develop the disease despite the complete absence of carriers. Then again, suppose, everyone is already infected so that no additional people are susceptible. The proposed equation would predict:

$$R_i = 0.01i = 0.01N$$

We would predict still more people coming down with the disease although there are none that can be infected! Clearly, the proposed equation or model of the epidemic is not realistic.

Let us try another line of approach. Let us assume that the number of newly infected per day will be some fraction of the product of the number of infected and the number of susceptible. Let us assume that the fraction is 1%, then the number of newly infected each day would be:

$$R_i = 0.01is$$

This equation or model is free of the objections of the initial model. If there are no infected persons in the population (i = 0), there will be no newly infected. Likewise, if there are no susceptible (s = 0), there will be no newly infected. We may assume that the population consists only of infected or susceptible people so that N = i + s. Our new model of an epidemic is in fact generally used, although it is a very crude mathematical model. With this model we can now study the spread of the contagious disease and predict the nature of the epidemic.

It is possible to represent this model using the scalor, adder, multiplier, and integrator operations discussed in earlier chapters. One formulation suitable for solution with an analog computer is shown in Fig. C-3.1.

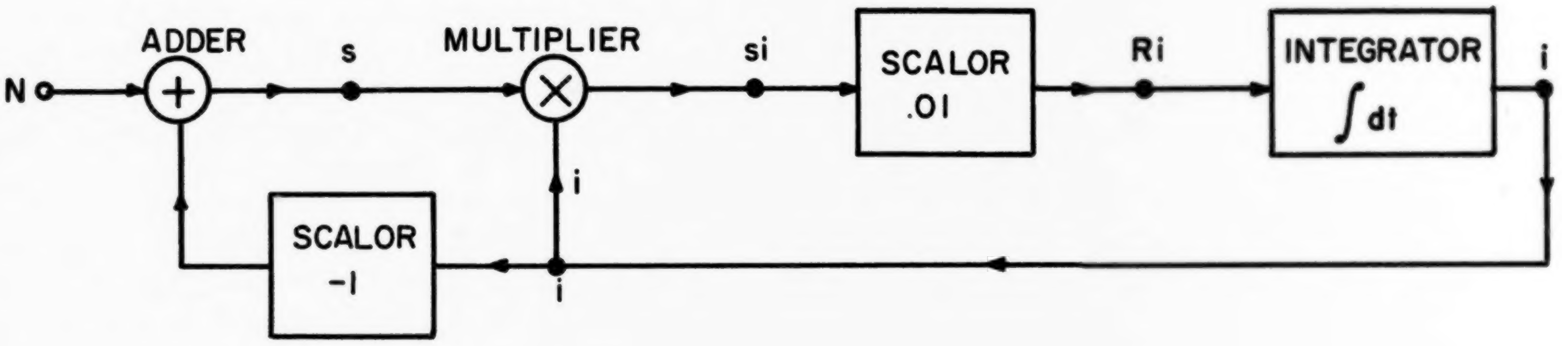


Fig. C-3.1 - Block diagram of Epidemic model.

This block diagram appears rather formidable at first glance, but we can grasp its meaning with a step by step examination of its structure. We should remember that we are searching for the day by day value of the number of infected persons, so that we can predict the growth of the epidemic and discover whether the turning point in its development will occur before conditions in the community become dangerous.

If the total population is N and the number of persons who have already been infected is i, then the number of people who remain to be infected is (N-i) or N + (-i). This represents the number of people who are susceptible, according to our model.

$$s = N + (-i)$$

Since the value of i (the number of infected people on any day) is exactly what the analogue computer will produce at its output, we loop the output back to a multiplier so that it is multiplied by (-1). The result (-1) (i) = (-i) is then combined with the value N (at the adder) to produce a numerical value for the number of people s who are susceptible to the disease.

Our mathematical model for the epidemic $R_i = .01si$ informs us that the rate of increase involves the product of s with i and the factor (.01). We therefore tsp the line which supplies the value of i to the scalor (-1) and feed this value into a multiplier along with the value s. This output from the multiplier represents the product si. Multiplication in the scalor unit by .01 produces the output $.01si = R_i$.

The factor R_i is the daily rate of increase in the number of infected cases. If we send this information into an integrator, we can sum up the total number of newly infected cases after any period of time. This is the factor i which we seek.

If we attach a meter to the output of the integrator and read it at intervals, of time, we can graph a curve which simulates the course of the epidemic.

We can also solve a numerical problem to illustrate how the epidemic will develop by performing calculations very similar to those we applied in chapter C-1 to find the output of an integrator with feedback. Comparison of Fig. C-3. 1 with Fig. C-1. 18, will reveal that the models are indeed rather similar. We can apply the same procedure of "inching" along in time, using the present number of infected people, i, to determine the rate, R_i at which the number of infected is increasing. Then, by multiplying R_i by the small time interval, Δt, we can estimate the increase Δi in the number of infected occuring during that time interval. By adding Δi to i, the number of infected at the end of the time interval is found. This calculation is then repeated over and over at successive instants of time.

Let us illustrate this procedure by considering a total population of $N = 100$, and using a time interval of one day for Δt. We shall assume that one person is infected on the first day, so that initially:

A) Let $N = 100$
Let $\Delta t = 1$
Let $t = 1$
Let $i = 1$

B) Let $s = N - i$
Let $R_i = (.01)(si)$
Let $\Delta i = (i R_i)(\Delta t)$

C) Let new i = old i + Δi
Let new t = old t + Δt
to be used for computation of the next day's changes.

We can set up these computations in tabular form

Day	i	$s = 100 - i$	si	$R_i = .01 si$	$\Delta i = R_i (\Delta t) = R_i (1)$
1	1	99	99	0.99	≈ 1

where we have rounded the 0.99 value for Δi to the nearest integer to obtain the number of newly infected. Then, on the second day, the number of infected (new i) will be the number from the first day (old i) plus the newly infected (Δi) or $1 + 1 = 2$. Hence, on the second day, we repeat the calculation to get:

Day	i	s = 100 - i	si	R_i = . -01si	$\Delta i = R_i(\Delta t) = R_i(1)$
2	2	98	196	1. 96	2

Then, at the beginning of the third day, the number of infected persons would be the sum of i and Δi, or 4 infected persons. A continuation of this calculation gives the entire course of the epidemic as shown in Tabe C-3. 1.

TABLE C-3. 1 EPIDEMIC CALCULATION

Day	i	s = 100 - i	si	i RATE = . 01si	Δi = (Ri) (1)
1	1	99	99	0. 99	1
2	2	98	196	1. 96	2
3	4	96	384	3. 84	4
4	8	92	736	7. 36	7
5	15	85	1275	12. 75	13
6	28	72	2016	20. 16	20
7	48	52	2496	24. 96	25
8	73	27	1971	19. 71	20
9	93	7	651	6. 51	7
10	100	0	-	-	-

Figure C-3. 2 shows the epidemic of Table C-3. 1 in graphical form.

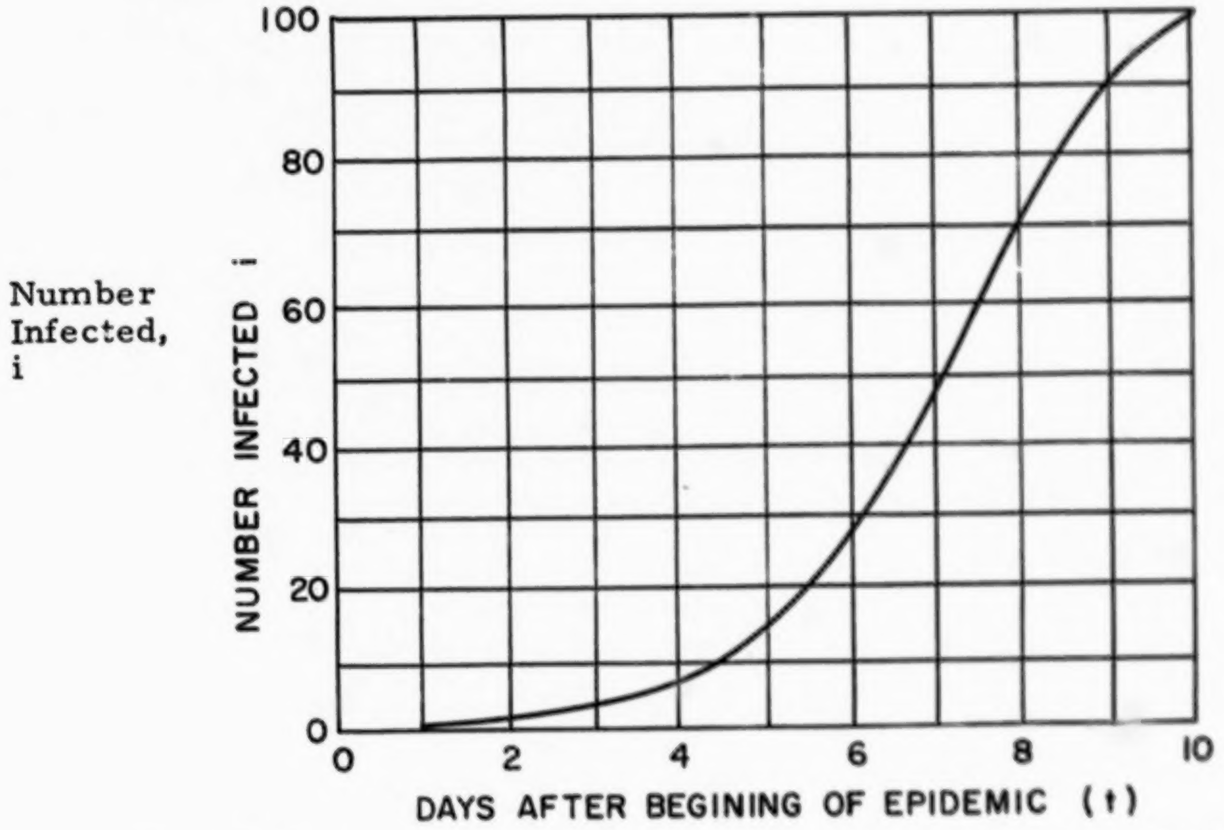


Fig. C-3. 2 - Plot of simple model epidemic given in Table C-3. 1.

The desirable state of a population is, of course, one in which no disease exists. We see from Table C-3. 1 or Fig. C-3. 2 that, according to our model, this desired state is unstable. A disturbance in the form of one infected person sets off a chain reaction which increases the movement away from the stable condition. In our example, this movement from the initial stable conditions continues until the 7th and 8th days, after which the rate at which people become infected begins to decrease as the entire population becomes infected.

Although the above model is a useful introduction, it is too simple to be realistic except under special circumstances. A more realistic model takes into account the fact that people also recover and become immune, or die, or can be isolated or leave the area. Thus the total population contains at any instant some number r of "recovered" people. This number r changes with time at some rate, R_r. The simplest assumption is that in a single day a certain fraction of those infected will be withdrawn from the ranks of those afflicted because of recovery, death, etc. For instance, if this fraction were 0. 1, we would have:

$$\text{Rate of recovery} = \text{number recovered/day} = R_r = (0.1)\, i \qquad \text{(C-3. 3)}$$

where we have used the term "recovered" to include all the effects mentioned: death, recovery, and insulation.

The total population, N, now consists of those who are infected, those who are recovered, and those who are susceptible.

$$N = i + r + s$$

As more people recover, they decrease the number of infected and increase the number of recovered. Thus, the net change Δi in the number of infected occurring in a time interval Δt is given by the newly infected minus the newly recovered,

$$\Delta i = R_r\, \Delta t - R_r\, \Delta t \qquad \text{(C-3. 4)}$$

and the change in the recovered is:

$$\Delta r = R_r\, \Delta t \qquad \text{(C-3. 5)}$$

Let us next modify the equations used previously. We must now specify that initially there are no recovered persons. Also to keep the calculations to reasonable length, let us assume that we initially have 50 susceptibles and one infected person in the population.

A) Let $N = 51$

Let $\Delta t = 1$

Let $t = 1$

Let $i = 1$

Let $r = 0$

Then,

B) Let $s = N - i - r$

Let $R_i = (.01)\ (si)$

Let $R_r = (0.1)\ i$

Let $\Delta i = R_i\ \Delta t - R_r\ \Delta t$

Let $\Delta r = R_r\ \Delta t$

C) Let new i = old $I + \Delta i$

Let new r = old $r + \Delta r$

Let new t = old $t + \Delta t$

Again this information permits us to repeat our computation for a point by point analysis. As before, we arrange the calculations in tabular form

Day	i	s	r	R_i	R_r	Δi	Δr
1	1	50	0	1	0	1	0

During the first day, the increase in infections will be 1, there will be a decrease in the susceptible persons of 1, and no further recoveries or deaths. Note that we have rounded off $0.01is$ and $0.1i$ to the nearest integer. This convention will be followed in the subsequent calculations, which are given in Table C-3.2.

Plots of i and s versus days are shown in Fig. C-3.3.

If the desired state is a disease-free population, then, in accordance with this refined model, such a state is unstable. One infected person will trigger a "snowballing" away from the desired state. But our model predicts that not all of the susceptible people will develop the disease. Two of the susceptible persons will avoid infection to the 19th day of the epidemic, subsequent to which no further change will occur (see Table C-3.2).

We know that the introduction of a single infected person into a population need not necessarily trigger an epidemic. Does our model reveal this possibility? What does our model predict about the conditions under which an epidemic will occur?

To explore these questions we return to Equations 3.4 ($\Delta i = Ri\ \Delta t - R_r\ \Delta t$) and Equation C-3.5 ($\Delta r = R_r\ \Delta t$). We permit f_i to represent the fraction of the product, (si) that become infected in a single day and f_r the fraction of infected persons who recover in a single day. We then write the equations of the model in the form:

(C-3.6) Change in recovered, $\Delta r = f_r i\, \Delta t$

(C-3.7) Change in infections, $\Delta i = (f_i si - f_r i)\ \Delta t = (f_i s - f_r)\ i\ \Delta t$

where we have factored out the term i in Equation (C-3.7).

We note that the sign of the right-hand side of Equation C-3.6 is always positive. This indicates that the number of persons who have recovered will always increase, and, as a consequence, the number of persons who are as yet susceptible, will always decrease. This also reflects the assumption of our model that all who recover do not again become susceptible. On the other hand, the sign of the second equation, C-3.7, varies with the sign of the term in parenthesis, $(f_i s - f_r)$.

If the number of susceptible persons in the population is large enough $(f_i s - f_r)$ will be positive, and an increase in the number of infected persons will result; however, if the number of susceptibles is small, the quantity $(f_i s - f_r)$ will be negative; and an epidemic will not be initiated by the addition of another infected person.

According to our model, the borderline between instability and stability, that is, the condition which separates the epidemic from the non epidemic response to an infection is that in which:

$$f_i s_{cr} - f_r = 0 \qquad \text{(C-3.8)}$$

where we have denoted the critical number of susceptibles by s_{cr}. For a large number of susceptible people, the rate at which people become infected will be greater than the rate at which people recover and become immune or die. If the number of susceptible people is sufficiently small, the rate at which people recover or die will exceed the rate at which they become infected, so that no epidemic will occur. Solving for s_{cr}:

$$s_{cr} = \frac{f_r}{f_i} \qquad \text{(C-3.9)}$$

which gives an explicit relationship for the critical number of susceptibles in terms of the factors f_i and f_r. In the above example, $f_i = 0.01$, $f_r = 0.1$ and therefore, by Equation C-3.9 $s_{cr} = 10$, and an epidemic developed. However, if we make $f_i = 0.01$, but $f_r = 0.6$, so that $s_{cr} = 0.6/0.01 = 60$. With an initial value of $s = 50$ we can compute the outcome of this condition in terms of the following table:

i	s	r	$f_i si$	$f_r i$	Δi	Δr
1	50	0	1	1	0	1
1	49	1	0	1	-1	1
0	49	2	0	0	0	0

no epidemic would result.

We note from Equation C-3.7 that our model has a self-limiting property. Suppose s initially exceeds s_{cr}. As i increases initially, s will necessarily decrease. The magnitude of the factor $(f_i s - f_r)$ becomes negative, after which the number of infected will begin to decrease. If this occurs with sufficient speed only a small proportion of those susceptible to the disease will become infected. In the numerical example, Table C-3.2, however, conditions were such that nearly all susceptible persons were afflicted.

TABLE C-3.2 IMPROVED EPIDEMIC CALCULATION

Day	i	s	r	R_i 0.01is	R_r 0.1i	Δi Infections	Δr recovered
1	1	50	0	1	0	1	0
2	2	49	0	1	0	1	0
3	3	48	0	1	0	1	0
4	4	47	0	2	0	2	0
5	6	45	0	3	1	2	1
6	8	42	1	3	1	2	1
7	10	39	2	4	1	3	1
8	13	35	3	5	1	4	1
9	17	30	4	5	2	3	2
10	20	25	6	5	2	3	2
11	23	20	8	5	2	3	2
12	26	15	10	4	3	1	3
13	27	11	13	3	3	0	3
14	27	8	16	2	3	-1	3
15	26	6	19	2	3	-1	3
16	24	5	22	1	2	-2	3
17	23	4	24	1	2	-1	2
18	22	3	26	0	2	-1	2
19	21	2	28	0	2	-2	2
20	19	2	30	0	2	-2	2

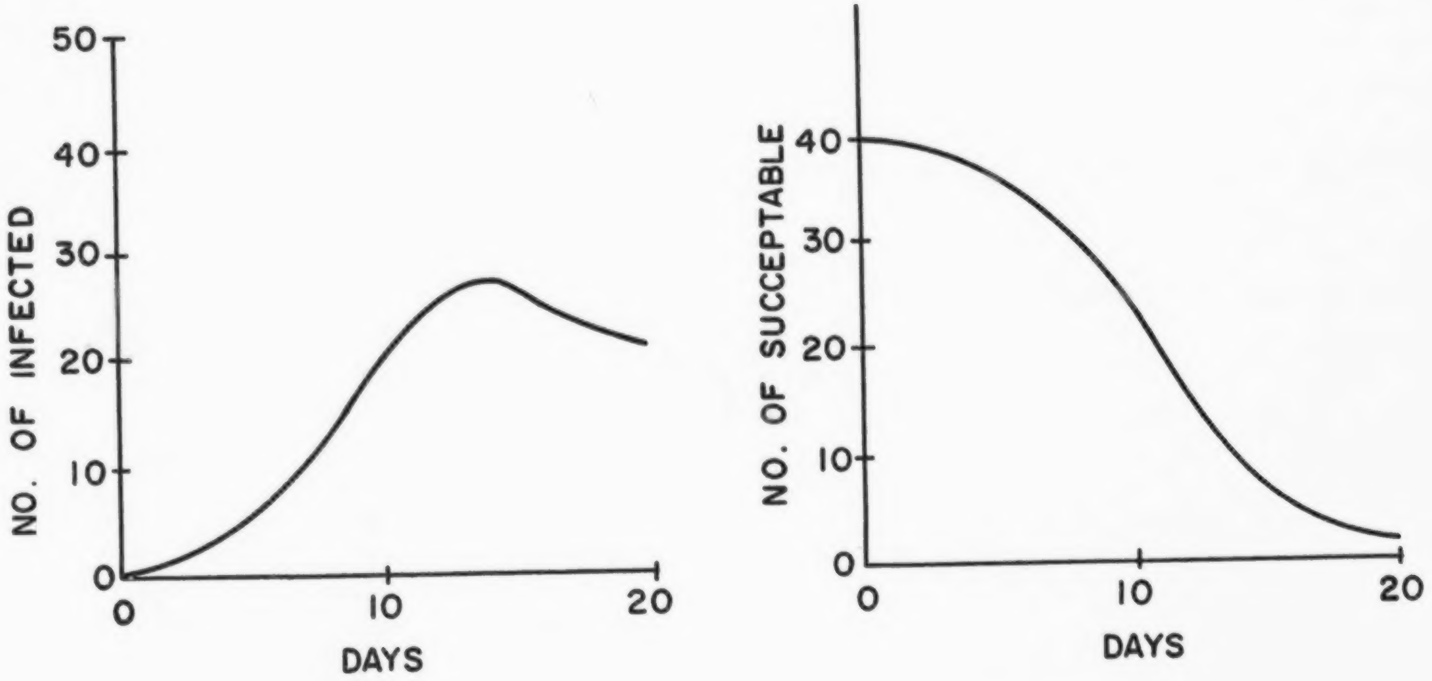


Fig. C-3. 3 - Plot of epidemic given by Table C-3. 2.

Equation C-3. 9 for the critical number of susceptibles quantitatively relates three factors that we know play important roles in the development of an epidemic. Reduction in the number of susceptibles s, to a value less than the critical value will prevent the growth of an epidemic. This often suggests the need for vaccination and immunization. In some South American countries, for example, the malaria-preventive drug, atabrine, is added to the table salt in small quantities. Although not everyone uses table salt, and different people use it in different amounts, it is hoped that enough immunity will be given to reduce s below the critical value. The rate at which the susceptible become infected, represented by the factor f_i, can be kept down by keeping susceptible persons away from contact with the infected ones through quarantine and isolation. Finally the rate at which people recover, f_r, is as seen by Equation C-3. 9 also to be important. We can expect to find that diseases which are contagious for long periods are more difficult to control than those with short contagious periods.

Mention should be made that we have considered the so-called deterministic models of epidemics in this section. In more refined models, the laws of chance are included. One considers, for example, various probabilities that an infected person will come in contact with another who is susceptible. Although such models are more realistic than those we have studied, they require the use of mathematics which is beyond the scope of our course.

C-3. 4 LAW OF SUPPLY AND DEMAND

It is a commonly accepted truism that the "supply always adjusts to meet the demand." For example, when medical evidence recently indicated that it was healthy to be slim, demand for weight reduction drugs increased. In response to this demand, the drug manufacturers increased the supply of such drugs.

The "law of supply and demand" clearly involves the concept of a static operating state. In the example of the weight reduction drug, we may imagine the existence of a number of "demanders" who consume the drug at the same rate at which it is supplied by the manufacturers. If for some reason the demand of the drug were to decrease, we believe that the supply will also decrease until a new equilibrium state is reached.

But as we have already seen, it is also necessary to know about the stability of the operating state. In economic systems, as in all other situations involving people, this is a very complex question and it is influenced by many factors. Complicated mathematical models have been devised to deal with problems such as the supply-demand drug example which we considered above. For simplicity of treatment, we shall focus attention on a small economic system consisting of a single supplier and a single buyer.

Imagine the following situation. Mr. I. M. Crafty is a toy manufacturer. Mr. Crafty makes one toy and his principal outlet is Discount Stores, Inc., a chain that depends on high-volume sales. To cut costs, Discount Stores stocks as little as possible in warehouses, where rental and deterioration add to expense. Discount's policy is to place orders every Tuesday for an amount that can be sold in a single week. Their estimate of the weekly sales volume is based on the current price quoted by the manufacturers.

Mr. Crafty on the other hand sets his price on the basis of the most recent order from Discount stores. Every Wednesday he sends letters to his customers announcing the new price.

What will happen? There is obviously a possible operating state where Discount orders the same amount Tuesday after Tuesday and Mr. Crafty sets the same price Wednesday after Wednesday; but will this operating state remain stable?

Suppose Discount decreases its toy order. Mr. Crafty may increase his price in an attempt to maintain his total profit. Discount may then feel that it cannot sell as many toys at the higher price and may further decrease its purchase order. On the other hand, Discount may place an even larger order than before to take advantage of the high-volume price discount. Mr. Crafty could then increase his price since that his toy is now in greater demand, which may result in a cut in the size of the order. Clearly the situation is not simple; a multitude of possibilities exist.

How can we apply modeling and analysis to gain an insight into this complicated affair?

We can first try to model Discount's strategy of buying. Probably Discount will buy a greater amount at the lower price set by Crafty. If we plot price versus the amount that Discount will buy, the simplest curve of this sort would be a straight line as shown in Fig. C-3.4 where for a given price, P, Discount will buy amount A.

Let us now examine and model one of Mr. Crafty's possible selling strategies. Recognizing that Discount Stores has increased its purchases because his price to them is low enough for gaining a good profit on the resale of the toys, Crafty may attempt to induce a larger sale to Discount by a further reduction in price. The larger his sales to Discount, the larger the profit he hopes to earn. A simple straight line model of this strategy is displayed in Fig. C-3.5.

On the basis of our straight-line models of Crafty's "supply policy" and Discount's "demand policy," what will happen? When will equilibrium of supply and demand occur? We can easily find this and also analyze for stability graphically. In Fig. C-3.6 we have plotted both Crafty's selling strategy, labeled C, and Discount's buying strategy, labeled D. Equilibrium will occur at the intersection point A_e, P. At this price, P_e, Crafty will be willing to supply amount, A_e, week after week, and Discount is willing to buy this amount at the stated price. For a smaller purchase Crafty will insist on a higher price. Discount will also expect to pay a higher price for a lower amount, but not necessarily the price Crafty expects. Likewise for an amount greater than A_e, Crafty's and Discount's expectations differ. Only for an amount A_e are they both in agreement on the price?

Suppose now that equilibrium A_e, P_e is disturbed. Perhaps a fire in the toy department of one of Discount's stores compels them to reduce their order to A_1 in Fig. C-3.7. For this quantity, Crafty will set his price at P_1 in Fig. C-3.7. At price P_1 Discount is willing to buy an amount A_2, as shown on the demand model in the graph. For this larger purchase (A_2), Crafty offers a lower price P_2, which encourages Discount to increase its purchase further. This will produce another cycle of adjustments until the point A_e is reached. This is true only if buyer and seller maintain their strategies consistently.

But now suppose that the strategies are those shown in Fig. C-3. 8 where the Crafty curve C is steeper than Discount's curve D. If Discount were now to place an order for an amount, A_1, which is less than the equilibrium purchase, A_e, the amounts purchased decrease, and Crafty will increase his price. At the new price Discount reduces his sales and this cycle will be repeated until Crafty will have priced himself out of business. If in any one week Discount were to increase its order above A_e, the prices will continue to drop until Crafty loses all profits. Clearly, the equilibrium is unstable in this case.

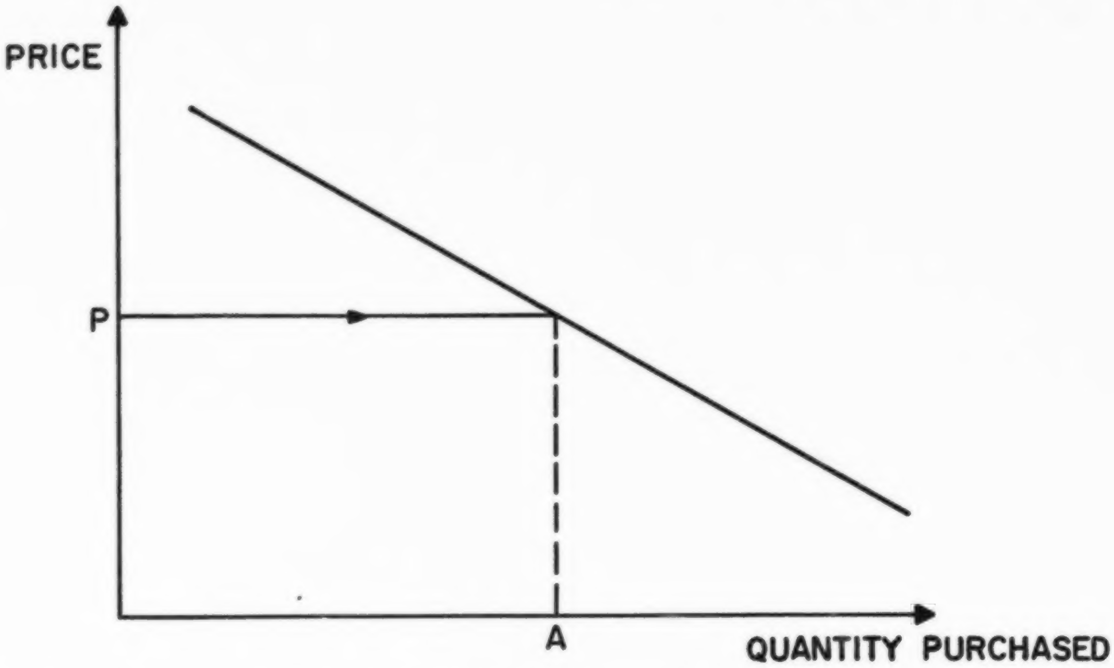


Fig. C-3. 4. Discount Stores' probable buying strategy.

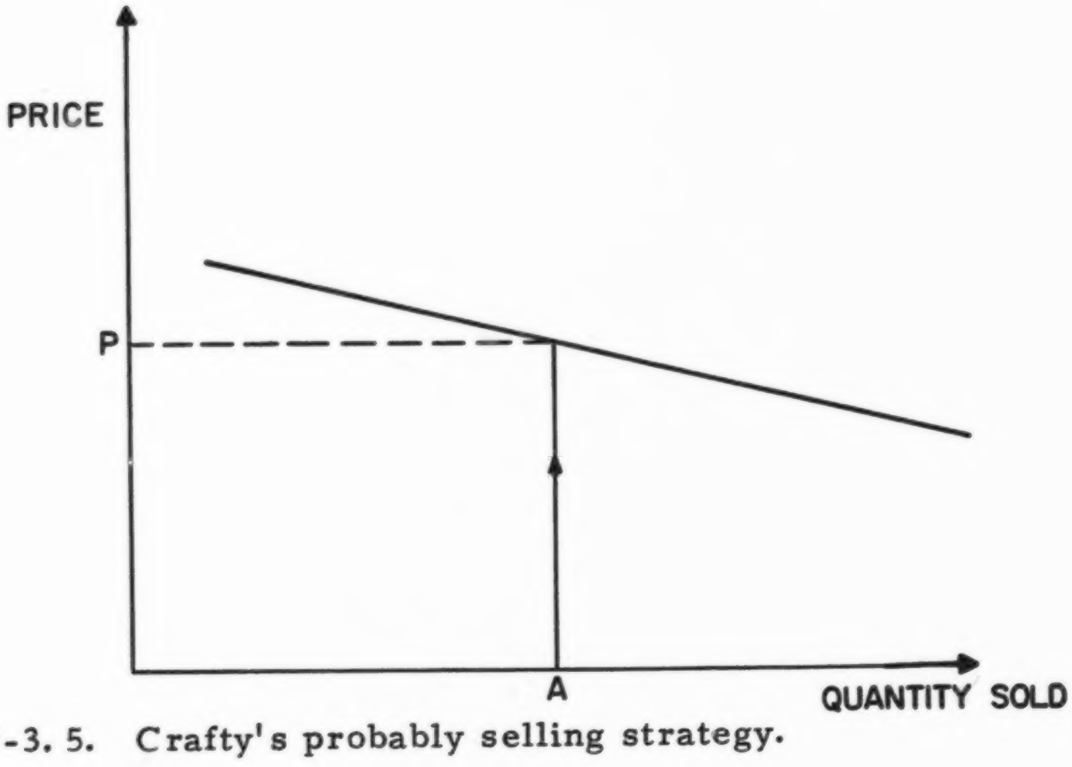


Fig. C-3. 5. Crafty's probably selling strategy.

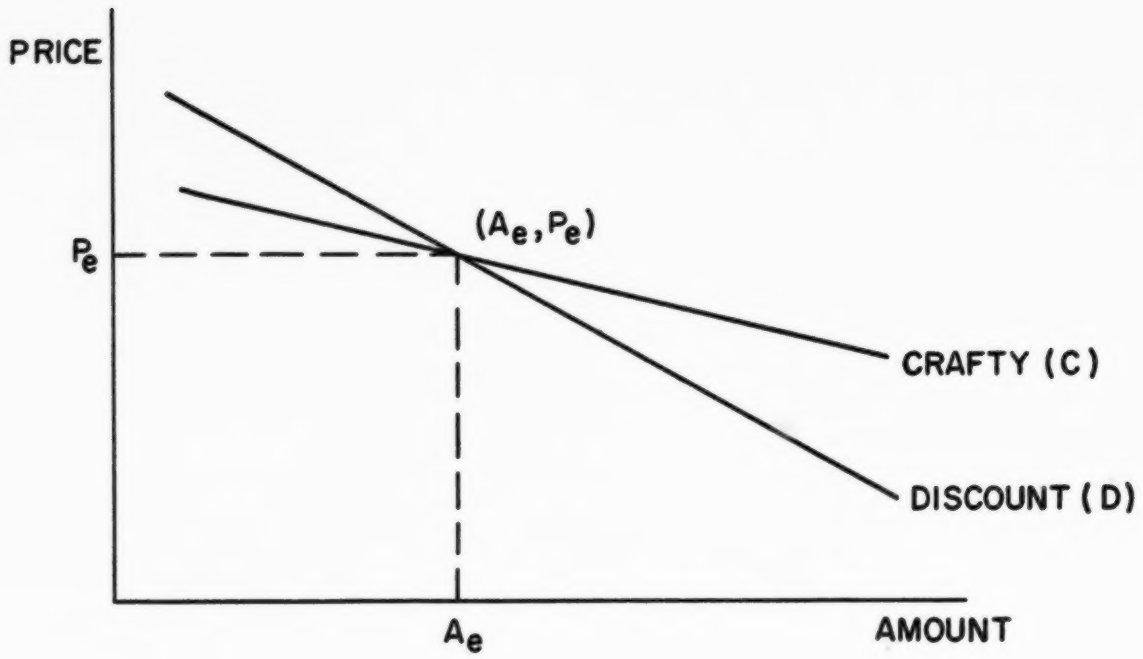


Fig. C-3.6. Equilibrium point for Crafty's and Discount's strategies.

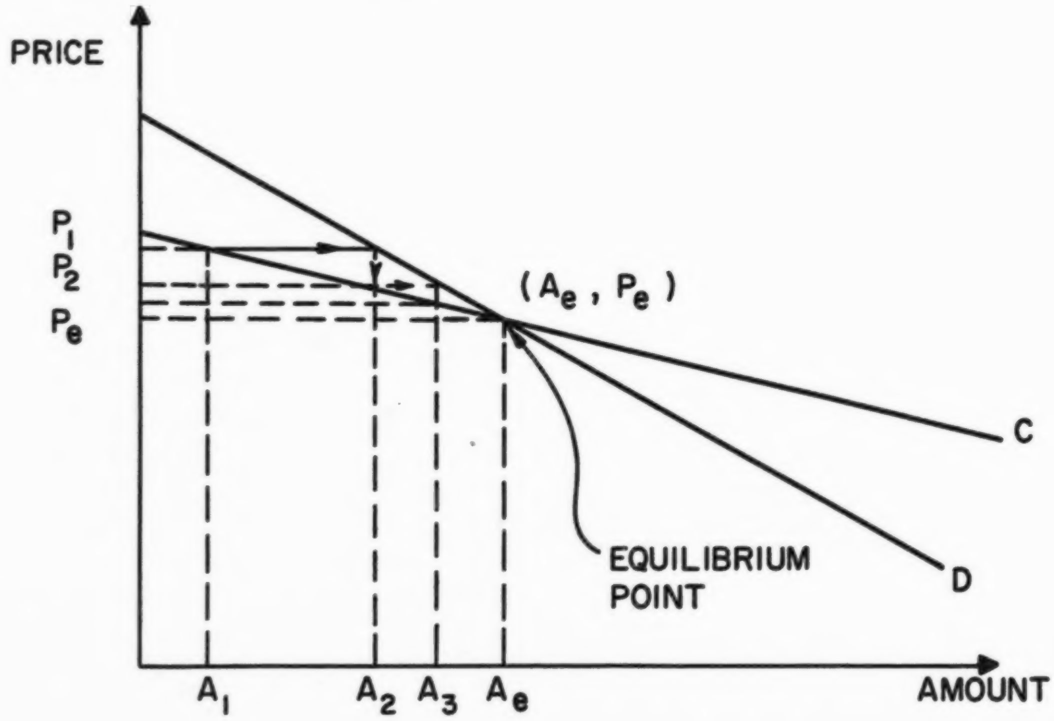


Fig. C-3.7. Stability of supply-demand point for Discount and Crafty.

will have priced himself out of business. If in any one week Discount were to increase its order above A_e, the prices will continue to drop until Crafty loses all profits. Clearly, the equilibrium is unstable in this case.

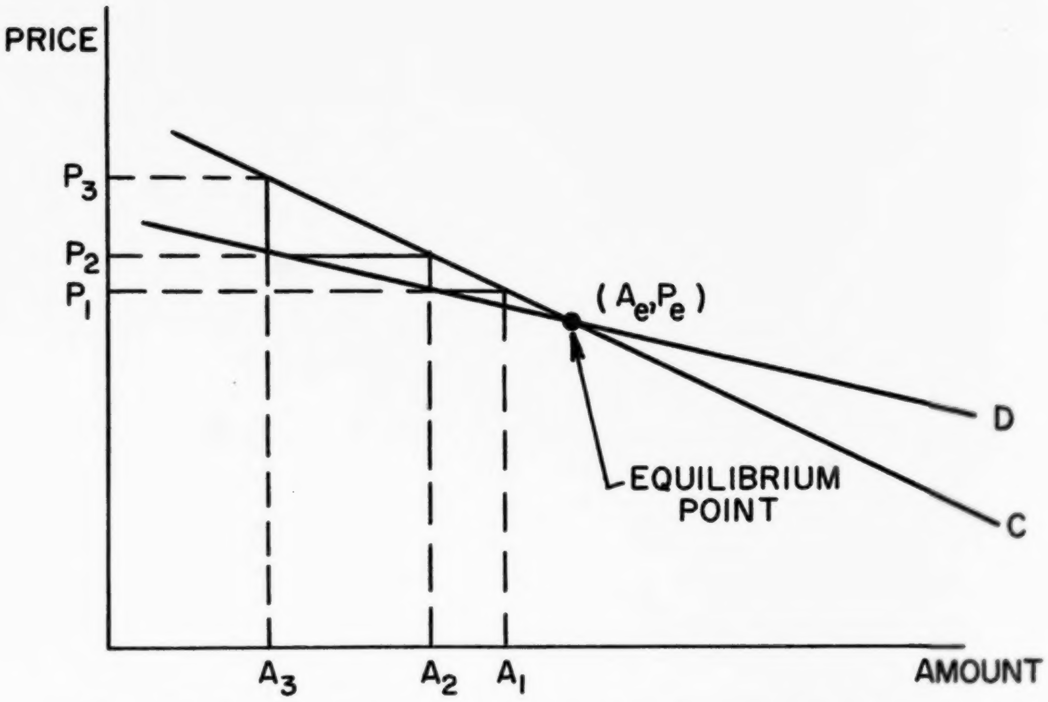


Fig. C-3. 8. Unstable "supply-demand" point.

Examination of Figs. C-3. 7 and C-3. 8 shows that the stability criterion is a very simple one in terms of our model. For stability, Discount's buying curve must be steeper than Crafty's selling curve.

The strategy that we considered is not the only one possible. Mr. Crafty may believe that when Discount increases his purchase, the toy sold out early during the week and is apparently in great demand. He may, in fact, ask a higher price. This is a less likely selling strategy and is shown in Fig. C-3. 9.

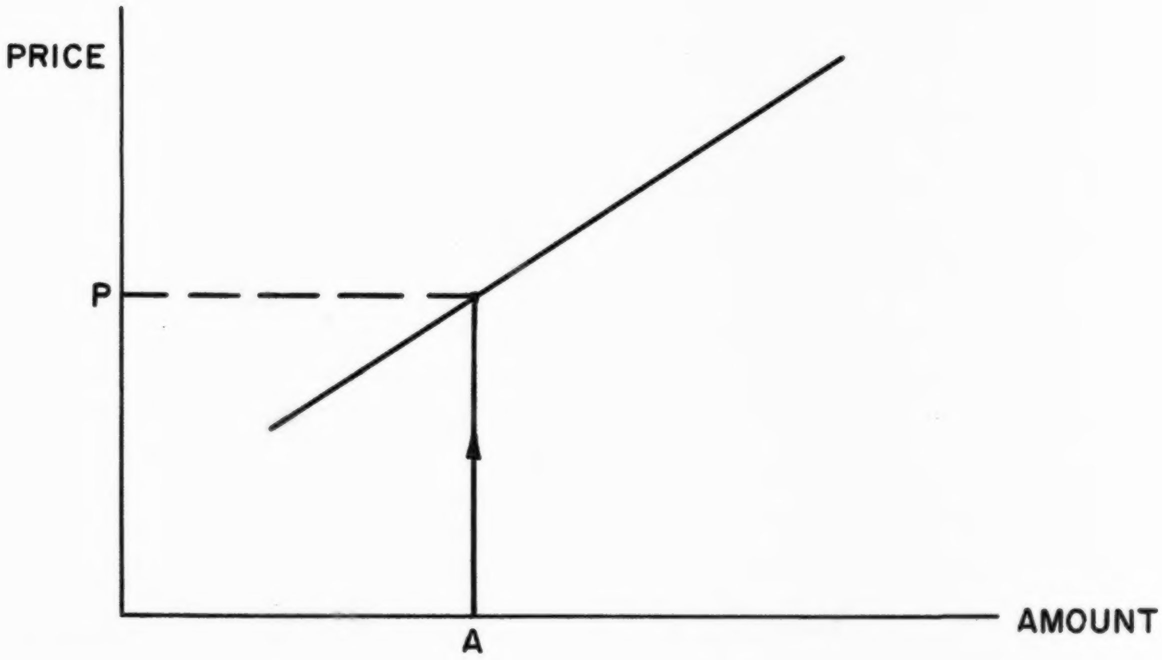


Fig. C-3. 9. Mr. Crafty's unlikely selling strategy.

Likewise, Discount may have observed that their customers have peculiar buying habits; the customers believe that when an item is high priced, the item must be of high quality and therefore they buy at an increased rate. This buying curve is shown in Fig. C-3. 10.

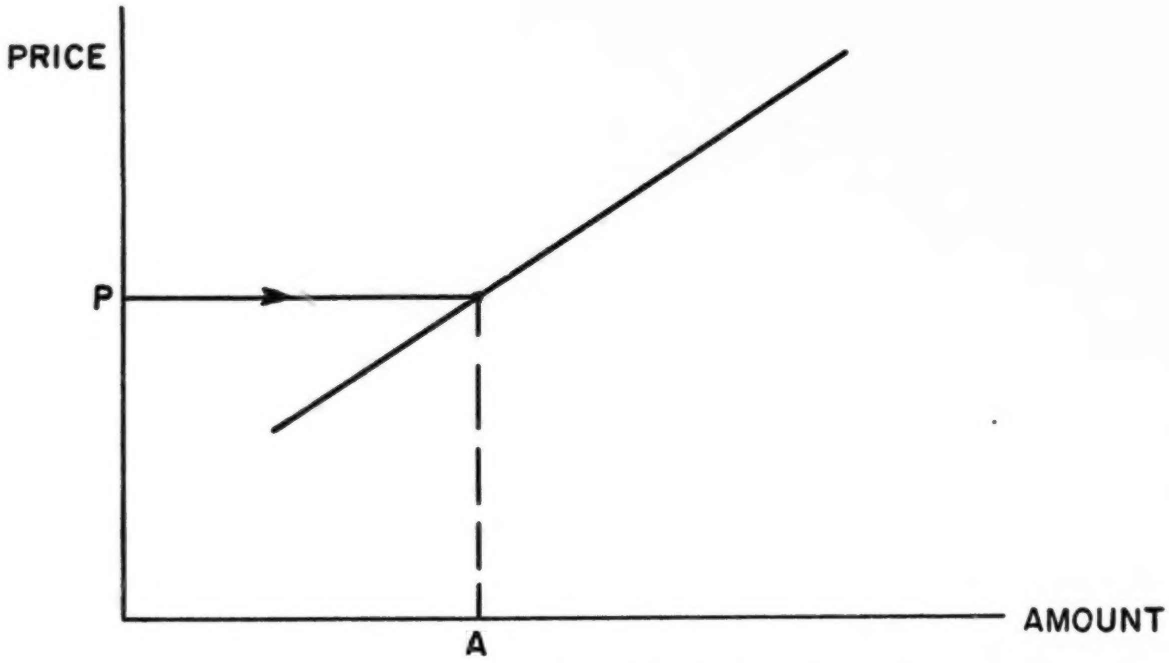


Fig. C-3. 10. Discount Stores' unlikely buying strategy.

Let us consider the stability of equilibrium when either Crafty or Discount adopt an unlikely strategy. Suppose, for example, that Crafty adopts his unlikely strategy and we have the case shown in Fig. C-3. 11. In this case a reduction to A_1 of Discount's order would cause a cyclic pattern of buying and selling. Can you explain why? The astute customer in Discount's store would notice that for some reason the price of Crafty's toy fluctuated between P_1 and P_2 every two weeks. This cycle would in fact be another equilibrium state, in addition to the point (A_e, P_e).

In Fig. C-3. 11 the slopes of the C and D curve are exactly the same magnitude but in opposite directions resulting in the cyclic behavior shown.

If the magnitude of the slope of the D curve be slightly greater than that of the C curve, then the situation shown in Fig. C-3. 12 occurs. (The equal slope case is shown in dotted lines for comparison.) Fig. C-3. 12 indicates that point A_e, P_e is stable.

Fig. C-3. 13 indicates that if the magnitude of the slope of the D curve is less than that of the C curve, the equilibrium point A_e, P_e is unstable.

We have thus considered two special cases. Two other possibilities remain: one in which Discount uses his unlikely strategy (Fig. C-3. 10) while Crafty uses his likely strategy (Fig. C-3. 5) or both Discount and Crafty go to their unlikely strategies.

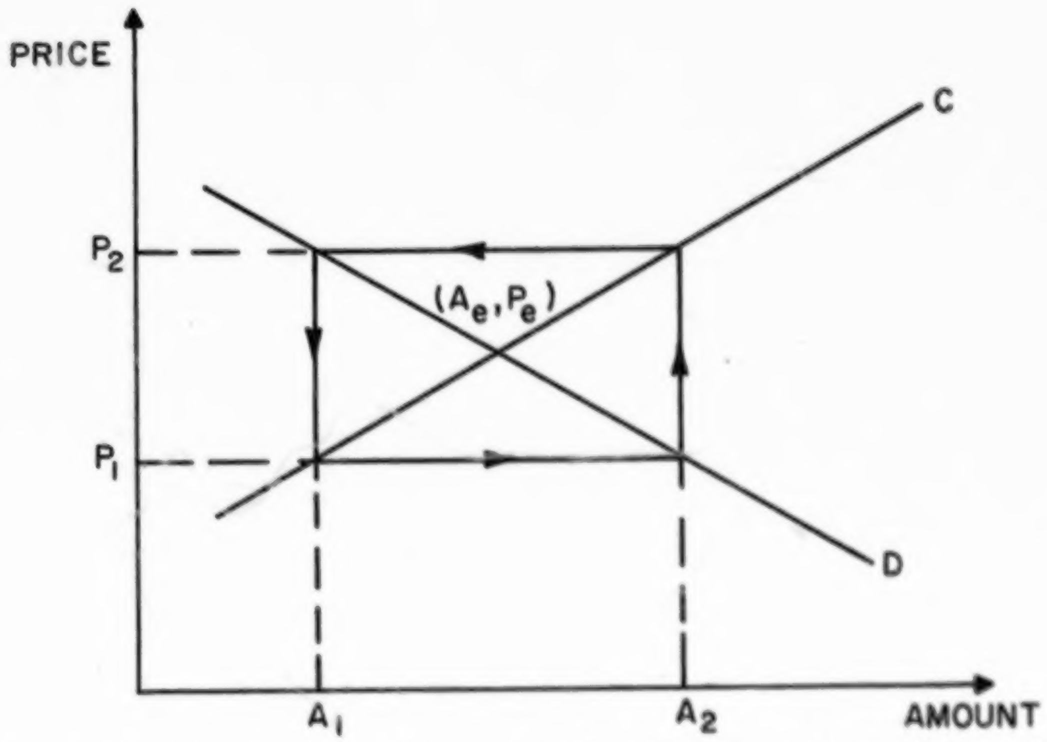


Fig. C-3.11. Supply-demand when Crafty adopts his unlikely strategy, and Discount adopts his likely strategy (equal slope magnitude)

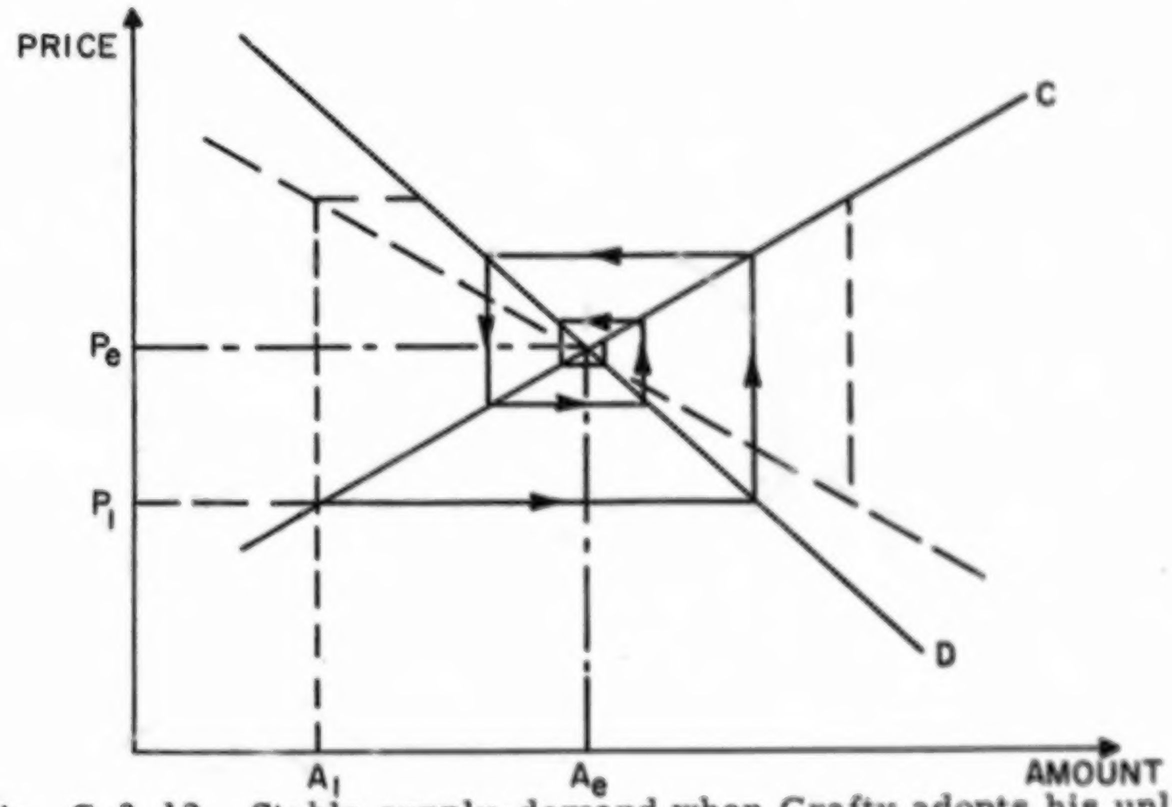


Fig. C-3.12. Stable supply-demand when Crafty adopts his unlikely strategy, and Discount adopts his likely strategy (increased slope magnitude for D)

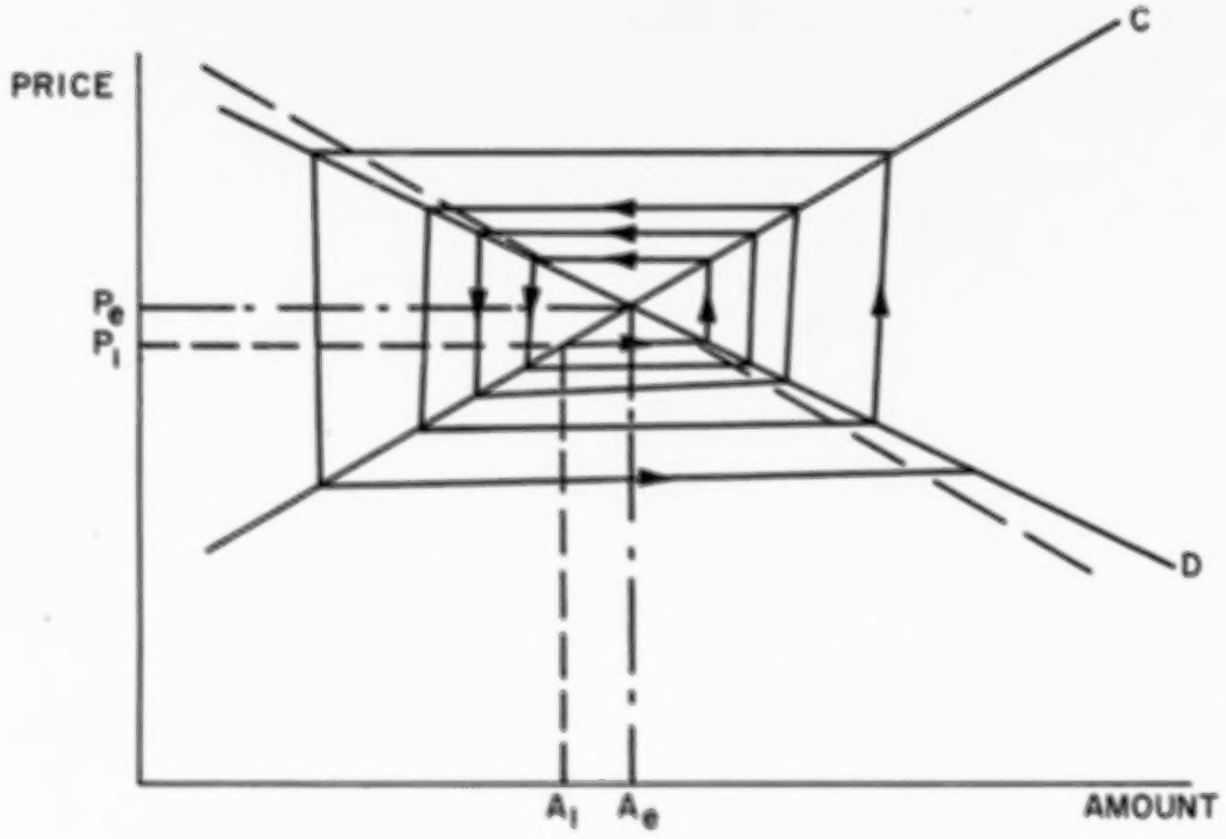


Fig. C-3.13. Unstable supply-demand when Crafty adopts his unlikely strategy. (decreased slope magnitude for D)

With a graphical analysis in the two remaining possibilities, similar to those outlined above, it can be shown that for all cases, the stability criterion depends upon a very simple rule: The equilibrium point, A_e, P_e, will be stable if the magnitude of the slope of Discount's buying strategy exceeds that of Crafty's selling strategy. Comparison of Fig. C-3.12 and C-3.3 will help to clarify the meaning of this statement. With the use of a simple model we have thus discovered a fact that might not otherwise be apparent, namely, the slope or rate at which Crafty and Discount change price with respect to the quantity purchased determines whether or not the equilibrium price is stable.

When the equilibrium point is stable, a slight disturbance will bring about "forces," in this case psychological or economic forces stemming from Discount's and Crafty's business sense, which will tend to restore equilibrium. On the other hand, for an unstable situation, a slight disturbance will bring about forces that will drive the price even further away from equilibrium.

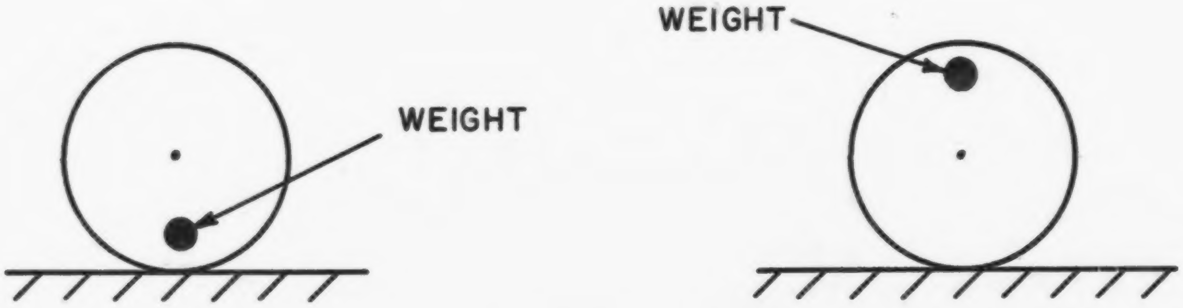
In closing this section, we again caution the reader that we are searching for insight into the stability phenomenon, rather than for a standard procedure that will predict the market-place happenings to within a few percent. Our model omits all probabilistic considerations. It does not consider that Crafty and Discount, will probably quickly guess each other's strategy and revise their original strategies accordingly.

Nevertheless, situations such as those pictured in our graphs are observed to occur in a free-market economy. The interested reader is referred to

Economics, 6th edition, by P. A. Samuelson, for a more extended discussion of the so-called "dynamic cobweb" of Figs. C-3.11 and C-3.12 and its relation, for example, to the statistics of corn and hogs and the observed "corn-hog cycle."

C-3.5. INSTABILITY IN PHYSICAL SYSTEMS

The detailed, quantitative study of instability has its roots in the exploitation of physical systems--bridges, buildings, electrical controls--for man's ends. The wheel with an off-center weight is a simple example. On level ground, two equilibrium conditions are obviously possible, as shown in Fig. C-3.14.



(a) Stable, weighted wheel (b) Unstable, weighted wheel

Fig. C-3.14.

From experience, we know that the equilibrium condition shown in Fig. C-3.14 (a) is stable and that shown in part (b) of the figure is unstable. A slight disturbance of the wheel when it is in the condition shown in (a) merely causes the wheel to rock back and forth until it settles into its original position. On the other hand, a slight disturbance of the wheel when the wheel is in condition (b) causes the wheel to turn further from its original position so that the weight moves to the bottom. The wheel then rocks back and forth until it comes to rest as in condition (a). Thus, state (a) represents stable equilibrium, while (b) depicts an unstable situation.

A similar situation exists in a less obvious form in the following situation. In bridge construction, the road deck must be permitted to expand and contract as the temperature changes. To permit this to occur safely, the bridge deck design allows the bridge to vary its length at the points of support. In one case, the bridge deck was supported by rockers placed under the two ends, as shown in Fig. C-3.15. While this bridge superstructure was in the process of construction collapse occurred, which killed several workmen and incurred costly damages. Collapse could have been avoided if the designer had been familiar with the principle illustrated in Fig. C-3.14. Here the bridge deck is supported by rockers which can tilt as the length of the deck changes. This is shown in Fig. C-3.15 and in detail in Fig. C-3.16. The bottom of the wedge shaped rocker is shown as a portion of a circular arc. The circle, from which the arc is cut, is centered at a point below the pin joint which connects the girders with the rocker. With no

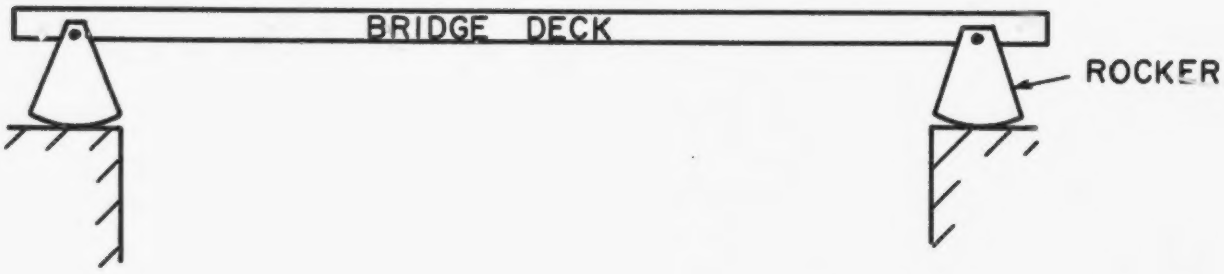


Fig. C-3. 15 Bridge deck on rockers to allow for thermal expansion.

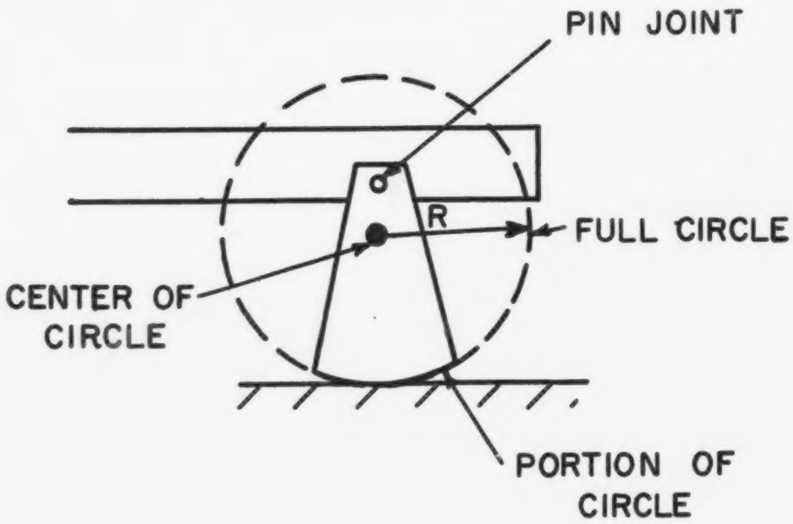


Fig. C-3. 16 Details of rocker and pin joint.

connection to the bridge deck the only forces acting on the rocker are its own weight Mg, and the reaction force from the earth (Fig. C-3. 17). The weight Mg acts from a point below the center of the circle, called the center of gravity of the rocker. The rocker itself is stable. When the deck is pinned to it, the much larger deck weight, W, acts above the center of the circle, to produce a situation similar to that in Fig. C-3. 14 (b), where the weight of the wheel is negligible in comparison to the applied weight W. It was the result of a condition of this type that a small disturbance sent thousands of dollars of material and effort crashing to the ground.

The effect of a disturbance on the equilibrium of the structure which has just been described can be examined in detail. For simplicity, we assume that the weight, of the deck, W can be considered as concentrated at a single point (the center of gravity). In Fig. C-3. 18, (a) shows a slight disturbance of the wheel when the center of gravity is below the center of wheel, (b) when it is at the center of the wheel, and (c) when it is above the center of the wheel. In all cases there will be a reaction force W^1 from the earth on the wheel.

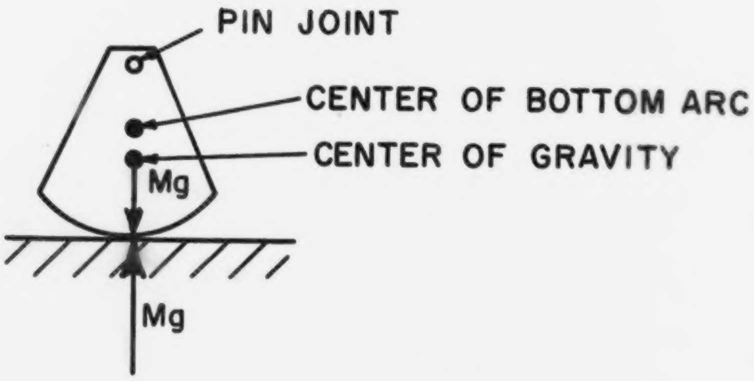


Fig. C-3. 17 Forces on the unloaded rocker.

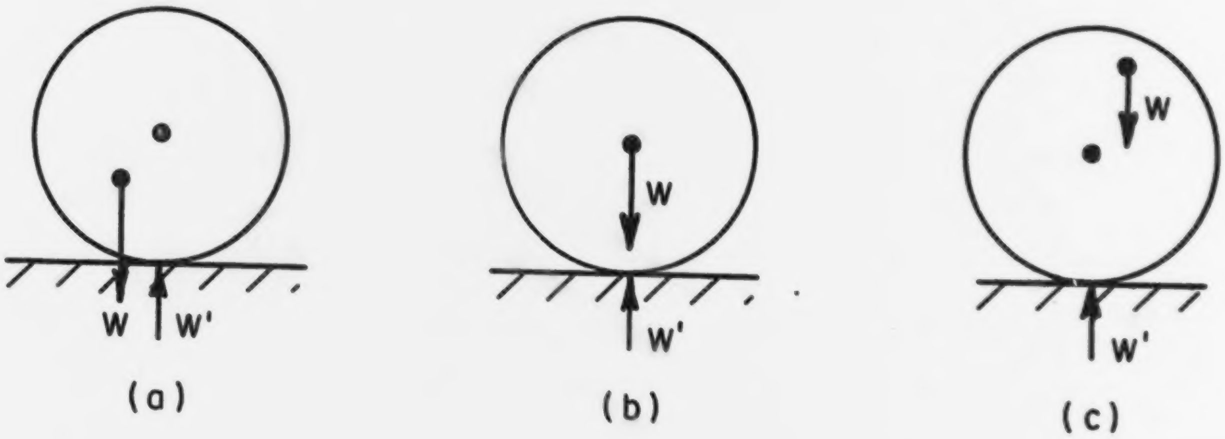


Fig. C-3. 18 Slightly disturbed wheel when the weight is below, at, and above the wheel's center.

In both (a) and (c) the forces W and W^1 create a torque which tends to turn the wheel. In (a) the unbalance tends to rotate the wheel counterclockwise to restore it to a position where the center of gravity is at its lowest position. On the other hand, in (c) the unbalance tends to rotate the weight clockwise to turn the wheel further from its original position. In the borderline or critical case (b), a disturbance creates no torque.

This simple example, which can be readily demonstrated with rockers made of wood and a wooden deck loaded with weights, displays the same characteristics that we have met in our epidemiology and economic examples. In the stable configuration (a), a disturbance away from the equilibrium state evokes torques that tend to restore the system to equilibrium. In the unstable state (c), the evoked torques tend to drive the system further from the equilibrium state.

As mass is added to the bridge deck, both the magnitude of the effective weight and the height of the center of gravity Fig. C-3. 18 is increased. With a sufficient added mass, the vertical position of the center of gravity is above the center of the wheel. At this point, the system passes suddenly and abruptly from a stable to an unstable condition.

An example of instability in a physical system which is of great engineering importance is aerodynamic or wind induced instability of structures. This sort of instability has come to the fore only relatively recently as the result of sophisticated structural design of bridges and airplanes.

This effect can be illustrated with a small electric fan and a spring supported half-round section, (Fig. C-3. 19). When the air stream from the fan is directed at the flat side of the section, oscillation of the section will begin and grow with ever increasing amplitude until the section either strikes the frame or the springs collapse.

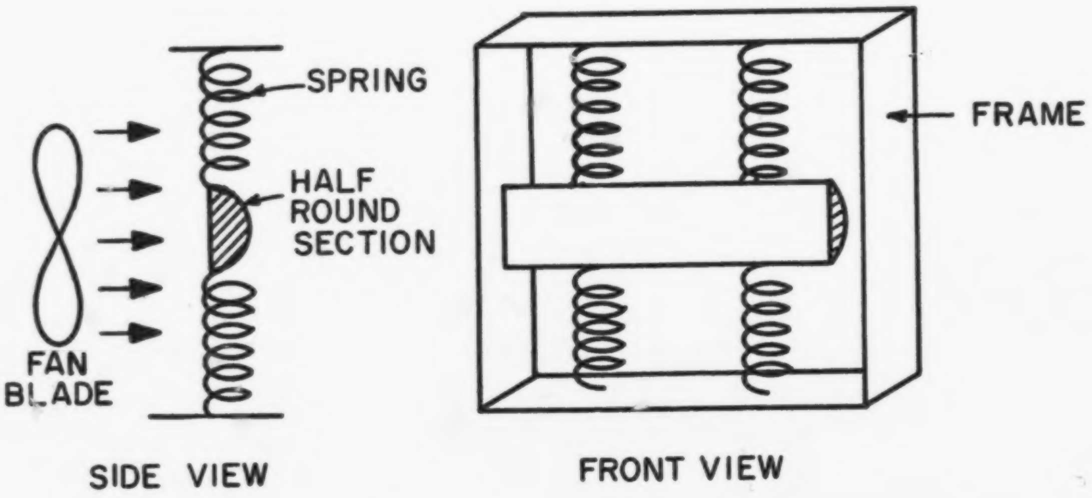


Fig. C-3. 19 Sketch of model showing aerodynamically induced instability.

This phenomenon can be explained. Wind tunnel tests show that the direction of the wind-induced (aerodynamic) forces on an object placed in an air stream vary with the shape of the object. For a hemicylindrical object, which exposes its flat surface to the air stream, the aerodynamic force always produces a component in the same direction as the motion of the object. Hence, when the object is moving upward, the wind-induced force tends to increase its upward motion; when it is moving downard, the wind-induced force tends to increase its downward motion. A slight disturbance of the hemicylinder which moves it from its position of rest results in a gradual increase of this until movement of the hemicylinder strikes the frame. It is interesting to note that when the air stream strikes the rounded surface of the section, violent oscillation does not occur.

Aerodynamic instability has been the cause of bridge and airplane failures in which millions of dollars and many human lives have been lost.

C-3.6 USES OF INSTABILITY

Thus far we have discussed the phenomena of stability and instability, with examples in which instability represented an undesireable feature. We cannot, however, draw general conclusions about the overall desireability of any phenomena. In older societies the social structure is often stratified. Individuals are not permitted to attain social positions which are considered as "above their natural station in life." A society of this type may be considered as highly stable and such stability was considered important. In the colonies which preceded the establishment of the United States, a handful of people were confronted with the task of developing enormous natural resources. A growing population was beneficial. In terms of the population model of Section B-1, this growth was a phenomenon typical of instability. Under these conditions instability is a desireable feature.

As we have seen, a system which moves further from its initial state when it is disturbed, is an unstable system. If this change is rapid, and produces a sharp demarcation between the initial and final condition, it may have useful applications. Perhaps the simplest application of this behaviour is displayed by a mechanical switch, such as a light switch, shown in Fig. C-3.21. In Fig. C-3.21 (a), the switch bar is in an unstable equilibrium. It is impossible to keep the switch bar in this position. Instead, the switch bar prefers one of the two equilibrium positions on either side, shown in (b) and (c). Thus by using the property of an unstable equilibrium we produce a device that is definitely either on or off.

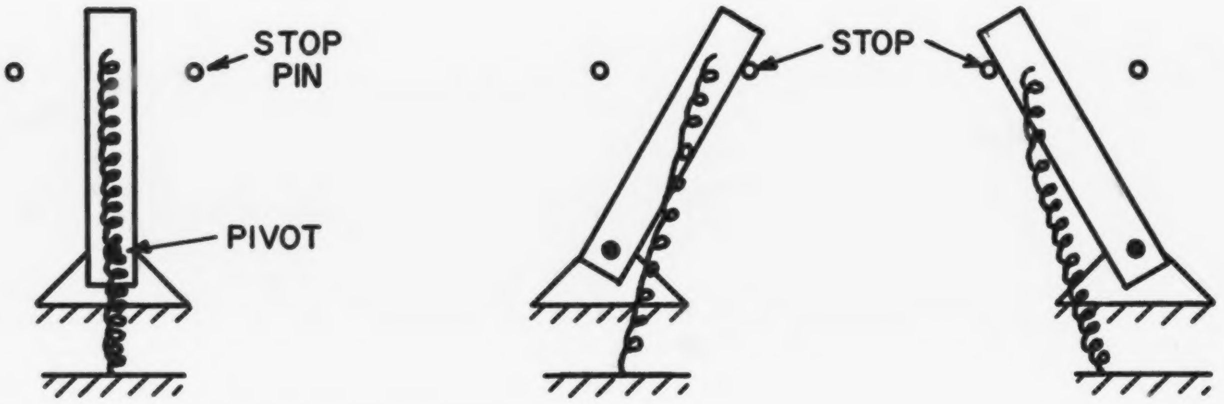


Fig. C-3.20 Schematic drawing of a light switch.

C-3.7 SUMMARY

In creating systems to serve man's ends or in coping with the world as we find it it is not enough to seek a desired operating state. We must also concern ourselves with the stability of such a state. If the state is unstable, a small disturbance will initiate a rapid shift from the desireable state. Forces--physical, psychological, economic, or biological--will be evoked which tend to move the system away from the desired state. On the other hand, if the desired state is stable, the evoked force will tend to restore the system to the desired state.

The phenomenon of stability surrounds us. Traditionally man has studied stability in connection with physical systems where the stability phenomenon could be examined in its simplest form. But as we have observed in such diverse examples as epidemiology and economics, stability plays an important role in much broader areas of human activity, although it is only recently that man has learned to apply the quantitative techniques of modeling to study the phenomenon of stability in other than physical systems.

CHAPTER C-3 -- PROBLEMS

C-3. 1. Corresponding to the adage "skyscrapers beget skyscrapers" is the saying "superhighways beget superhighways." Explain the meaning of this saying. Interpret it as an instability phenomenon, giving some of the factors that have led to the accelerated rate of superhighway construction.

C-3. 2. "Homogeneous grouping" is a plan for increasing the efficiency of education. According to this plan, students are grouped in separate classes according to ability. Proponents of the plan envision an equilibrium state where each pupil performs at his maximum ability, and is neither held back by the slower students nor frustrated by faster students. Opponents claim this equilibrium is unstable. They say, "Suppose for some reason, that a student in the fastest class performs slightly below his ability. He is then assigned to a second class which is slower and where he is bored, after which. . ." Finish the opponent's argument. State reasons why you agree or disagree with it.

C-3. 3. Two nations, A and B, are glowering at each other, each threatening war. The commanding general of nation A reads his agent's intelligence reports each week and for the next week calls up an additional number of troops equal to 10% of B's strength. The commanding general of nation B does likewise. Formulate this situation as a system of equations, letting N_A and NB be the number of A and B's troops (ans. $\Delta N_A = 0.10 N_B$, $\Delta N_B = 0.10 N_A$).

C-3. 4. Assume in problem C-3. 4 that A and B each start with 1, 000 troops at the border. How many troops will each have after 10 weeks? (ans. 2, 356).

C-3. 5. Calculate the course of an epidemic for the simple model that does not include the effect of recovery. Assume $s = 150$, $i = 1$, and $f_i = 0.02$.

C-3. 6. Calculate the course of the epidemic given in Table C-3. 2 under the assumption that $f_r = 0.2$.

C-3. 7. What is the critical number of susceptible if $f_i = 10^{-3}$ and $f_r = 0.6$?

C-3. 8. Suppose you had determined f_i and f_r in the epidemic model by examining statistics for a smallpox epidemic in Asia. Could these values be used to predict the course of a smallpox epidemic in New York City? Give reasons for your answer.

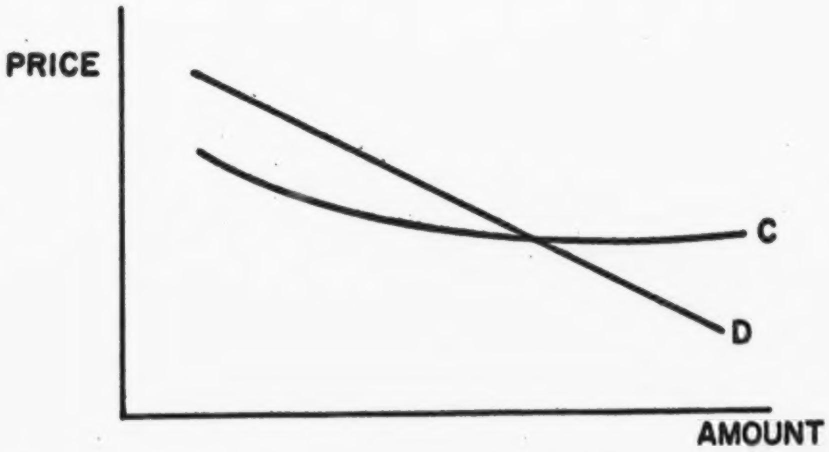
C-3. 9. Which is more effective as far as the critical number of susceptibles S_{cr} is concerned - increasing the recovery-isolation factor f_r by 50% or decreasing the factor f_i by 50%? (Decreasing f_i by 50%).

C-3. 10. Assume that you are the president of Discount Stores. You believe that it is important to present an image of stable prices. You find that Crafty's toy prices have been fluctuating widly. The accounting department has provided you with the following data:

Date	Crafty's Price	No. Bought
7/1	$2.00	250
7/8	2.50	188
7/15	1.88	265
7/22	2.65	169
7/29	1.69	289

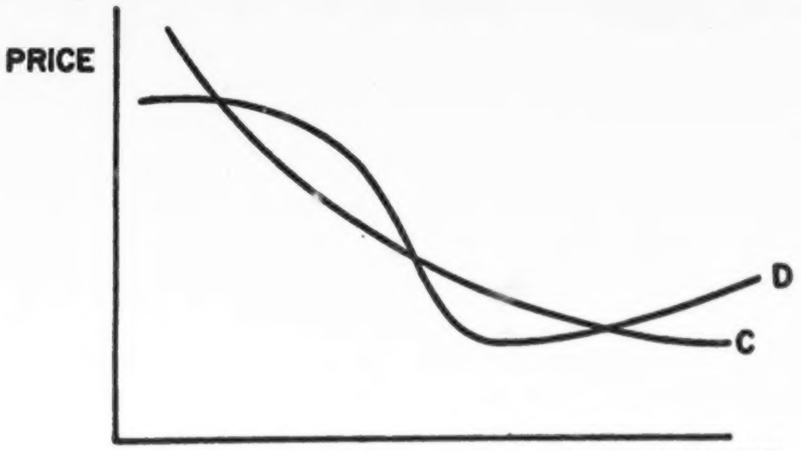
How would you instruct your purchasing department in order to stabilize Crafty's price?

C-3.11. Discuss the stability of the following Crafty-Discount supply-demand curve.



Problem C-3.11.

C-3.12. Find the equilibrium points and discuss the stability of the following Crafty-Discount supply-demand curves.



Probelm C-3.12.